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On Pedal and Contrapedal Curves of Bézier Curves

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Abstract

The aim of this paper is to charaterize pedal and contrapedal curves of a Bézier curve which has many applications in computer graphics and related areas. Especially, the pedal and contrapedal curve of a planar Bézier curve at the starting and the ending points are investigated. In addition, the origin is taken as a pedal point.

Keywords: *Bézier curve; Contrapedal curve; Pedal curve; Pedal point.*
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1. Introduction

Geometry of curves is very essential because it has many important applications in many different areas. Therefore, various curves and surfaces have been studied by many authors for many years. Recently, due to its different structure, Bézier curves have attracted the attention of many researchers. Bézier curves are the most significant mathematical representations of curves which are applied to computer graphics and related areas.

The plane curves in the Euclidean plane are one of the most important subjects in differential geometry. In this respect, examining the pedal and contrapedal curves is an fundamental issue. Among them, the pedal curves of regular curves have an importance and are studied by many authors in different areas of mathematics. A pedal curve (a contrapedal curve) of a regular plane curve is the locus of the feet of the perpendiculars from a point to the tangents (normals) to the curve. In the recent studies, [3] and [7] studied pedal and contrapedal curves of fronts in the Euclidean plane. In the CAGD field, a classical family of sinusoidal spirals was introduced by Ueda [8] and [9] via a pedal-point construction, and later identified as belonging to the special subset of rational Bézier curves called p-Bézier curves [6].

The rest part of the paper is given as follows: Section 2 gives some basic notations and definitions for needed throughout the study. Section 3 gives the Serret-Frenet frame of a planar Bézier curve. Section 4 characterizes pedal curve of a planar Bézier curve and investigate at end points. Section 5 constructs the contrapedal curve of a planar Bézier curve and investigate at the starting and the ending points. In the final section, we conclude our work.

2. Preliminaries

A classical Bézier curve of degree n with control points p_j is defined as

$$B(t)=\sum_{j=0}^{n}B_j^n(t)p_j, t\in[0,1] \tag{2.1}$$

$$B_{i,n}(t)=\begin{cases}\frac{n!}{(n-i)!i!}(1-t)^{n-i}t^i, & \text{if } 0\leq i\leq n\\ 0, & \text{otherwise}\end{cases} \tag{2.2}$$

are called the Bernstein basis functions of degree n. The polygon formed by joining the control points $p_0, p_1, ..., p_n$ in the specified order is called the Bézier control polygon.
It is well-known that, if a curve differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve, is called Frenet apparatus of the curves.

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Definition 2.1. *The first derivative $B'(t)$ of a degree-n Bézier curve $B(t)$ is clearly a degree $n-1$ curve. Such a curve can be written in Bézier form as*

$$B'(t)=\sum_{j=0}^{n-1}B_j^{n-1}(t)\triangle p_j \tag{2.3}$$

where $\triangle p_j = p_{j+1}-p_j$, $j=0,1,...,n-1$ are the control points of $B^{'}(t)$[4].

Definition 2.2. *Let $J:E^2\to E^2$ be a linear transformation defined by*

$$J(P_1,P_2)=(-P_2,P_1)[2]. \tag{2.4}$$

Definition 2.3. *Let $\alpha: I\to E^2$ be a non-unit speed planar curve. The Serret-Frenet frame $\{T,N\}$ and curvature κ of α for $\forall t\in I$ are defined by the following equations [2]:*

$$T(t)=\frac{\alpha'(t)}{\|\alpha'(t)\|} \qquad N(t)=\frac{J\alpha'(t)}{\|\alpha'(t)\|} \qquad \kappa(t)=\frac{<\alpha''(t),J\alpha'(t)>}{\|\alpha'(t)\|^3}. \tag{2.5}$$

Definition 2.4. *The pedal curve of a regular curve $\beta:(a,b)\to R^{^2}$ with respect to a point $p\in R^2$ is defined by*

$$\beta^*[\beta,p](t)=p+\frac{<\beta(t)-p,J\beta'(t)>}{\|\beta'(t)\|^2}J\beta'(t)[2]. \tag{2.6}$$

Definition 2.5. *The contrapedal curve of a regular curve $\beta:(a,b)\to R^{^2}$ with respect to a point $p\in R^2$ is defined by*

$$\beta^*[\beta,p](t)=p+\frac{<\beta(t)-p,\beta'(t)>}{\|\beta'(t)\|^2}\beta'(t)[2]. \tag{2.7}$$

From now on, we will say a Bézier curve instead of a non-unit speed planar Bézier curve of degree n with control points $p_0,p_1,...,p_n$ throughout the paper.

3. The Serret-Frenet Frame of a planar Bézier curve

In this section, the Serret-Frenet frame and curvature of Bézier curve are given.

Theorem 3.1. *A Bézier curve has the following Serret-Frenet frame $\{T,N\}$ and curvature κ of Bézier curve defined by (2.1) for $\forall t\in\mathbb{R}$ are*

$$T(t)=\frac{\sum\limits_{j=0}^{n-1}B_j^{n-1}(t)\triangle p_j}{\left(\sum\limits_{j,i=0}^{n-1}B_j^{n-1}(t)B_i^{n-1}(t)<\triangle p_j,\triangle p_i>\right)^{\frac{1}{2}}}, \qquad N(t)=\frac{\sum\limits_{j=0}^{n-1}B_j^{n-1}(t)J\triangle p_j}{\left(\sum\limits_{j,i=0}^{n-1}B_j^{n-1}(t)B_i^{n-1}(t)<\triangle p_j,\triangle p_i>\right)^{\frac{1}{2}}} \tag{3.1}$$

and

$$\kappa(t)=\frac{n-1}{n}\frac{\sum\limits_{j=0}^{n-2}B_j^{n-2}(t)\sum\limits_{i=0}^{n-1}B_i^{n-1}(t)<\triangle^2 p_j,J\triangle p_i>}{\left(\sum\limits_{j,i=0}^{n-1}B_j^{n-1}(t)B_i^{n-1}(t)<\triangle p_j,\triangle p_i>\right)^{\frac{3}{2}}} \tag{3.2}$$

where $\triangle^2 p_j=p_{j+2}-2p_{j+1}+p_j$ [1].

4. Pedal Curve of a planar Bézier curve

In this section, we characterize pedal curve of a planar Bézier curve and investigate this curve at $t=0$ and $t=1$.

Theorem 4.1. *The pedal curve $B^*(t)$ of a Bézier curve defined by (2.1) for $\forall t\in R$ and pedal point p is*

$$B^*[B,p](t)=p+\frac{\langle\sum\limits_{j=0}^{n}B_j^n(t)p_j-p,\sum\limits_{i=0}^{n-1}B_i^{n-1}(t)J\triangle p_i\rangle}{\sum\limits_{j,i=0}^{n-1}B_j^{n-1}(t)B_i^{n-1}(t)\langle\triangle p_j,\triangle p_i\rangle}\sum_{k=0}^{n-1}B_k^{n-1}(t)J\triangle p_k. \tag{4.1}$$

Proof. The pedal curve $B^*(t)$ of a Bézier curve is obtained using (2.1), (2.3), (2.6) as follows:

$$\beta^*[\beta,p](t) = p + \frac{<\beta(t)-p, J\beta'(t)>}{\|\beta'(t)\|^2} J\beta'(t) \tag{4.2}$$

$$= p + \frac{\langle \sum_{j=0}^{n} B_j^n(t)p_j - p, \sum_{i=0}^{n-1} B_i^{n-1}(t)J\triangle p_i\rangle}{\sum_{j,i=0}^{n-1} B_j^{n-1}(t)B_i^{n-1}(t)\langle\triangle p_j,\triangle p_i\rangle} \sum_{k=0}^{n-1} B_k^{n-1}(t)J\triangle p_k. \tag{4.3}$$

□

Remark 4.2. *The pedal curve couple $B^*(t)$ of a Bézier curve which is defined by (2.1) and pedal point p is*

$$B^*[B,p](0) = p + \frac{\langle p_0 - p, J\triangle p_0\rangle}{\langle\triangle p_0,\triangle p_0\rangle} J\triangle p_0 \tag{4.4}$$

at $t=0$.

Remark 4.3. *The pedal curve couple $B^*(t)$ of a Bézier curve which is defined by (2.1) and pedal point p is*

$$B^*[B,p](1) = p + \frac{\langle p_n - p, J\triangle p_{n-1}\rangle}{\langle\triangle p_{n-1},\triangle p_{n-1}\rangle} J\triangle p_{n-1} \tag{4.5}$$

at $t=1$.

Corollary 4.4. *The pedal curve couple $B^*(t)$ of a Bézier curve which is defined by (2.1) and pedal point $p=p_0$ is*

$$B^*[B,p_0](0) = p_0 + \frac{\langle p_0 - p_0, J\triangle p_0\rangle}{\langle\triangle p_0,\triangle p_0\rangle} J\triangle p_0 = p_0 \tag{4.6}$$

at $t=0$ and

$$B^*[B,p_0](1) = p_0 + \frac{\langle p_n - p_0, J\triangle p_{n-1}\rangle}{\langle\triangle p_{n-1},\triangle p_{n-1}\rangle} J\triangle p_{n-1} \tag{4.7}$$

at $t=1$.

Corollary 4.5. *The pedal curve couple $B^*(t)$ of a Bézier curve which is defined by (2.1) and pedal point $p=p_n$ is*

$$B^*[B,p_n](0) = p_n + \frac{\langle p_0 - p_n, J\triangle p_0\rangle}{\langle\triangle p_0,\triangle p_0\rangle} J\triangle p_0 \tag{4.8}$$

at $t=0$ and

$$B^*[B,p_n](1) = p_n + \frac{\langle p_n - p_n, J\triangle p_{n-1}\rangle}{\langle\triangle p_{n-1},\triangle p_{n-1}\rangle} J\triangle p_{n-1} = p_n \tag{4.9}$$

at $t=1$.

Theorem 4.6. *The pedal curve $B^*(t)$ of a Bézier curve defined by (2.1) for $\forall t \in R$ and pedal point $p=(0,0)=\boldsymbol{0}$ is*

$$B^*[B,\boldsymbol{0}](t) = \frac{\sum_{i=0}^{n} B_i^n(t) \sum_{j=0}^{n-1} B_j^{n-1}(t)\langle p_i, J\triangle p_j\rangle}{\sum_{i,j=0}^{n-1} B_i^{n-1}(t)B_j^{n-1}(t)\langle\triangle p_i,\triangle p_j\rangle} \sum_{k=0}^{n-1} B_k^{n-1}(t)J\triangle p_k. \tag{4.10}$$

Proof. Put the value $p=(0,0)$ in equation (4.1), it can be seen easily. □

Remark 4.7. *The pedal curve couple $B^*(t)$ of a Bézier curve which is defined by (2.1) and pedal point $p=(0,0)=\boldsymbol{0}$ is*

$$B^*[B,\boldsymbol{0}](0) = \frac{\langle p_0, J\triangle p_0\rangle}{\langle\triangle p_0,\triangle p_0\rangle} J\triangle p_0 \tag{4.11}$$

at $t=0$.

Remark 4.8. *The pedal curve couple $B^*(t)$ of a Bézier curve which is defined by (2.1) and pedal point $p=(0,0)=\boldsymbol{0}$ is*

$$B^*[B,\boldsymbol{0}](1) = \frac{\langle p_n, J\triangle p_{n-1}\rangle}{\langle\triangle p_{n-1},\triangle p_{n-1}\rangle} J\triangle p_{n-1} \tag{4.12}$$

at $t=1$.

5. Contrapedal Curve of a planar Bézier curve

In this section, we characterize contrapedal curve of a planar Bézier curve and investigate this curve at $t=0$ and $t=1$.

Theorem 5.1. *The contrapedal curve $B^*(t)$ of a Bézier curve defined by (2.1) for $\forall t\in R$ and pedal point p is*

$$B^*\left[B,p\right](t)=p+\frac{\langle \sum\limits_{j=0}^{n}B_j^n(t)\,p_j-p,\sum\limits_{i=0}^{n-1}B_i^{n-1}(t)\triangle p_i\rangle}{\sum\limits_{j,i=0}^{n-1}B_j^{n-1}(t)B_i^{n-1}(t)\langle\triangle p_j,\triangle p_i\rangle}\sum_{k=0}^{n-1}B_k^{n-1}(t)\triangle p_k. \tag{5.1}$$

Proof. The pedal curve $B^*(t)$ of a Bézier curve is obtained using (2.1), (2.3), (2.5) as follows:

$$\beta^*[\beta,p](t)=p+\frac{<\beta(t)-p,\beta'(t)>}{\|\beta'(t)\|^2}\beta'(t) \tag{5.2}$$

$$=p+\frac{\langle \sum\limits_{j=0}^{n}B_j^n(t)p_j-p,\sum\limits_{i=0}^{n-1}B_i^{n-1}(t)\triangle p_i\rangle}{\sum\limits_{j,i=0}^{n-1}B_j^{n-1}(t)B_i^{n-1}(t)\langle\triangle p_j,\triangle p_i\rangle}\sum_{k=0}^{n-1}B_k^{n-1}(t)\triangle p_k. \tag{5.3}$$

□

Remark 5.2. *The contrapedal curve couple $B^*(t)$ of a Bézier curve which is defined by (2.1) and pedal point p is*

$$B^*\left[B,p\right](0)=p+\frac{\langle p_0-p,\triangle p_0\rangle}{\langle\triangle p_0,\triangle p_0\rangle}\triangle p_0 \tag{5.4}$$

at $t=0$.

Remark 5.3. *The contrapedal curve couple $B^*(t)$ of a Bézier curve which is defined by (2.1) and pedal point p is*

$$B^*\left[B,p\right](1)=P+\frac{\langle p_n-p,\triangle p_{n-1}\rangle}{\langle\triangle p_{n-1},\triangle p_{n-1}\rangle}\triangle p_{n-1} \tag{5.5}$$

at $t=1$.

Corollary 5.4. *The contrapedal curve couple $B^*(t)$ of a Bézier curve which is defined by (2.1) and pedal point $p=p_0$ is*

$$B^*\left[B,p_0\right](0)=p_0+\frac{\langle p_0-p_0,\triangle p_0\rangle}{\langle\triangle p_0,\triangle p_0\rangle}\triangle p_0=p_0 \tag{5.6}$$

at $t=0$ and

$$B^*\left[B,p_0\right](1)=p_0+\frac{\langle p_n-p_0,\triangle p_{n-1}\rangle}{\langle\triangle p_{n-1},\triangle p_{n-1}\rangle}\triangle p_{n-1} \tag{5.7}$$

at $t=1$.

Corollary 5.5. *The contrapedal curve couple $B^*(t)$ of a Bézier curve which is defined by (2.1) and pedal point $p=p_n$ is*

$$B^*\left[B,p_n\right](0)=p_n+\frac{\langle p_0-p_n,\triangle p_0\rangle}{\langle\triangle p_0,\triangle p_0\rangle}\triangle p_0 \tag{5.8}$$

at $t=0$ and

$$B^*\left[B,p_n\right](1)=p_n+\frac{\langle p_n-p_n,\triangle p_{n-1}\rangle}{\langle\triangle p_{n-1},\triangle p_{n-1}\rangle}\triangle p_{n-1}=p_n \tag{5.9}$$

at $t=1$.

Theorem 5.6. *The contrapedal curve $B^*(t)$ of a Bézier curve defined by (2.1) for $\forall t\in R$ and pedal point $p=(0,0)=\boldsymbol{0}$ is*

$$B^*\left[B,\boldsymbol{0}\right](t)=\frac{\sum\limits_{i=0}^{n}B_i^n(t)\sum\limits_{j=0}^{n-1}B_j^{n-1}(t)\langle p_i,\triangle p_j\rangle}{\sum\limits_{i,j=0}^{n-1}B_i^{n-1}(t)B_j^{n-1}(t)\langle\triangle p_i,\triangle p_j\rangle}\sum_{k=0}^{n-1}B_k^{n-1}(t)\triangle p_k. \tag{5.10}$$

Proof. Put the value $p=(0,0)$ in equation (5.1), it can be seen easily. □

Remark 5.7. *The contrapedal curve couple $B^*(t)$ of a Bézier curve which is defined by (2.1) and pedal point $p=(0,0)=\boldsymbol{0}$ is*

$$B^*\left[B,\boldsymbol{0}\right](0)=\frac{\langle p_0,\triangle p_0\rangle}{\langle\triangle p_0,\triangle p_0\rangle}\triangle p_0 \tag{5.11}$$

at $t=0$.

Remark 5.8. *The contrapedal curve couple $B^*(t)$ of a Bézier curve which is defined by (2.1) and pedal point $p=(0,0)=\boldsymbol{0}$ is*

$$B^*\left[B,\boldsymbol{0}\right](1)=\frac{\langle p_n,\triangle p_{n-1}\rangle}{\langle\triangle p_{n-1},\triangle p_{n-1}\rangle}\triangle p_{n-1} \tag{5.12}$$

at $t=1$.

6. Conclusion

In this paper pedal and contrapedal curves of a Bézier curve are characterized. Especially, these couples of planar Bézier curve at the end points are shown. Moreover, the origin is taken as a pedal point.

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