DOI:10.25092/baunfbed. 984440 **J. BAUN Inst. Sci. Technol., 24(2), 462-482, (2022)**

New approximate-analytical solutions to nonlinear time-fractional partial differential equations via homotopy perturbation Elzaki transform method

Halil ANAÇ*,

Department of Mathematical Engineering, Faculty of Engineering and Natural Sciences, Gumushane University, Gumushane, Turkey.

Geliş Tarihi (Received Date): 18.08.2021
Kabul Tarihi (Accepted Date): 24.01.2022

Abstract

Some nonlinear time-fractional partial differential equations are solved by homotopy perturbation Elzaki transform method. The fractional derivatives are defined in the Caputo sense. The applications are examined by homotopy perturbation Elzaki transform method. Besides, the graphs of the solutions are plotted in the MAPLE software. Also, absolute error comparison of homotopy perturbation Elzaki transform method and homotopy perturbation Sumudu transform method solutions with the exact solution of nonlinear time-fractional partial differential equations is presented. In addition, this absolute error comparison is indicated in the tables. The novelty of this article is the first analysis of both the gas dynamics equation of Caputo fractional order and the Klein-Gordon equation of Caputo fractional order via this method. Thus, homotopy perturbation Elzaki transform method is quick and effective in obtaining the analytical solutions of time-fractional partial differential equations.

Keywords: *Klein-gordon equation, homotopy perturbation elzaki transform method, mittag-leffler function, caputo fractional derivative.*

* Halil ANAÇ, halilanac0638@gmail.com, https://orcid.org/0000- 0002-1316-3947

Homotopi pertürbasyon Elzaki dönüşümü yöntemi ile doğrusal olmayan zaman-kesirli kısmi diferansiyel denklemler için yeni yaklaşık analitik çözümler

Öz

Bazı doğrusal olmayan zaman-kesirli mertebeden kısmi diferansiyel denklemler, homotopi pertürbasyon Elzaki dönüşümü yöntemi ile çözülmüştür. Kesirli türevler Caputo anlamında tanımlanmıştır. Uygulamalar homotopi pertürbasyon Elzaki dönüşümü yöntemi ile incelenmiştir. Bunun yanında, çözümlerin grafikleri MAPLE yazılımında çizdirilmiştir. Ayrıca homotopi pertürbasyon Elzaki dönüşümü yöntemi ve homotopi pertürbasyon Sumudu dönüşümü yöntemi çözümlerinin, lineer olmayan zaman-kesirli mertebeden kısmi diferansiyel denklemlerin tam çözümü ile mutlak hata karşılaştırması sunulmaktadır. Ek olarak, bu mutlak hata karşılaştırması tablolarda belirtilmiştir. Bu makalenin yeniliği, hem Caputo kesir dereceli gaz dinamiği denkleminin hem de Caputo kesir dereceli Klein-Gordon denkleminin bu yöntemle ilk analizidir. Bu nedenle, homotopi pertürbasyon Elzaki dönüşümü yöntemi, zaman-kesirli mertebeden kısmi diferansiyel denklemlerin analitik çözümlerinin elde edilmesinde hızlı ve etkilidir.

Anahtar kelimeler: *Klein-gordon denklemi, homotopi pertürbasyon elzaki dönüşüm metodu, mittag-leffler fonksiyonu, caputo kesirli türevi.*

1. Introduction

Recently, fractional differential equations (FDEs) have been attracted in many scientific fields [1-4]. The fractional approach is generally used to model many problems and it is largely applied a lot of problems in mechanics, anomalous diffusion, wave propagation and turbulence, etc. [5-6]. Thus, many scientists intensively study on the fractional calculus and improve this calculus. One of the most important advantages of using FDEs which include non-local property is that they demonstrate new properties for many problems. There are the difficulties in solving nonlinear FDEs. Since most of the FDEs cannot be analytically solved, the effective and powerful numerical methods have been developed. However, the number of these methods is quite insufficient.

Nevertheless, the fractional order can vary according to time and space. This case directs to another quick improving area of FPDEs which have variable order fractional operators [7-12]. Some powerful numerical techniques have been developed in the literature and many leading researchers have made contributions in this field. A few of these methods are adomian decomposition method (ADM) [13], homotopy perturbation method (HPM) [14-16], collocation method [17-19], Sumudu transform method (STM) [20-21], differential transformation method (DTM) [22-26], variational iteration method (VIM) [27].

Elzaki transform method (ETM) which proposed by Elzaki has been applied to constant coefficients linear ordinary differential equations [28]. Also, Elzaki used differential transform method with Elzaki transform (ET) to solve some nonlinear differential equations [29]. Homotopy perturbation Elzaki transform method (HPETM) is firstly established by Elzaki and Hilal [30]. Also, three nonlinear PDEs have been solved by the HPETM [30]. Elzaki and Kim solved to radial diffusivity and shock wave equations by a new hybrid method which is combined ET and the new modified variational iteration method [31]. Aggarwal et al. used ET to acquire the solutions of linear Volterra integral equations of first kind [32]. Jena and Chakraverty applied HPETM to acquire the solution for the system of time-fractional Navier-Stokes equations [33]. There are many recent studies on partial differential equations or fractional partial differential equations in different methods [34-42].

The purpose of this work is to give the applications of homotopy perturbation Elzaki transform method to obtain the numerical solutions of Gas-dynamics and Klein-Gordon equations. Thus, it is observed that numerical solutions of FPDEs obtain both fastly and efficiently via an current method. Since there are very few studies in the literature in which this method is applied to fractional partial differential equations, it was necessary to conduct this study. This study has been carried out with the aim of both giving a new perspective to the existing solutions of the equations and benefiting them in future studies. The novelty in this paper is that, these equations will be handled for the first time by including Caputo fractional derivative with Mittag-Leffler type kernel to inquire more about the nature of them. Besides, the effectiveness and validity of the HPETM for FPDEs is explained.

The rest of the study is listed in the following. In Section 2, basic definitions of fractional derivatives, Elzaki transforms of the partial derivatives are presented. The homotopy perturbation Elzaki transform method is presented in Section 3. In Section 4, the applications are demonstrated for nonlinear Gas-dynamics and Klein-Gordon equations. The conclusion is introduced in Section 5.

2. Preliminaries

A few basic definitions are given in this section.

Definition 2. 1. [21] The Riemann-Liouville fractional integral operator of order $a \geq 0$, of a function $f \in C_\mu, \mu \geq -1$ is as follows

$$I^a f(x) = \begin{cases} \frac{1}{\Gamma(a)} \int_0^x (x-t)^{a-1} f(t)dt, & a > 0, x > 0, \\ I^0 f(x) = f(x), & a = 0, \end{cases} \tag{1}$$

where $\Gamma(.)$ is Gamma function.

Let $f \in C_\mu, \mu, \gamma \geq -1, \alpha, \beta \geq 0$. Then, there are two properties of the operator I^a as follows **[31]**

(1) $I^a I^\beta f(x) = I^\beta I^a f(x) = I^{a+\beta} f(x)$,

(2) $I^a x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

Definition 2. 2. [21] The Caputo fractional derivative (CFD) of $f(x)$ is as follows

$$D^{a}f(x)=I^{a-n}D^{n}f(x)=\frac{1}{\Gamma(n-a)}\int_{0}^{x}(x-t)^{n-a-1}f^{(n)}(t)dt, \tag{2}$$

where $n-1<a\leq n,\ n\in N,\ x>0, f\in C_{-1}^{n}$.

Two features of the operator D^{a} are given as **[21]:**

(1) $D^{a}I^{a}f(x)=f(x)$,

(2) $I^{a}D^{a}f(x)=f(x)-\sum_{k=0}^{n-1}f^{(k)}(0^{+})\frac{x^{k}}{k!}, x>0$.

Definition 2. 3. [21] The Mittag-Leffler function E_{a} is given by

$$E_{a}(z)=\sum_{n=0}^{\infty}\frac{z^{a}}{\Gamma(na+1)}, a>0. \tag{3}$$

Definition 2. 4. [28] The ET of the function $f(t)$ is defined by

$$T(v)=E[f(t)]=v\int_{0}^{\infty}f(t)e^{-\frac{t}{v}}dt, t>0\,. \tag{4}$$

Definition 2. 5. [30] The Elzaki transforms of the partial derivatives are as follows

$$E\left[\frac{\partial f(x,t)}{\partial t}\right]=\frac{1}{v}T(x,v)-vf(x,0), E\left[\frac{\partial^{2}f(x,t)}{\partial t^{2}}\right]=\frac{1}{v^{2}}T(x,v)-f(x,0)-v\frac{\partial f(x,0)}{\partial t},$$
$$E\left[\frac{\partial f(x,t)}{\partial x}\right]=\frac{d}{dx}[T(x,v)], E\left[\frac{\partial^{2}f(x,t)}{\partial x^{2}}\right]=\frac{d^{2}}{dx^{2}}[T(x,v)]. \tag{5}$$

Definition 2.6. [33] If $T(v)$ is the ET of the function $f(t)$, then ET of CFD is defined by

$$E[D^{\alpha}f(t)]=\frac{T(v)}{v^{\alpha}}-\sum_{k=0}^{n-1}v^{k-\alpha+2}f^{(k)}(0), n-1<\alpha\leq n. \tag{6}$$

3. Homotopy Perturbation Elzaki Transform Method

Consider the nonlinear partial differential equation with the initial condition (IC)

$$\begin{cases} D_{t}{}^{\alpha}u(x,t)+Ru(x,t)+Nu(x,t)=g(x,t), \\ \qquad 0<\alpha\leq 1, \\ \qquad u(x,0)=h(x), \end{cases} \tag{7}$$

where $D_{t}^{\alpha}u(x,t)$ is the CFD of the function $u(x,t)$, R, N are respectively linear and nonlinear differential operators, $g(x,t)$ is the source term **[33]**.

If the ET is applied to both side of Eq. (7), then it is obtained as **[33]**

$$E[D_t{}^{\alpha}u(x,t) + Ru(x,t) + Nu(x,t)] = E[g(x,t)]. \quad (8)$$

If the differential property of ET and IC are used, then Eq. (9) is obtained **[33]**

$$E[u(x,t)] = \sum_{k=0}^{n-1} v^{k+2} f^{(k)}(0) + v^{\alpha} E[g(x,t)] - v^{\alpha} E[Ru(x,t) + Nu(x,t)]. \quad (9)$$

If the inverse Elzaki transform is applied to Eq. (9), then it is obtained as **[33]**

$$u(x,t) = G(x,t) - E^{-1}\{v^{\alpha} E[Ru(x,t) + Nu(x,t)]\}. \quad (10)$$

where $G(x,t)$ shows the term which appeared from the source term and IC.

Alos, if HPM is applied to Eq. (10), it is obtained as

$$u(x,t) = G(x,t) - p(E^{-1}\{v^{\alpha} E[Ru(x,t) + Nu(x,t)]\}). \quad (11)$$

Besides, HPM

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t), \quad (12)$$

is applied and then the nonlinear term is decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \quad (13)$$

where $H_n(u)$ is He's polynomials and is given by

$$H_n(u_0, u_1, \dots, u_n) = \frac{1}{n!} \frac{\partial}{\partial p^n} \left[N\left(\sum_{i=0}^{\infty} p^i u_i \right) \right]_{p=0}, n = 0,1,2, \dots \quad (14)$$

Eq.s (12)-(13) are substituted in Eq. (11), it is obtained as

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p\left\{ E^{-1}\left\{ v^{\alpha} E\left[R \sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u) \right] \right\} \right\}. \quad (15)$$

Eq. (15) is the coupled of ET and HPM.

If the coefficients of like powers of p are compared, then the approximations are obtained:

$p^0: u_0(x,t) = G(x,t),$
$p^1: u_1(x,t) = -E^{-1}\{v^{\alpha} E[Ru_0(x,t) + H_0(u)]\},$
$p^2: u_2(x,t) = -E^{-1}\{v^{\alpha} E[Ru_1(x,t) + H_1(u)]\},$
$p^3: u_3(x,t) = -E^{-1}\{v^{\alpha} E[Ru_2(x,t) + H_2(u)]\},$

$\vdots$

Hence, the HPETM solution is obtained as

$$u(x,t) = \lim_{N\to\infty} \sum_{m=0}^{N} u_m(x,t)\,.$$

This series solutions converge too quickly in a few terms.

Now, two examples are given.

.

4. Applications

Example 4. 1. Let us consider the nonlinear time-fractional gas dynamics equation

$$\begin{cases} D_t{}^{\alpha} u(x,t) + u(x,t)u_x(x,t) = u(x,t)\big(1-u(x,t)\big), \\ \qquad 0<\alpha\le 1, \\ \qquad u(x,0) = e^{-x}. \end{cases} \tag{16}$$

If ET is implemented to Eq. (16) and the differential property of ET is used, then Eq. (17) is obtained as

$$E[u(x,t)] = v^2 e^{-x} + v^{\alpha} E\big[u(x,t)\big(1-u(x,t)\big) - u(x,t)u_x(x,t)\big]. \tag{17}$$

If the inverse ET is applied to Eq. (17), then Eq. (18) is obtained as

$$u(x,t) = e^{-x} + E^{-1}\big\{v^{\alpha} E\big[u(x,t)\big(1-u(x,t)\big) - u(x,t)u_x(x,t)\big]\big\}. \tag{18}$$

Now, HPM is applied, then it is obtained as

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = e^{-x} + p\left[E^{-1}\left\{v^{\alpha} E\left[\sum_{n=0}^{\infty} p^n H_n(u)\right]\right\}\right],$$

where $H_n(u)$ are He's polynomials which show the nonlinear terms.

The first few components of $H_n(u)$ are found as follows.

$H_0(u) = u_0(1-u_0) - u_0 u_{0x}$,
$H_1(u) = u_1 - 2u_0u_1 - u_0u_{1x} - u_1u_{0x}$,
$H_2(u) = u_2 - 2u_0u_2 - {u_1}^2 - u_0u_{2x} - u_2u_{0x} - u_1u_{1x}$,
$\vdots$

If the coefficients of like powers of p are compared, then they are obtained as

$p^0: u_0(x,t) = e^{-x}$,
$H_0(u) = e^{-x}(1-e^{-x}) + e^{-x}e^{-x} = e^{-x}$,
$p^1: u_1(x,t) = E^{-1}\big[v^{\alpha} E[H_0(u)]\big]$

$= E^{-1}\left[v^{\alpha}E[e^{-x}]\right] = e^{-x}E^{-1}[v^{\alpha+2}]$
$= e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)},$
$H_1(u) = e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)} - 2e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)}e^{-x} + e^{-x}\frac{e^{-x}t^{\alpha}}{\Gamma(\alpha+1)} + e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)}e^{-x}$
$= e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)},$
$p^2: u_2(x,t) = E^{-1}\left[v^{\alpha}E[H_1(u)]\right]$
$= E^{-1}\left[v^{\alpha}E\left[e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right]\right]$
$= e^{-x}E^{-1}\left[v^{\alpha}E\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right]\right]$
$= e^{-x}E^{-1}\left[v^{\alpha}\left[\frac{\Gamma(\alpha+1)v^{\alpha+2}}{\Gamma(\alpha+1)}\right]\right]$
$= e^{-x}E^{-1}[v^{2\alpha+2}]$
$=e^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.$
$\vdots$

Hence the solution of this problem is found as follows.

$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$
$= e^{-x} + e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + e^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots$
$= e^{-x}\left(1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right)$
$= e^{-x}E_{\alpha}[t^{\alpha}].$ (19)

In Table 1, it is compared between the HPETM and HPSTM solution. Thus, it is observed that the HPETM solution and the HPSTM solution give the same result in example 4.1. In Table 2, it is shown that the comparison of absolute error between results obtained from HPETM, HPSTM and exact solution for variable values of x and t. It is seen in Tables 1 and 2 that HPETM and HPSTM are the similar methods.

Table 1. Comparison of HPETM, HPSTM and the exact solutions, when $\alpha = 1$.

x	t	HPETM	HPSTM [43]	Exact solution
0.1	0.1	0.9999961534	0.9999961534	1.0000000000
0.2	0.1	0.9048339376	0.9048339376	0.9048374180
0.3	0.1	0.8187276038	0.8187276038	0.8187307531
0.4	0.1	0.7408153711	0.7408153711	0.7408182207
0.5	0.1	0.6703174676	0.6703174676	0.6703200460
0.6	0.1	0.6065283267	0.6065283267	0.6065306597
0.7	0.1	0.5488095251	0.5488095251	0.5488116361
0.8	0.1	0.4965833936	0.4965833936	0.4965853038
0.9	0.1	0.4493272357	0.4493272357	0.4493289641
1.0	0.1	0.4065680959	0.4065680959	0.4065696597

Table 2. Absolute error comparison of HPETM and HPSTM for Example 4.1, when $\alpha = 1$

		t					
	x	**0.001**	**0.002**	**0.003**	**0.004**	**0.005**	**0.006**
HPETM	**0.01**	$4.20x10^{-14}$	$6.73x10^{-13}$	$3.41x10^{-12}$	$1.07x10^{-11}$	$2.63x10^{-11}$	$5.46x10^{-11}$
HPSTM		$4.20x10^{-14}$	$6.73x10^{-13}$	$3.41x10^{-12}$	$1.07x10^{-11}$	$2.63x10^{-11}$	$5.46x10^{-11}$
HPETM	**0.02**	$4.25x10^{-14}$	$6.80x10^{-13}$	$3.44x10^{-12}$	$1.08x10^{-11}$	$2.65x10^{-11}$	$5.51x10^{-11}$
HPSTM		$4.25x10^{-14}$	$6.80x10^{-13}$	$3.44x10^{-12}$	$1.08x10^{-11}$	$2.65x10^{-11}$	$5.51x10^{-11}$
HPETM	**0.03**	$4.29x10^{-14}$	$6.87x10^{-13}$	$3.47x\ 10^{-12}$	$1.10x10^{-11}$	$2.68x10^{-11}$	$5.57x10^{-11}$
HPSTM		$4.29x10^{-14}$	$6.87x10^{-13}$	$3.47x10^{-12}$	$1.10x10^{-11}$	$2.68x10^{-11}$	$5.57x10^{-11}$
HPETM	**0.04**	$4.33x10^{-14}$	$6.94x10^{-13}$	$3.51x10^{-12}$	$1.11x10^{-11}$	$2.71x10^{-11}$	$5.62x10^{-11}$
HPSTM		$4.33x10^{-14}$	$6.94x10^{-13}$	$3.51x10^{-12}$	$1.11x10^{-11}$	$2.71x10^{-11}$	$5.62x10^{-11}$
HPETM	**0.05**	$4.38x10^{-14}$	$7.01x10^{-13}$	$3.55x10^{-12}$	$1.12x10^{-11}$	$2.74x10^{-11}$	$5.68x10^{-11}$
HPSTM		$4.38x10^{-14}$	$7.01x10^{-13}$	$3.55x10^{-12}$	$1.12x10^{-11}$	$2.74x10^{-11}$	$5.68x10^{-11}$
HPETM	**0.06**	$4.42x10^{-14}$	$7.08x10^{-13}$	$3.58x10^{-12}$	$1.13x10^{-11}$	$2.76x10^{-11}$	$5.74x10^{-11}$
HPSTM		$4.42x10^{-14}$	$7.08x10^{-13}$	$3.58x10^{-12}$	$1.13x10^{-11}$	$2.76x10^{-11}$	$5.74x10^{-11}$

Also, the graphs of the solutions of this problem for $\alpha = 1, \alpha = 0.9, \alpha = 0.8, \alpha = 0.7, \alpha = 0.6$ values have been respectively obtained in MAPLE software are presented in Figs. 1-5.

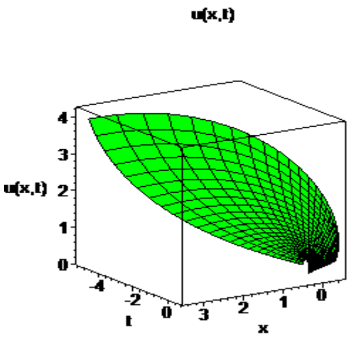

Figure 1. For $\alpha = 1$, the graph of time-dependent of Eq. (19).

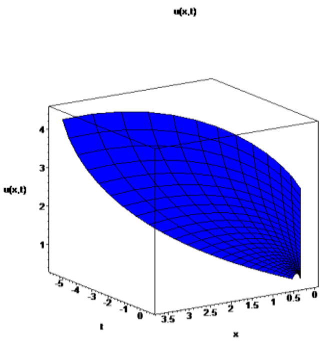

Figure 2. For $\alpha = 0.9$, the graph of time-dependent of Eq. (19).

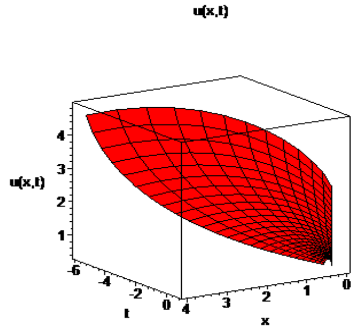

Figure 3. For $\alpha = 0.8$, the graph of time-dependent of Eq. (19).

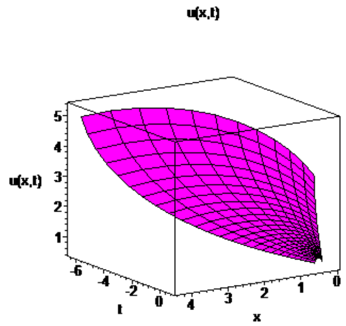

Figure 4. For $\alpha = 0.7$, the graph of time-dependent of Eq. (19).

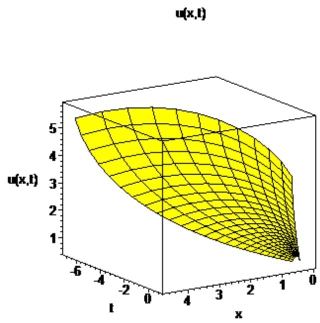

Figure 5. For $\alpha = 0.6$, the graph of time-dependent of Eq. (19).

Example 4. 2. Let us consider the nonlinear time-fractional Klein-Gordon equation

$$\begin{cases} D_t{}^{\alpha} u(x,t) - u_{xx}(x,t) + u^2(x,t) = 0, \\ \qquad 0 < \alpha \leq 1, t \geq 0, \\ \qquad u(x,0) = 1 + sinx. \end{cases} \tag{20}$$

If ET is implemented to Eq. (20) and the differential property of ET is used, then Eq. (20) is obtained as

$$E[u(x,t)] = v^2(1 + sinx) + v^{\alpha} E[u_{xx}(x,t) - u^2(x,t)]. \tag{21}$$

If the inverse ET is applied to Eq. (21), then Eq. (22) is obtained as

$$u(x,t) = (1 + sinx) + E^{-1}\{v^{\alpha} E[u_{xx} - u^2]\}. \tag{22}$$

Now, HPM is applied to Eq.(22), it is obtained as

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = (1+sinx) + p\left[E^{-1}\left\{v^\alpha E\left[\sum_{n=0}^{\infty} p^n H_n(u)\right]\right\}\right],$$

where $H_n(u)$ are He's polynomials which show the nonlinear terms.

The first few components of $H_n(u)$ are found as follows:

$$H_0(u) = u_{0xx} - u_0{}^2,$$
$$H_1(u) = u_{1xx} - 2u_1u_0,$$
$$H_2(u) = u_{2xx} - u_2u_0 - u_1{}^2,$$
$$\vdots$$

If the coefficients of like powers of p are compared, then they are obtained:

$$p^0: u_0(x,t) = 1 + \sin x,$$
$$H_0(u) = -\sin x - (1+\sin x)^2,$$
$$p^1: u_1(x,t) = E^{-1}\left[v^\alpha E[H_0(u)]\right]$$
$$= [-\sin x - (1+\sin x)^2]\frac{t^\alpha}{\Gamma(\alpha+1)},$$
$$H_1(u) = (11\sin x - 2\cos 2x + 8sin^2x + 2 + 2sin^3x)\frac{t^\alpha}{\Gamma(\alpha+1)},$$
$$p^2: u_2(x,t) = E^{-1}\left[v^\alpha E[H_1(u)]\right]$$
$$= E^{-1}\left[v^\alpha E\left[(11\sin x - 2\cos 2x + 8sin^2x + 2 + 2sin^3x)\frac{t^\alpha}{\Gamma(\alpha+1)}\right]\right]$$
$$= (11\sin x - 2\cos 2x + 8sin^2x + 2 + 2sin^3x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$\vdots$$

Hence, the solution of Eq. (20) is found as:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$
$$= 1 + \sin x + [-\sin x - (1+\sin x)^2]\frac{t^\alpha}{\Gamma(\alpha+1)}$$
$$+(11\sin x - 2\cos 2x + 8sin^2x + 2 + 2sin^3x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \quad (23)$$

In Table 3, it is compared between the HPETM and HPM solutions. Thus, it is seen that the HPETM and HPSTM solutions give the same result for constant t and variable x values in example 4.2. It is observed in Table 3 that HPETM and HPM are the similar methods.

Table 3. Comparison of HPETM and HPM solutions, when $\alpha = 1$.

x	t	**HPETM**	**HPM [44]**
0.01	0.01	0.9997043983	0.9997043983
0.02	0.01	1.0094059472	1.0094059472
0.03	0.01	1.0191036779	1.0191036779
0.04	0.01	1.0287966227	1.0287966227
0.05	0.01	1.0384838147	1.0384838147
0.06	0.01	1.0481642886	1.0481642886
0.07	0.01	1.0578370798	1.0578370798
0.08	0.01	1.0675012255	1.0675012255
0.09	0.01	1.0771557638	1.0771557638
0.10	0.01	1.0867997349	1.0867997349

Also, the following graphs of the solutions of this problem for $\alpha = 1, \alpha = 0.9, \alpha = 0.8, \alpha = 0.7, \alpha = 0.6$ values have been respectively obtained in MAPLE software.

Also, the graphs of the solutions of this problem for $\alpha = 1, \alpha = 0.9, \alpha = 0.8, \alpha = 0.7, \alpha = 0.6$ values have been respectively obtained in MAPLE software are presented in Figs. 6-10.

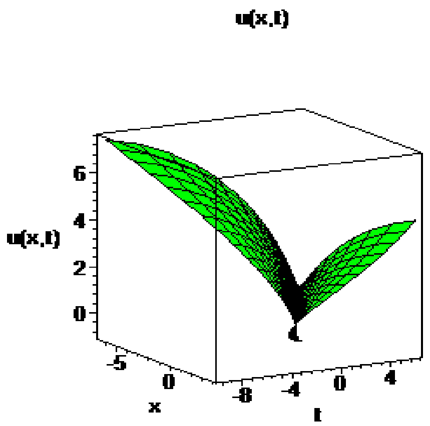

Figure 6. For $\alpha = 1$, the graph of time-dependent of Eq. (23).

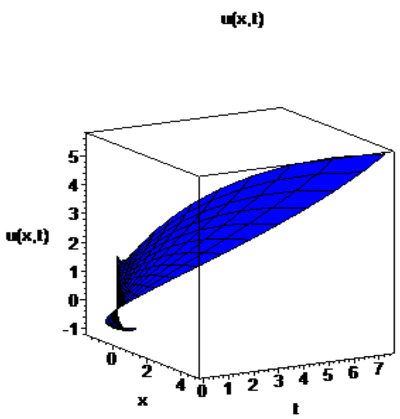

Figure 7. For $\alpha = 0.9$, the graph of time-dependent of Eq. (23).

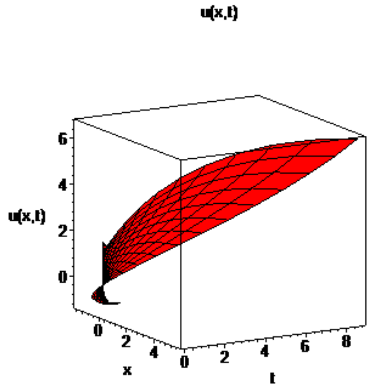

Figure 8. For $\alpha = 0.8$, the graph of time-dependent of Eq. (23).

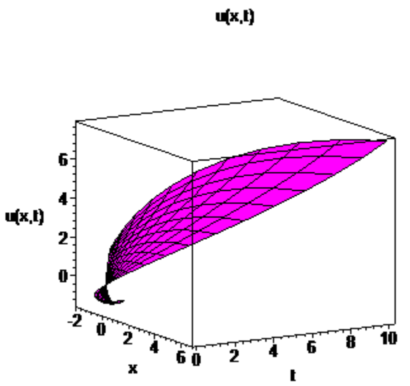

Figure 9. For $\alpha = 0.7$, the graph of time-dependent of Eq. (23).

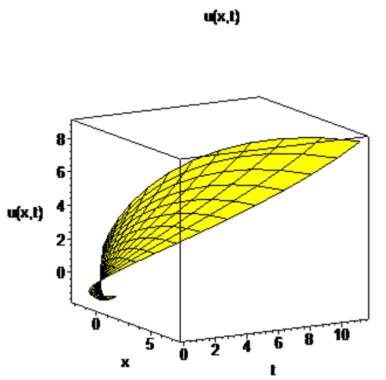

Figure 10. For $\alpha = 0.6$, the graph of time-dependent of Eq. (23).

5. Results and discussions

The graphs of the temperature $u(x,t)$ of HPETM solutions for Examples 4.1 and 4.2. Figs. 1-5 show the results obtained by using HPETM for Example 4.1. It has been observed that the solutions increase as the alpha values move away from 1. Also, it has been observed from Table 1 that HPETM and HPSTM give the same results for variable x and t values. In addition, the absolute error calculations were made for HPETM and HPSTM in Table 2. Figs. 6-10 show the results obtained by using HPETM for Example 4.2. It has been observed that the solutions increase as the alpha values move away from

1. Also, it has been observed from Table 3 that HPETM and HPSTM give the same results for variable x and constant t values.

5. Conclusion

In this paper, these nonlinear FPDEs are analyzed by HPETM. Besides, the graphs of the solutions of these equations for the different alpha values have been obtained in MAPLE software. It is seen that the general construction of the surface graphs plotted in Maple software differ for Example 4.1. Also, it is seen that the general construction of the surface graphs plotted in Maple software differ for Example 4.2. The numerical solutions of FPDEs have been fastly and successfully obtained. Therefore, it is inferred that HPETM is fast, efficient and powerful to obtain the numerical solutions for different nonlinear fractional partial differential equations.

References

[1] Hilfer, R., **Application of Fractional Calculus in Physics**, World Scientific Publishing Company, Singapore, (2000).

[2] Kilbas, A., Srivastava, H. and Trujillo, J., **Theory and Applications of Fractional Differential Equations,** Elsevier, Amsterdam, (2006).

[3] Miller, K. S. and Ross, B., **An Introduction to the Fractional Calculus and Fractional Differential Equations,** Wiley, New York, (1993).

[4] Oldham, K. B. and Spanier, J., **The Fractional Calculus,** Academic Press, New York, (1974).

[5] Metzler, R. and Nonnenmacher, T. F., Space-and time-fractional diffusion and wave equations, fractional Fokker-Planck equations, and physical motivation, **Chemical Physics,** 284, 1-2, 67-90, (2002).

[6] Morgado, M. L. and Rebelo, M., Numerical approximation of distributed order reaction–diffusion equations, **Journal of Computational and Applied Mathematics**, *275*, 216-227, (2015).

[7] Baleanu, D., Jajarmi, A., Bonyah, E. and Hajipour, M., New aspects of poor nutrition in the life cycle within the fractional calculus, **Advances in Difference Equations,** 2018, 1, 1-14, (2018).

[8] Jajarmi, A. and Baleanu, D., Suboptimal control of fractional-order dynamic systems with delay argument, **Journal of Vibration and Control,** 24, 12, 2430-2446, (2018).

[9] Jajarmi, A. and Baleanu, D., A new fractional analysis on the interaction of HIV with CD4+ T-cells, **Chaos, Solitons & Fractals,** 113, 221-229, (2018).

[10] Klimek, M., Fractional sequential mechanics-models with symmetric fractional derivative, **Czechoslovak Journal of Physics,** 51, 12, 1348-1354, (2001).

[11] Laskin, N., Fractional quantum mechanics, **Physical Review E,** 62, 3, 3135-3145, (2000).

[12] Mainardi, F., **Fractional Calculus and Waves in Linear Viscoelasticity,** Imperial College Press, London, (2010).

[13] Wazwaz, A. M., A reliable modification of Adomian decomposition method, **Applied Mathematics and Computation,** 102, 1, 77-86, (1999).

[14] He, J. H., Homotopy perturbation method: a new nonlinear analytical technique, **Applied Mathematics and Computation,** 135, 1, 73-79, (2003).

[15] He, J. H., Homotopy perturbation method for solving boundary value problems, **Physics Letters,** 350, 1-2, 87-88, (2006).

[16] He, J. H., Addendum: new interpretation of homotopy perturbation method, **International Journal of Modern Physics B,** 20, 18, 2561-2568, (2006).

[17] Yüzbaşı, Ş., A collocation method for numerical solutions of fractional-order logistic population model, **International Journal of Biomathematics,** 9, 02, 1650031, (2016).

[18] Yüzbaşı, Ş., A numerical method for solving second-order linear partial differential equations under Dirichlet, Neumann and Robin boundary conditions, **International Journal of Computational Methods,** 14, 2, 1750015, (2017).

[19] Yüzbaşı, Ş., A collocation approach for solving two-dimensional second-order linear hyperbolic equations, **Applied Mathematics and Computation,** 338, 101-114, (2018).

[20] Merdan, M., Anaç, H. and Kesemen, T., The new Sumudu transform iterative method for studying the random component time-fractional Klein-Gordon equation, **Sigma,** 10, 3, 343-354, (2019).

[21] Wang, K. and Liu, S., A new Sumudu transform iterative method for time-fractional Cauchy reaction-diffusion equation, **Springer Plus,** 5, 1, 865, (2016).

[22] Anaç, H., Merdan, M., Bekiryazıcı, Z. and Kesemen, T., Bazı Rastgele Kısmi Diferansiyel Denklemlerin Diferansiyel Dönüşüm Metodu ve Laplace-Padé Metodu Kullanarak Çözümü, **Gümüşhane Üniversitesi Fen Bilimleri Enstitüsü Dergisi,** 9, 1, 108-118, (2019).

[23] Ayaz, F., Solutions of the system of differential equations by differential transform method, **Applied Mathematics and Computation,** 147, 2, 547-567, (2004).

[24] Kangalgil, F. and Ayaz, F., Solitary wave solutions for the KdV and mKdV equations by differential transform method, **Chaos, Solitons & Fractals,** 41, 1, 464-472, (2009).

[25] Merdan, M., A new applicaiton of modified differential transformation method for modeling the pollution of a system of lakes, **Selçuk Journal of Applied Mathematics,** 11, 2, 27-40, (2010).

[26] Zhou, J. K., **Differential Transform and Its Applications for Electrical Circuits,** Huazhong University Press, Wuhan, (1986).

[27] He, J. H., Variational iteration method-a kind of non-linear analytical technique: some examples, **International Journal of Non-linear Mechanics,** 34, 4, 699-708, (1999).

[28] Elzaki, T. M., Applications of new transform "Elzaki transform" to partial differential equations, **Global Journal of Pure and Applied Mathematics,** 7, 1, 65-70, (2011).

[29] Elzaki, T. M., Solution of nonlinear differential equations using mixture of Elzaki transform and differential transform method, **In International Mathematical Forum,** 7, 13, 631-638, (2012).

[30] Elzaki, T. M. and Hilal, E. M. A., Homotopy perturbation and Elzaki transform for solving nonlinear partial differential equations, **Mathematical Theory and Modeling,** 2, 3, 33-42, (2012).

[31] Elzaki, T. M. and Kim, H., The solution of radial diffusivity and shock wave equations by Elzaki variational iteration method, **International Journal of Mathematical Analysis,** 9, 22, 1065-1071, (2015).

[32] Aggarwal, S., Chauhan, R. and Sharma, N., Application of Elzaki transform for solving linear Volterra integral equations of first kind, **International Journal of**

Research in Advent Technology, 6, 12, 3687-3692, (2018).

[33] Jena, R. M. and Chakraverty, S., Solving time-fractional Navier–Stokes equations using homotopy perturbation Elzaki transform, **SN Applied Sciences,** 1, 1, 1-16, (2019).

[34] Akgül, E. K., Akgül, A. and Yavuz, M., New illustrative applications of integral transforms to financial models with different fractional derivatives, **Chaos, Solitons & Fractals,** 146, 110877, (2021).

[35] Jena, R. M., Chakraverty, S., Yavuz, M. and Abdeljawad, T., A new modeling and existence–uniqueness analysis for Babesiosis disease of fractional order, **Modern Physics Letters B,** 35(30), 2150443, (2021).

[36] Yavuz, M. and Sene, N., Fundamental calculus of the fractional derivative defined with Rabotnov exponential kernel and application to nonlinear dispersive wave model, **Journal of Ocean Engineering and Science,** 6(2), 196-205, (2021).

[37] Veeresha, P., A numerical approach to the coupled atmospheric ocean model using a fractional operator, **Mathematical Modelling and Numerical Simulation with Applications (MMNSA)**, 1(1), 1-10, (2021).

[38] Yokuş, A., Construction of different types of traveling wave solutions of the relativistic wave equation associated with the Schrödinger equation, **Mathematical Modelling and Numerical Simulation with Applications (MMNSA),** 1(1), 24-31, (2021).

[39] Yavuz, M., Characterizations of two different fractional operators without singular kernel, **Mathematical Modelling of Natural Phenomena,** 14(3), 302, (2019).

[40] Zada, L., Nawaz, R., Nisar, K. S., Tahir, M., Yavuz, M., Kaabar, M. K. and Martínez, F., New approximate-analytical solutions to partial differential equations via auxiliary function method, **Partial Differential Equations in Applied Mathematics,** 4, 100045, (2021).

[41] Yavuz, M. and Sene, N., Approximate solutions of the model describing fluid flow using generalized ρ-laplace transform method and heat balance integral method, **Axioms,** 9(4), 123, (2020).

[42] Yavuz, M. and Özdemir, N., Numerical inverse Laplace homotopy technique for fractional heat equations, **Thermal Science,** 22(1), 185-194, (2018).

[43] Singh, J., Kumar, D. and Kılıçman, A., Homotopy perturbation method for fractional gas dynamics equation using Sumudu transform, **In Abstract and Applied Analysis**, 2013, (2013), DOI: 10.1155/2013/934060.

[44] Golmankhaneh, A. K., Golmankhaneh, A. K. and Baleanu, D., On nonlinear fractional Klein–Gordon equation, **Signal Processing,** 91, 3, 446-451, (2011).