A NOTE ON ITERATION SEQUENCES FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS OF BANACH SPACE

Yonghong Yao and Rudong Chen
Department of Mathematics,
Tianjin Polytechnic University,
Tian jin 300160, P. R. China.

Haiyun Zhou
Department of Mathematics,
Shijiazhuang Mechanical Engineering College,
Shijiazhuang 050003, P. R. China.

Abstract: In this paper, we extend the result due to Liu Qihou and prove some sufficient and necessary conditions for modified Ishikawa iterative sequences of asymptotically quasi-nonexpansive mappings with error member to converge to fixed points.

2000 AMS Subject Classification: 47H05, 47H10.

Key Words: Asymptotically quasi-nonexpansive mapping, Modified Ishikawa iterative sequence, Banach space.

1. Introduction

Let E be a subset of normed space X, and let T be a self-map of $E.T$ is said to be an asymptotically quasi-nonexpansive map, if there is $u_n \in [0, +\infty), \lim_{n\to\infty} u_n = 0$,such that $\|T^n x - p\| \leq (1 + u_n)\|x - p\|, \forall x \in E, \forall p \in F(T)$ ($F(T)$ denotes the set of fixed points).

T is an asymptotically nonexpansive map if $\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\|, \forall x, y \in E$.

The corresponding author: yuyanrong@tjpu.edu.cn (Yonghong Yao).

Petryshyn and Williamson [1], in 1973, proved a sufficient and necessary condition for Picard iterative sequences and Mann iterative sequences to converge to fixed points for quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [2]extend the result of [1] and gave the sufficient and necessary condition for Ishikawa iterative sequences to converge to fixed points for quasi-nonexpansive mappings. In 2001, Liu [3] extend the above result and obtained some sufficient and necessary condition for Ishikawa iterative sequence of asymptotically quasi-nonexpansive mappings with error member to converge to fixed points. In this manuscript, we will extend the result of [3] to the modified Ishikawa iterative sequences with errors and will prove some sufficient and necessary conditions for modified Ishikawa iterative sequences of asymptotically quasi-nonexpansive mappings with error member to converge to fixed points.

2. Main Results

Theorem 2.1. *Let E be a nonempty closed convex subset of Banach space, and $T : E \to E$ an asymptotically quasi-nonexpansive mapping of E (T need not be continuous), and $F(T)$ nonempty. $\forall x_1 \in E$, let*

$$x_{n+1} = a_n x_n + b_n T^{m_n} y_n + c_n v_n$$

$$y_n = \bar{a}_n x_n + \bar{b}_n T^{k_n} x_n + \bar{c}_n w_n, \forall n \in N,$$

where $v_n, w_n \in E$ and $(\|v_n\|)_{n=1}^{\infty}$,$(\|w_n\|)_{n=1}^{\infty}$ are bounded,m_n, k_n are two any positive integer sequences; $0 \leq a_n, \bar{a}_n, b_n, \bar{b}_n, c_n, \bar{c}_n \leq 1, a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1, \forall n \in N$,$\sum_{n=1}^{\infty} b_n u_{m_n} < +\infty$,$\sum_{n=1}^{\infty} b_n u_{k_n} < +\infty$, $\sum_{n=1}^{\infty} c_n < +\infty$,$\sum_{n=1}^{\infty} \bar{c}_n < +\infty$. Then $(x_n)_{n=1}^{\infty}$ converges to a fixed point if and only if $\lim_{n\to\infty} inf d(x_n, F(T)) = 0$,where $d(y, C)$ denotes the distance of y to set C;i.e., $d(y, C) = inf_{\forall x \in C} d(y, x)$.

Theorem 2.2. *Let E be a nonempty closed convex subset of Banach space, and $T : E \to E$ an asymptotically nonexpansive mapping of E (T need not be continuous), and $F(T)$ nonempty. $\forall x_1 \in E$, let*

$$x_{n+1} = a_n x_n + b_n T^{m_n} y_n + c_n v_n$$

$$y_n = \bar{a}_n x_n + \bar{b}_n T^{k_n} x_n + \bar{c}_n w_n, \forall n \in N,$$

where $v_n, w_n \in E$ and $(\|v_n\|)_{n=1}^{\infty}, (\|w_n\|)_{n=1}^{\infty}$ are bounded, m_n, k_n are two any positive integer sequences;$0 \leq a_n, \bar{a}_n, b_n, \bar{b}_n, c_n, \bar{c}_n \leq 1$,$a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1, \forall n \in N$. $\sum_{n=1}^{\infty} b_n u_{m_n} < +\infty, \sum_{n=1}^{\infty} b_n u_{k_n} < +\infty$,$\sum_{n=1}^{\infty} c_n < +\infty, \sum_{n=1}^{\infty} \bar{c}_n < +\infty$.Then $(x_n)_{n=1}^{\infty}$ converges to a fixed point if and only if $\lim_{n\to\infty} inf d(x_n, F(T)) = 0$, where $d(y, C)$ denotes the distance of y to set C;i.e., $d(y, C) = inf_{\forall x \in C} d(y, x)$.

Theorem 2.3. *Let E be a nonempty closed convex subset of Banach space, and $T : E \to E$ an asymptotically quasi-nonexpansive mapping of E (T need not be continuous),and $F(T)$ nonempty. $\forall x_1 \in E$, let*

$$x_{n+1} = a_n x_n + b_n T^{m_n} y_n + c_n v_n$$

$$y_n = \bar{a}_n x_n + \bar{b}_n T^{k_n} x_n + \bar{c}_n w_n, \forall n \in N.$$

where $v_n, w_n \in E$ and $(\|v_n\|)_{n=1}^{\infty}, (\|w_n\|)_{n=1}^{\infty}$ are bounded, m_n, k_n are two any positive integer sequences; $0 \leq a_n, \bar{a}_n, b_n, \bar{b}_n, c_n, \bar{c}_n \leq 1, a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1, \forall n \in N$, $\sum_{n=1}^{\infty} b_n u_{m_n} < +\infty, \sum_{n=1}^{\infty} b_n u_{k_n} < +\infty, \sum_{n=1}^{\infty} c_n < +\infty, \sum_{n=1}^{\infty} \bar{c}_n < +\infty$. Then $(x_n)_{n=1}^{\infty}$ converges to a fixed point p of T if and only if there exists some infinite subsequence of $(x_n)_{n=1}^{\infty}$ which converges to p.

In order to prove the above theorem, we will first prove the following lemmas.

Lemma 1. *Let E be a nonempty convex subset of linear normed space, T an asymptotically quasi-nonexpansive mapping of E, and $F(T)$ nonempty.$\forall x_1 \in E$,let*

$$x_{n+1} = a_n x_n + b_n T^{m_n} y_n + c_n v_n$$

$$y_n = \bar{a}_n x_n + \bar{b}_n T^{k_n} x_n + \bar{c}_n w_n, \forall n \in N,$$

where $v_n, w_n \in E$ and $(\|v_n\|)_{n=1}^{\infty}, (\|w_n\|)_{n=1}^{\infty}$ are bounded,m_n, k_n are two any positive integer sequences with $\sum_{n=1}^{\infty} b_n u_{m_n} < +\infty, \sum_{n=1}^{\infty} b_n u_{k_n} < +\infty; a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1, 0 \leq a_n, \bar{a}_n, b_n, \bar{b}_n, c_n, \bar{c}_n \leq 1, \forall n \in E$. Then

(a)$\|x_{n+1} - p\| \leq (1 + r_n)\|x_n - p\| + t_n, \forall n \in N, \forall p \in F(T)$,

where $r_n = b_n(u_{m_n} + u_{k_n} + L u_{m_n}), L = sup_{n \geq 0} u_n, t_n = b_n(1 + u_{m_n})\bar{c}_n\|w_n - p\| + c_n\|v_n - p\|$.

(b)There exists a constant $M > 0$, such that $\|x_{n+m} - p\| \leq M\|x_n - p\| + M \sum_{k=n}^{\infty} t_k, \forall n, m \in N, \forall p \in F(T)$,where $M = e^{\sum_{i=n}^{\infty} b_i(u_{m_i} + u_{k_i} + L u_{m_i})}$.

Proof of (a). For all $p \in F(T)$,

$$\begin{aligned} \|x_{n+1} - p\| &= \|a_n x_n + b_n T^{m_n} y_n + c_n v_n - p\| \\ &\leq a_n\|x_n - p\| + b_n\|T^{m_n} y_n - p\| + c_n\|v_n - p\| \\ &\leq a_n\|x_n - p\| + b_n(1 + u_{m_n})\|y_n - p\| + c_n\|v_n - p\|, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \|y_n - p\| &\leq \bar{a}_n\|x_n - p\| + \bar{b}_n\|T^{k_n} x_n - p\| + \bar{c}_n\|w_n - p\| \\ &\leq \bar{a}_n\|x_n - p\| + \bar{b}_n(1 + u_{k_n})\|x_n - p\| + \bar{c}_n\|w_n - p\| \\ &\leq (1 + \bar{b}_n u_{k_n})\|x_n - p\| + \bar{c}_n\|w_n - p\|. \end{aligned} \tag{2}$$

substituting (2) into (1),it can be obtain that

$$\begin{aligned}\|x_{n+1}-p\| &\leq a_n\|x_n-p\|+b_n(1+u_{m_n})(1+\bar{b}_n u_{k_n})\|x_n-p\| \\ &\quad +b_n(1+u_{m_n})\bar{c}_n\|w_n-p\|+c_n\|v_n-p\| \\ &\leq [1+b_n(u_{m_n}+u_{k_n}+u_{m_n}u_{k_n})]\|x_n-p\| \\ &\quad +b_n(1+u_{m_n})\bar{c}_n\|w_n-p\|+c_n\|v_n-p\| \\ &\leq (1+r_n)\|x_n-p\|+t_n,\end{aligned}$$

where $r_n = b_n(u_{m_n}+u_{k_n}+Lu_{m_n}), L = sup_{n\geq 0}u_n, t_n = b_n(1+u_{m_n})\bar{c}_n\|w_n-p\| + c_n\|v_n-p\|$. This completes the proof of (a).

Proof of (b).From (a) it can be obtained that

$$\begin{aligned}\|x_{n+m}-p\| &\leq (1+r_{n+m-1})\|x_{n+m-1}-p\|+t_{n+m-1} \\ &\leq e^{r_{n+m-1}}\|x_{n+m-1}-p\|+t_{n+m-1} \\ &\leq e^{(r_{n+m-1}+r_{n+m-2})}\|x_{n+m-2}-p\|+e^{r_{n+m-1}}t_{n+m-2}+t_{n+m-1} \\ &\leq \cdots \\ &\leq e^{\sum_{i=n}^{n+m-1} r_i}\|x_n-p\|+e^{\sum_{i=n}^{n+m-1} r_i}\sum_{i=n}^{n+m-1} t_i \\ &\leq M\|x_n-p\|+M\sum_{i=n}^{n+m-1} t_i, whereM = e^{\sum_{i=n}^{\infty} b_i(u_{m_i}+u_{k_i}+Lu_{m_i})}.\end{aligned}$$

This completes the proof of (b).

Lemma 2[3]. *Let the number of sequences* $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}, and (r_n)_{n=1}^{\infty}$ *satisfy that* $a_n \geq 0, b_n \geq 0, r_n \geq 0, \sum_{n=1}^{\infty} b_n < +\infty, \sum_{n=1}^{\infty} r_n < +\infty$ *and* $a_{n+1} \leq (1+r_n)a_n + b_n, \forall n \in N.$ *Then*

(a) $\lim_{n\to\infty} a_n$ *exist.*

(b)If $\lim_{n\to\infty} inf a_n = 0$*, then* $\lim_{n\to\infty} a_n = 0$.

Proof of the Theorem 2.1. From Lemma 1,we have

$$\|x_{n+1}-p\| \leq (1+r_n)\|x_n-p\|+t_n, \forall p \in F(T), \forall n \in N, \tag{3}$$

Since $\sum_{n=1}^{\infty} b_n u_{m_n} < +\infty$, $\sum_{n=1}^{\infty} b_n u_{k_n} < +\infty$, $\sum_{n=1}^{\infty} c_n < +\infty$, $\sum_{n=1}^{\infty} \bar{c}_n < +\infty$, $(\|v_n\|)_{n=1}^{\infty}, (\|w_n\|)_{n=1}^{\infty}$ are bounded; thus we know $\sum_{n=1}^{\infty} r_n < +\infty, \sum_{n=1}^{\infty} t_n < +\infty$.From (3),we obtain

$$d(x_{n+1}, F(T)) \leq (1+r_n)d(x_n, F(T))+t_n,$$

Since $\lim_{n\to\infty} inf d(x_n, F(T)) = 0$ and from Lemma 2,we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

It will be proven that $(x_n)_{n=1}^{\infty}$ is a Cause sequence.

For all $\epsilon_1 > 0$, from Lemma 1,it can be known there must exist a constant $M > 1$,such that

$$\|x_{n+m} - p\| \leq M\|x_n - p\| + M \sum_{k=n}^{n+m-1} t_k, \forall p \in F(T), \forall n, m \in N. \tag{4}$$

Because $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\sum_{k=1}^{\infty} t_k < +\infty$,there must exist a constant N_1, such that when $n \geq N_1$,

$$d(x_n, F(T)) \leq \frac{\epsilon_1}{3M} and \sum_{k=n}^{\infty} t_k \leq \frac{\epsilon_1}{6M},$$

so there must exist $p_1 \in F(T)$, such that $d(x_{N_1}, p_1) \leq \frac{\epsilon_1}{3M}$.

From (4),it can be obtained that when $n \geq N_1$,

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|x_n - p_1\| \\ &\leq M\|x_{N_1} - p_1\| + M\|x_{N_1} - p_1\| + 2M \sum_{k=N_1}^{\infty} t_k \\ &\leq \epsilon_1. \end{aligned}$$

This implies $(x_n)_{n=1}^{\infty}$ is a Cause sequence. The space is complete; thus $\lim_{n\to\infty} x_n$ exists.

Let $\lim_{n\to\infty} x_n = p$. It will be prove that p is a fixed point.

For all $\epsilon_2 > 0, \lim_{n\to\infty} x_n = p$; thus, there exist a natural number N_2 such that when $n \geq N_2$,

$$\|x_n - p\| \leq \frac{\epsilon_2}{4 + 2u_1}. \tag{5}$$

$\lim_{n\to\infty} d(x_n, F(T)) = 0$ implies that there exists a natural number $N_3 \geq N_2$, such that

$$d(x_{N_3}, F(T)) \leq \frac{\epsilon_2}{4 + 2u_1}.$$

Thus, there exists a $p_2 \in F(T)$, such that

$$\|x_{N_3} - p_2\| = d(x_{N_3}, p_2) \leq \frac{\epsilon_2}{4 + 2u_1}. \tag{6}$$

From (5) and (6),

$$\begin{aligned}
\|Tp - p\| &= \|Tp - p_2 + p_2 - x_{N_3} + x_{N_3} - p\| \\
&\leq \|Tp - p_2\| + \|x_{N_3} - p_2\| + \|x_{N_3} - p\| \\
&\leq (1 + u_1)\|p - p_2\| + \|x_{N_3} - p_2\| + \|x_{N_3} - p\| \\
&\leq (1 + u_1)\|x_{N_3} - p_2\| + (1 + u_1)\|x_{N_3} - p\| + \|x_{N_3} - p_2\| + \|x_{N_3} - p\| \\
&= (2 + u_1)\|x_{N_3} - p\| + (2 + u_1)\|x_{N_3} - p_2\| \\
&\leq \epsilon_2.
\end{aligned}$$

ϵ_2 is an arbitrary positive number. Thus $Tp = p$; i.e., p is a fixed point of T. This completes the proof of Theorem 2.1. Using the same method, Theorem 2.2 can be proven. Theorem 2.3 can be proven by Theorem 2.1.

Remark. *Theorem 2.1-2.3 extend the result of [3] to the modified Ishikawa iterative sequences with errors.*

References

[1]. W.V.Petryshyn and T.E.Williamson, *Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings*, J. Math. Anal. Appl. **43** (1973), 459–497.

[2]. M,K,Ghosh and L.Debnath, *Convergence of Ishikawa iterations of quasi-nonexpansive mappings*, J. Math. Anal. Appl. **207** (1997), 96–103.

[3]. Liu Qihou, *Iterative sequences for asymptotically quasi-nonexpansive mappings with error member*, J. Math. Anal. Appl. **259** (2001), 18–24.