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Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 3, Pages 650–665 (2022)
DOI:10.31801/cfsuasmas.988076
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: August 28, 2021; Accepted: January 27, 2022

TAIL DEPENDENCE ESTIMATION BASED ON SMOOTH ESTIMATION OF DIAGONAL SECTION

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ABSTRACT. This paper is mainly developed around the diagonal section which is strongly related to tail dependence coefficients as defined in Nelsen [19]. Hence, we propose a flexible method for estimating tail dependence coefficients based on the new smooth estimation of the diagonal section based on the Bernstein polynomial approximation. To assess the performance of the new estimators we conduct the Monte-Carlo simulation study. As a result of the simulation study, both estimators perform satisfactory performance. Also, the estimation methods are illustrated by real data examples.

1. INTRODUCTION

Let X and Y be the random variable having the joint distribution function H and the marginals F and G, respectively. The copula C is the function that links the multivariate joint distribution function to its marginal distributions due to the following relationship proposed by Sklar [24]:

$$H(x, y) = C(F(x), G(y)).$$

Copula C is unique if and only if marginals F and G are continuous. Also, it satisfies the following properties

(1) $C(0, u) = C(u, 0) = 0$ for all $u \in [0, 1]$
(2) $C(1, u) = C(u, 1) = u$ for all $u \in [0, 1]$
(3) for all $u,\ u',\ v,\ v' \in [0, 1]$ with $u < u'$ and $v < v'$

$$V_C([u, u'] \times [v, v']) = C(u, u') - C(u, v') - C(u', v) + C(u, v) \geq 0$$

where $V_C([u, u'] \times [v, v'])$ is the $C - volume$ of the rectangle $[u, u'] \times [v, v']$.

2020 *Mathematics Subject Classification.* 62H20, 62G05.
Keywords. Copula, tail dependence, Bernstein polynomial, diagonal section.
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The contribution of this study is two-fold: first, we are proposing a smooth estimation of diagonal section of the copula. Second, we estimate the tail dependence coefficients using the smooth estimation of the diagonal section.

The diagonal section of copulas is an important aspect in the field of dependence modelling. Especially, the diagonal section provides some pieces of information about the tail dependence behaviour of the bivariate random variables (Joe [18]). Thus, estimation of the diagonal section is a crucial part of the estimation of tail dependence coefficients between bivariate random variables. For this reason, we propose a non-parametric smooth estimation of the diagonal section based on the Bernstein polynomial approximation. The proposed estimation method is a continuous approximation of the classical estimation which has jump discontinuous. In a copula framework, estimation procedures based on the polynomial approximation is not new: see, e.g., Susam and Hudaverdi [21]- [22], Dimitrova et al. [6], Durante and Okhrin [8], Ambrard and Girard [1].

The second aim of this paper is tail dependence estimation based on the plug-in method. Because there is a direct relationship between the diagonal section and tail dependence coefficients as defined in Nelsen [19], the tail dependence estimation method is mainly developed around the smooth estimation of the diagonal section. The use of the Bernstein estimator of the diagonal section reduced the complexity of the tail dependence estimation coefficients. Moreover, the proposed estimation method of the tail dependence coefficient is flexible according to its polynomial degree, hence the error of the estimation may be reduced by increasing the degree of the Bernstein polynomial. There are some papers which introduces the tail dependence estimation in the literature e.g., Susam and Erdogan [23], Ferreira [13], Schmidt and Stadtmuller [20], Ferreira and Ferreira [14], Frahm et al. [16], Caillault and Guégan [4], Goegebeur and Guillou [17]. The plug-in estimation of tail dependence based on Bernstein polynomial approximation is not a new idea. Susam and Erdogan [23] proposed a tail dependence estimation using the plug-in principle. Their tail dependence estimation is mainly developed around the smooth estimation of the Kendall distribution function of Archimedean copula family. The main difference of this article from the Susam and Erdogan [23] is that our proposed tail dependence estimator is applicable to all copula families such as Elliptical, Extreme value, etc.

The paper is organized as follows. In section 2, we propose the smooth estimation of the diagonal section using Bernstein polynomial approximation and investigate its properties. Also, we conduct a simulation study to measure its performance. In section 3, we deal with the estimation of tail dependence coefficients using the smooth estimation of the diagonal section. Moreover, we investigate its performance. As an illustration, we apply the proposed tail dependence estimation method to the Danube data set. Finally, the conclusion is given in the last section.

2. Estimation of Diagonal Section

In this section, firstly, we review basic definitions and properties about diagonal section of copulas, which can be found, for instance, in Durante et al. [9] and Durante et al. [10]. Then, we investigated the smooth estimation of diagonal section of copulas based on the Bernstein polynomial approaximation.

$\delta_C : [0,1] \to [0,1]$, called diagonal section of copula, is the function defined by $\delta(t) = C(t,t)$. Let us consider that X and Y are uniformly distributed on the unit interval. Moreover, suppose that $W = \max(X,Y)$ is distributed according to the cumulative distribution function (cdf) H. The behaviour of the random variable W is determined by the diagonal section of the copula $C_{X,Y}$, such that $\delta_C(t) = H_W(t)$ (Durante et al. [10]). Diagonal section of the copula has the following properties:

(D1) $\delta_C(0) = 0$ and $\delta_C(1) = 1$;
(D2) $\delta_C(t) \leq t$ for all $t \in [0,1]$;
(D3) $\delta_C(t)$ is non-decreasing function;
(D4) δ_C is $2 - Lipschitz$, such that $\left|\delta_C(t_2) - \delta_C(t_1)\right| \leq 2\left|t_2 - t_1\right|$ for all $t_2, t_1 \in [0,1]$.

Let $\{(X_1,Y_1),\ldots,(X_n,Y_n)\}$ be a random sample of (X,Y) from cdf $H(x,y)$. The inference is then based on the pseudo-samples defined as

$$U_i = \frac{R(X_i)}{n},\; V_i = \frac{R(Y_i)}{n},\; i = 1,\ldots,n;$$

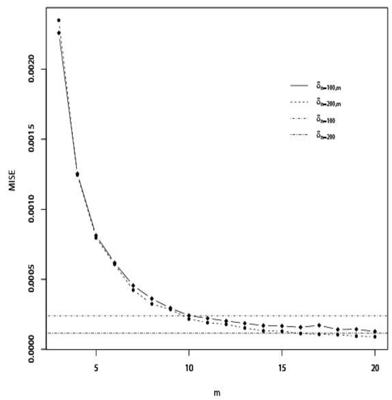
where $R(.)$ is the rank of random variable. Hence, the pair of random variables (U,V) yield an approximate sample from the copula $C(u,v)$. The non-parametric estimation of diagonal section relies on the pseudo-observations $w_i = \max\big(u_i, v_i\big), i = 1,\ldots,n$ which have the distribution function $C(w,w)$. It is natural to non-parametric estimate the diagonal section given by

$$\delta_n(t) = \frac{1}{n}\sum_{i=0}^{n} \mathbf{I}\big(w_i \leq t\big),\; t \in [0,1]; \tag{1}$$

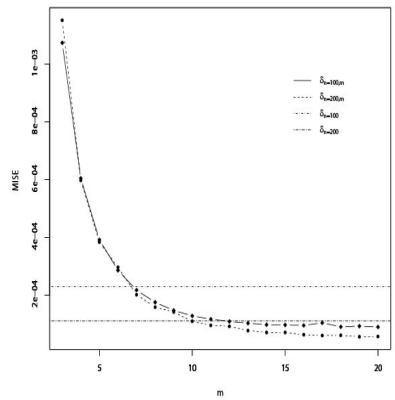
which by the Glivenko–Cantelli lemma converges to the true cdf. Erderly [12] investigated properties of empirical diagol section δ_n. An empirical diagonal section can be written by following:

$$\delta_n(t) = C_n(t,t),\; t \in [0,1]$$

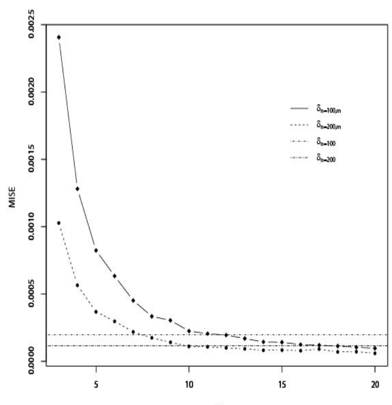
where C_n is the empirical copula defined by Deheuvels [5]. Hence, the properties of δ_n may be imvestigated using the properties of empirical copula and empirical



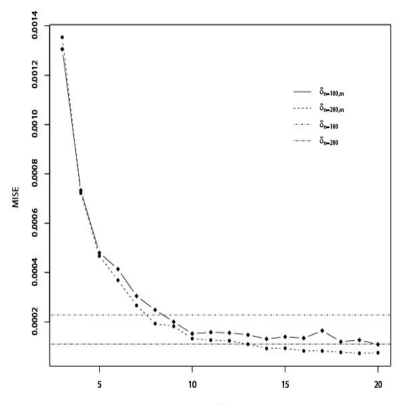
(A) Gumbel Copula with $\tau = 0.25$



(B) Gumbel Copula with $\tau = 0.50$



(C) Clayton Copula with $\tau = 0.25$

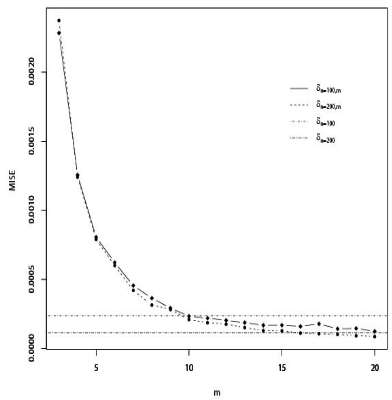


(D) Clayton Copula with $\tau = 0.50$

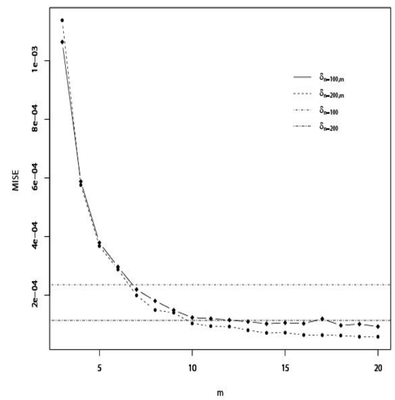
Figure 1. MISE values for some Archimdean copulas with $\tau = 0.25, 0.5$

cdf. It is clear that $\delta_n(0) = 0$, $\delta_n(1) = 1$ and $\delta_n(t)$ is non-decreasing function. Moreover, by the Fréchet–Hoeffding bounds for empirical copula:

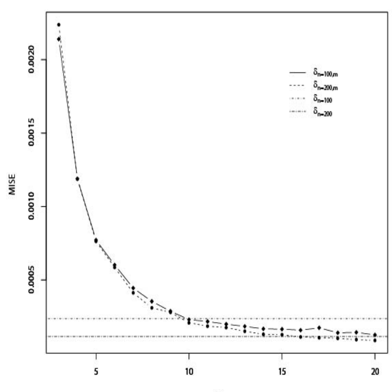
$$\max(2t - 1, 0) \le \delta_n(t) \le t, \; t \in [0, 1],$$



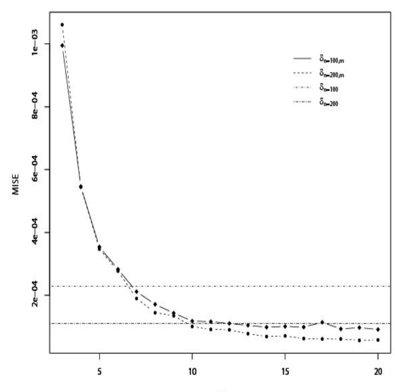
(A) Normal Copula with $\tau = 0.25$



(B) Normal Copula with $\tau = 0.50$



(C) Student-t Copula with $\tau = 0.25$



(D) Student-t Copula with $\tau = 0.50$

FIGURE 2. MISE values for some Elliptical copulas with $\tau = 0.25, 0.5$

hence property D3 is also satisfied. Erderly [12] also proved the propoerty given by:

$$\delta_n(\frac{i+1}{n}) - \delta_n(\frac{i}{n}) \in \{0, \frac{1}{n}, \frac{2}{n}\},\ i = 1, \ldots, n. \tag{2}$$

Because of the δ_n has jump discontinuties, estimating a continuous distribution function may not be a good choice. Hence, in this paper, we propose a smooth estimation of δ_n using Bernstein polynomial approximation. The Bernstein estimator of order $(m > 0)$ of the diagonal section δ is defined as,

$$\delta_{m,n}(t) = \sum_{k=0}^{m} \delta_n(\frac{k}{m})P_{k,m}(t),\ t \in [0, 1]$$

where $P_{k,m}(t) = \binom{m}{k} t^k(1-t)^{m-k}$ is the binomial probability. The following theorem defined in Feller [15] helps us to prove the consistency of the Bernstein empirical diagonal section.

Theorem 1. *If $f(t)$ is a bounded and continuous function on the interval $[0, 1]$, then as $m \to \infty$*

$$f^*_m(t) = \sum_{k=0}^{m} f(\frac{k}{m})P_{k,m}(t) \to f(t)$$

The following theorem states that the Bernstein empirical diagonal section is a consistent estimator of $\delta(t)$.

Theorem 2. *Let δ be a continuous diagonal section on the interval $[0, 1]$. If $m, n \to \infty$, then $\sup_{t\in[0,1]} |\delta_{n,m}(t) - S(t)| \to 0$ a.s.*

Proof. Recall Theorem 1 of f^*_m for any f.

$$\sup_{t\in[0,1]} |\delta_{n,m}(t) - \delta(t)| \leq \sup_{t\in[0,1]} |\delta_{n,m}(t) - \delta^*_m(t)| + \sup_{t\in[0,1]} |\delta^*_m(t) - \delta(t)|.$$

As

$$\delta_{n,m}(t) - \delta^*(t) = \sum_{k=0}^{m} \big(\delta_n(t) - \delta(t)\big)P_{k,m}(t)$$

we have

$$\sup_{t\in[0,1]} |\delta_{n,m}(t) - \delta^*_m(t)| \leq \max_{0\leq k\leq m} |\delta_n(\frac{k}{m}) - \delta(\frac{k}{m})| \leq \sup_{t\in[0,1]} |\delta_n(t) - \delta(t)|$$

Then, $\sup_{t\in[0,1]} |\delta_n(t) - \delta(t)| \to 0$ a.s as $n \to \infty$. See also, Babu et al. [2]. □

The next proposition investigates the properties of the Bernstein empirical diagonal section:

Proposition 1. *The Bernstein empirical diagonal section with order $m > 0$ has the following properties:*

(P1) $\delta_{n,m}(0) = 0$ *and* $\delta_{n,m}(1) = 1$;
(P2) $\delta_{n,m}(t) \leq t$ *for all* $t \in [0, 1]$;
(P3) $\delta_{n,m}(t)$ *is non-decreasing function;*
(P4) $\delta_{n,m}$ *is* $2 - $*Lipschitz.*

Proof. From the endpoint property of Berntein ploynomial, $\delta_{m,n}(1) = \delta_n(1) = 1$ and $\delta_{m,n}(0) = \delta_n(0) = 0$. See Duncan [7]. We know that $\delta_n(t) \leq t$, hence we can write $\delta_n(\frac{i}{m}) = \frac{i}{m} - r_i,\ i = 1, \ldots, m$ then

$$\begin{aligned}
\delta_{m,n}(t) &= \sum_{i=0}^{m} \delta_n(\frac{i}{m})\binom{m}{k} t^k (1-t)^{m-k} \\
&= \sum_{k=0}^{m} (\frac{k}{m} - r_k)\binom{m}{k} t^k (1-t)^{m-k} \\
&= \sum_{k=0}^{m} (\frac{k}{m})\binom{m}{i} t^k (1-t)^{m-k} - \sum_{k=0}^{m} r_k \binom{m}{k} t^k (1-t)^{m-k} \\
&= t\sum_{k=1}^{m} \binom{m-1}{k-1} t^{k-1} (1-t)^{m-k} - \sum_{k=0}^{m} r_k \binom{m}{k} t^k (1-t)^{m-k} \\
&= t\sum_{l=0}^{m-1} t^l (1-t)^{m-l-1} \binom{m-1}{l} - \sum_{k=0}^{m} r_k \binom{m}{k} t^k (1-t)^{m-k} \\
&= t - \sum_{k=0}^{m} r_k \binom{m}{k} t^k (1-t)^{m-k} < t.
\end{aligned}$$

Thus $\delta_{m,n}(t) \leq t$ is satisfied for all $t \in [0, 1]$. The first derivative of the $\delta_{m,n}$ can be obtained as

$$\delta'_{m,n}(t) = m \sum_{k=0}^{m-1} \big(\delta_n(\frac{k+1}{m}) - \delta_n(\frac{k}{m})\big) t^k (1-t)^{m-k-1} \binom{m-1}{k}$$

, see Duncan [7]. Becasue δ_n is non-decreasing function such that

$$\delta_n(\frac{k+1}{m}) - \delta_n(\frac{k}{m}) \geq 0,\ k = 0, \ldots, m-1;$$

then $\delta'_{m,n}(t) \geq 0,\ t \in [0, 1]$. We note that a function $f : [a, b] \to \Re$ is said to be a Lipschitz if there is a constant L such that

$$\big|f(x_2) - f(x_1)\big| \leq L\big|x_2 - x_1\big|,\ \ \forall\, x_2, x_1 \in [a, b],$$

where Lipschitz constant of f equals to $\sup_{x\in[0,1]} \big|f'(t)\big|$. Brown et al. [3] showed that Bernstein polynomial approximation defined as

$$B(t) = \sum_{k=0}^{m} f(\frac{k}{m}) P_{k,m}(t),\ t \in [0, 1]$$

is $L-$*Lipschitz* function. Hence, the Lipschitz constant L eqauls to $L = \sup_{t\in[0,1]} \big|B'(t)\big|$. We know that $\delta_{m,n}(t)$ is non-decreasing function and $\delta'_{m,n}(1)$ equals to $m\Big(\delta_n(\frac{m}{m})-$

$\delta_n(\frac{m-1}{m})\Big) \in \{0,1,2\}$. Hence, the Lipschitz constant of $\delta_{m,n}(t)$ can be calculated as

$$L = \sup_{t\in[0,1]} \left|\delta'_{m,n}(t)\right| = 2.$$

□

To measure the performance of the proposed estimator, we conduct Monte-Carlo simulation study. Gumbel, Clayton (Archimedean) and Normal, Student-t (Elliptical) copulas that have parameters coressponding to Kendall's tau as $\tau = 0.25$, 0.50 are used to generate the data. Specifically, 10.000 Monte-Carlo samples of size $n = 100, 200$ are generated from each copula, and the performance of the Bernstein empirical diagonal section with order $m = 3, \ldots, 30$ are measured by means of the Mean Integrated Squared error (MISE) defined as

$$MISE(\delta) = E\Big(\int_0^1 \big(\delta_{m,n}(t) - \delta(t)\big)^2 dt\Big).$$

Simulation results are shown in Figures 1 and 2 for the Archimedean and Elliptical copulas, respectively. From these figures, it is clear that MISE scores of the Bernstein empirical diagonal section gets closure to the true cdf when both order m and sample size n are increased for all copula classes. Moreover, the Bernstein empirical diagonal section $\delta_{m,n}$ outperforms to classical one δ_n for all possible situations.

3. Tail Dependence Estimation

In this section, firstly, we will be introducing the tail dependence concept. Then, we investigate the plug-in estimators for the upper and lower tail dependence coefficients based on the smooth estimation of the diagonal section discussed in Section 2.

An crucial part of the dependence between the variables in the upper-right quadrant and in the lower-left quadrant of $\mathbf{I^2}$. In general, most dependence measures associate the entire distribution of two or more random variables. However, the dependence between the upper part of the distribution may be different than the mid-range and/or lower part of the distribution (Embrechts et al. [11]). Let X and Y be continuous random variables with margins F and G, respectively. Nelsen [19] shows that the tail dependence coefficients depend on the derivative of diagonal section are given by following:

$$\lambda_U = 2 - \lim_{t\to 1^-} \frac{1 - C(t,t)}{1-t} = 2 - \delta'_C(1^-), \;\; \lambda_L = \lim_{t\to 0^+} \frac{C(t,t)}{t} = \delta'_C(0^+). \quad (3)$$

In general, the tail dependence between variables may strongly depend on the choice of model or estimation technique (Frahm et al. [16]). For this reason, to estimate the tail dependence coefficients we prefer to use smooth estimation of diagonal section of copula which outperforms the classical estimator as shown in section 2. The estimation of the tail dependence coeffcients investigated in next propsition.

Proposition 2. *Let $\delta_{m,n}(.)$ be the estimator of diagonal section based on Bernstein polynomial approximation and $\delta_n(.)$ be empirical diagonal section. The estimation of the lower tail and the upper tail dependence for copulas are obtained by*

$$\hat{\lambda}_L = m\Big(\delta_n(\frac{1}{m})\Big)$$
$$\hat{\lambda}_U = 2 - m\Big((1 - \delta_n(\frac{m-1}{m})\Big)$$

The proof of the Proposition 2 can be easley done using the properties of Bernstein polynomials. It is obvious that there are clear link beetwen the tail dependence estimations $\hat{\lambda}_L$, $\hat{\lambda}_U$ and the Bernstein polynomial degree m.

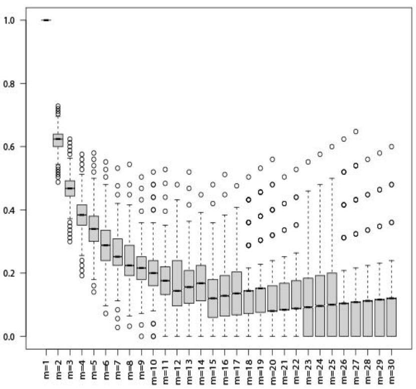
Table 1. The values of the tail dependence and dependence parameter for Gumbel, Clayton, Normal and Student-t copula for different level of dependence

Copula	τ	θ	λ_U	λ_L
Gumbel	0.25	1.3333	0.3182	0
	0.50	2	0.5857	0
Clayton	0.25	0.6666	0	0.3535
	0.50	2	0	0.7071
Normal	0.25	0.3826	0	0
	0.50	0.7071	0	0
Student	0.25	0.3826	0.1953	0.1953
	0.50	0.7071	0.3968	0.3968

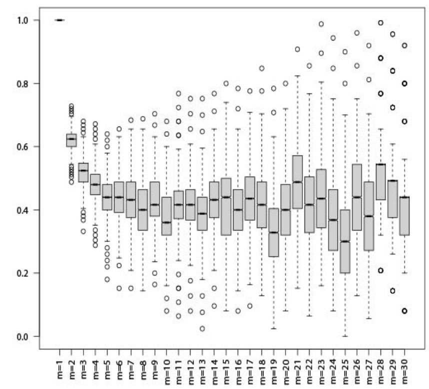
To asses the performance of the tail dependence estimation, we simulate K = 10.000 times bivariate random of sample size n = 250, 750, respectively, from Gumbel, Clayton, Normal and Student-t copulas with Kendall's tau $\tau = 0.25$, 0.50. The value of the upper tail dependence (λ_U), lower tail dependence (λ_L) and the dependence parameter (θ) for Gumbel, Clayton, Normal and Student-t copulas with $\tau = 0.25$, 0.50 are given in Table 1.

The boxplots of the results of the tail dependence estimations obtained after K Monte-Carlo samples of size n = 250, 750 from Gumbel, Clayton, Normal and Student-t copulas for varying Kendall's tau values $\tau = 0.25$, 0.50 are displayed in Figs. 3-6.

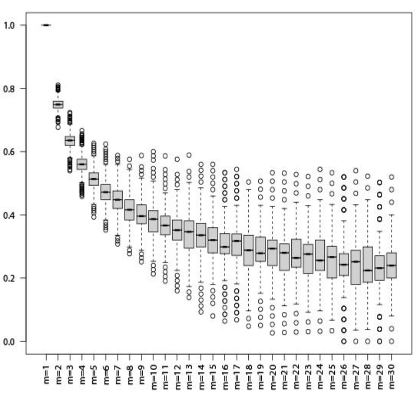
The following results can be obtained:



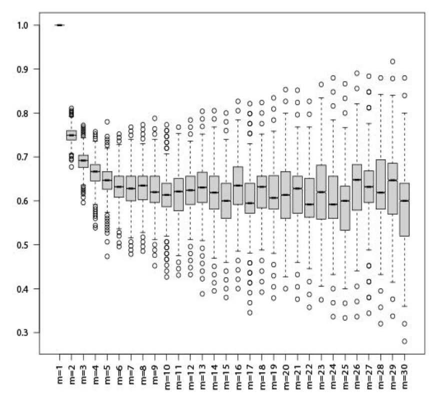
(A) λ_L estimation for $\tau = 0.25$ and $n = 250$



(B) λ_U estimation for $\tau = 0.25$ and $n = 250$



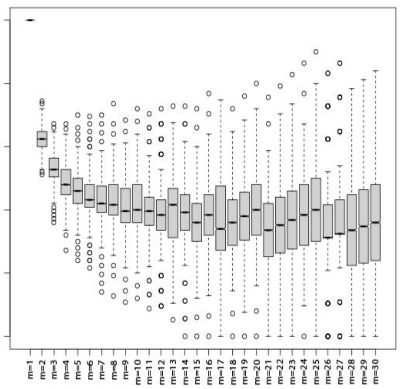
(C) λ_L estimation for $\tau = 0.5$ and $n = 750$



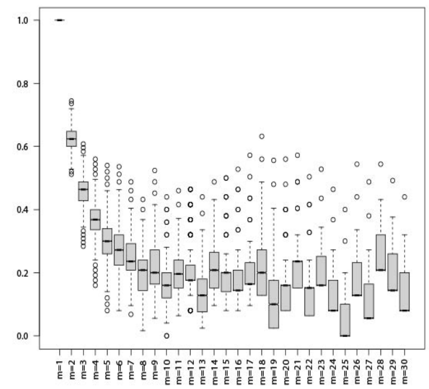
(D) λ_U estimation for $\tau = 0.5$ and $n = 750$

FIGURE 3. Box-plots of the estimation of the tail dependence coefficients of Gumbel copula

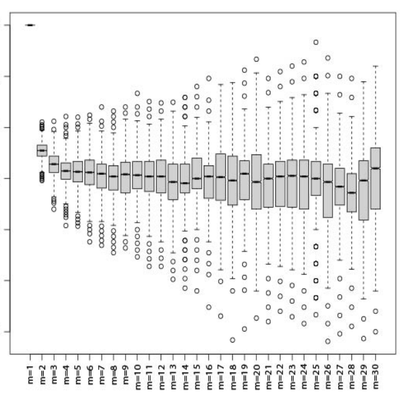
(1) For copulas studied in this paper, the upper tail dependence and lower tail dependence estimation converge to its true value defined in Table 1, regardless of Kendall's tau and sample size.



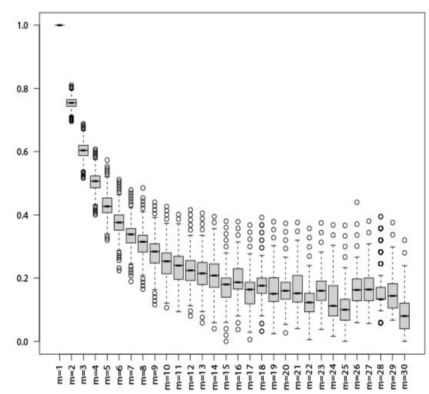
(A) λ_L estimation for $\tau = 0.25$ and $n = 250$



(B) λ_U estimation for $\tau = 0.25$ and $n = 250$



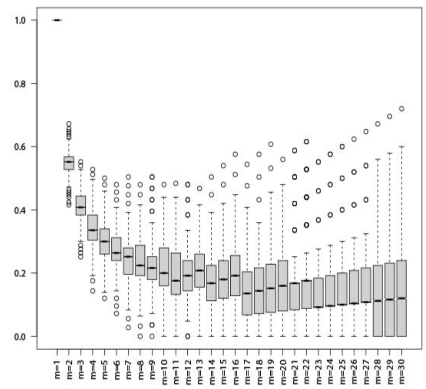
(C) λ_L estimation for $\tau = 0.5$ and $n = 750$



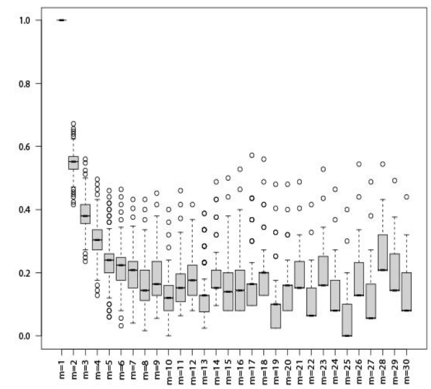
(D) λ_U estimation for $\tau = 0.5$ and $n = 750$

FIGURE 4. Box-plots of the estimation of the tail dependence coefficients of Clayton copula

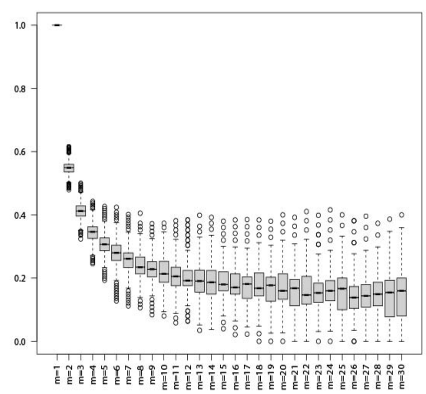
(2) It is obvious that variance of the upper tail dependence and lower tail dependence estimation increases when the Bernstein polynomial degree increases in all situations.



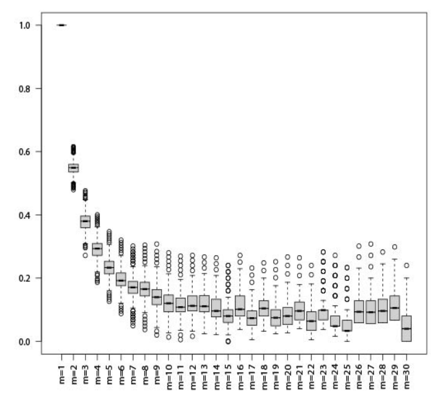
(A) λ_L estimation for $\tau =$ 0.25 and $n = 250$



(B) λ_U estimation for $\tau =$ 0.25 and $n = 250$



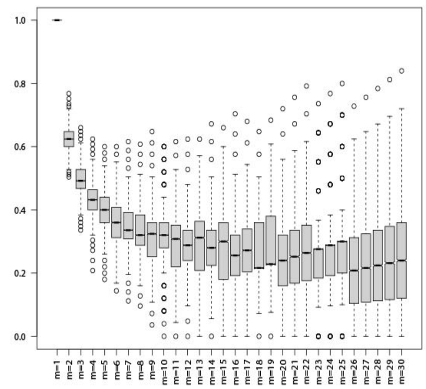
(C) λ_L estimation for $\tau =$ 0.5 and $n = 750$



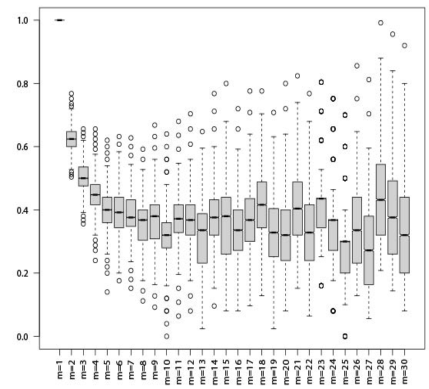
(D) λ_U estimation for $\tau =$ 0.5 and $n = 750$

FIGURE 5. Box-plots of the estimation of the tail dependence coefficients of Normal copula

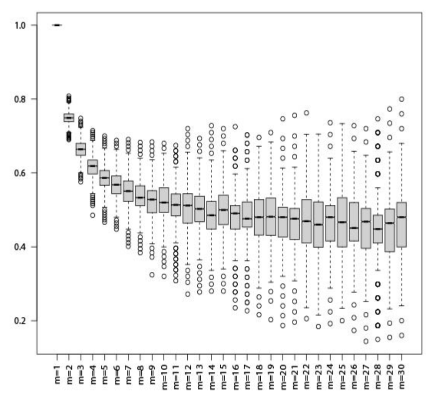
(3) For Gumbel copula with sample size $n = 750$ and Kendall's tau $\tau = 0.50$, to estimate the λ_L approaches its true value, the polynomial degree of estimation should be chosen higher than 30.



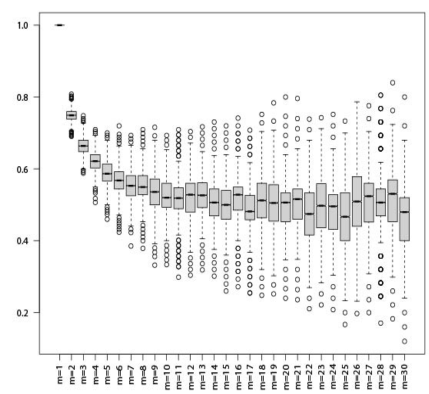
(A) λ_L estimation for $\tau = 0.25$ and $n = 250$



(B) λ_U estimation for $\tau = 0.25$ and $n = 250$



(C) λ_L estimation for $\tau = 0.5$ and $n = 750$



(D) λ_U estimation estimation for $\tau = 0.5$ and $n = 750$

Figure 6. Box-plots of the estimation of the tail dependence coefficients of Student-t copula

4. Case Study

In this section, in order to demonstrate the estimation methods of diagonal section and tail dependence coefficients, we use the Danube data set which is available in the R package copula. According to this package, the Danube data set contains ranks of base flow observations from the Global River Discharge project of the Oak Ridge National Laboratory Distributed Active Archive Centre (ORNL DAAC), a NASA data centre. The measurements are the monthly average flow rate for two stations situated at Scharding (Austria) on the Inn River and Nagymaros (Hungary) on the Danube.

The scatter plot of the pseudo-observations of the Danube data set is displayed in Figure 7. In this figure, symmetrical dependence structures are observed. From this figure, it seems that the Danube data set has a heavy right tail dependence structure and mild left tail dependence structure. Figure 8 represents the estimation of upper tail dependence and lower tail dependence coefficient for polynomial degree $m = 1, \ldots, 30$. As it is expected estimation of upper tail dependece is greater than the lower tail dependence estimation for all poynomial degrees. From figure 8, $\hat{\lambda}_U$ approaximates to 0.5 and $\hat{\lambda}_L$ approaximates to 0.20.

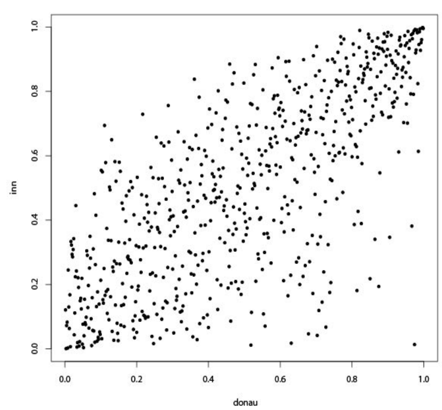
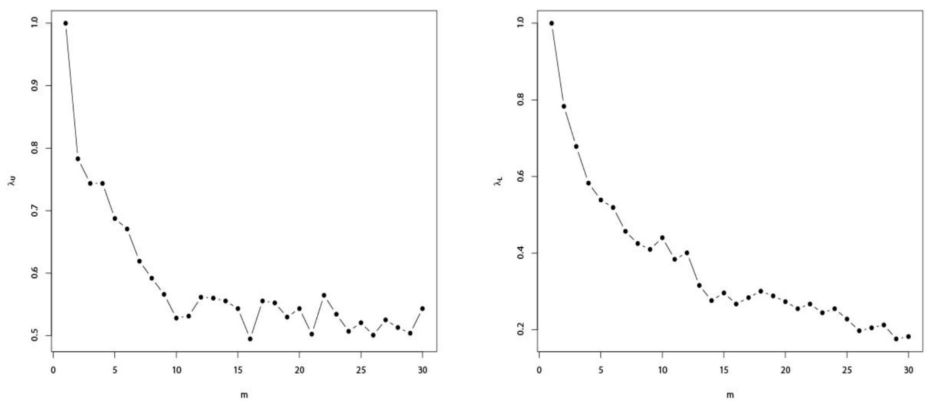


Figure 7. Scatter plot of Danube data set

5. Conclusion

In this paper, we have presented a smooth estimation of the diagonal section based on the Bernstein polynomial approximation. The new estimator is flexible according to its polynomial degree; the error of the estimation may be decreased



(A) Estimation of λ_U (B) Estimation of λ_L

FIGURE 8. Estimation of λ_U and λ_L for Danube data set

when the polynomial degree increases. Moreover, Bernstein diagonal section outperforms the empirical diagonal section for the higher polynomial degrees. Also, considering the strong relationship between the diagonal section and the tail dependence coefficient, we propose the tail dependence coefficients estimation method via Bernstein diagonal section. According to the simulation results and real data example, the tail dependence coefficients estimation method has a satisfactory performance.

Declaration of Competing Interests The authors declare that they have no known competing financial interest or personal relationships that could have apeperaed to influence the work reported in this paper.

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