KMU Journal of Engineering and Natural Sciences
https://dergipark.org.tr/en/pub/kmujens
3(2), 74-78, (2021) © KMUJENS
e-ISSN: 2687-5071

Küme ailelerinin kümülatif graf temsilleri üzerine

On cumulative graph representations of set-families

Kadirhan POLAT[1]

[1]Department of Mathematics, Faculty of Science and Letter, Agri Ibrahim Cecen University, Agri, Turkey

(Received: 29 August 2021; Accepted: 21 September 2021)

Özet. Bu makalede, önce kümülatif graf tanımını tanıtıyoruz. Ardından, bir küme ailesinden başlayarak, o küme ailesini temsil eden kümülatif grafı elde etmek için izlenecek adımları veriyoruz. Ayrıca, bu adımların örnek bir uygulamasını gösteriyoruz.

Anahtar Kelimeler: kümülatif graf, küme ailesi, graf temsili.

Abstract. In this paper, we first introduce the definition of cumulative graph. Then, starting from a set-family, we give the steps to follow to obtain the cumulative graph representing that set-family. Also, we show an example implementation of these steps.

Keywords: cumulative graph, set family, graph representation.

2010 Mathematics Subject Classification: 05C62;05C20;03E75.

1. Introduction

We denote the sets of all endpoints, all tails and all heads of a digraph G by $V(G), V_t(G)$ and $V_h(G)$, respectively, which implies $V(G) = V_t(G) \cup V_h(G)$ for any digraph G. A subgraph H of G is denoted by $G[H]$. A vertex-induced subgraph by $W \subseteq V$ of a graph $G = (V, A)$ is denoted by $G[W, \cdot]$ while an edge-induced subgraph by $B \subseteq A$ of G is denoted by $G[\cdot, B]$ (see [1–5] for detailed information).

Let $\mathcal{P}(X)$ represent the power set of a set X. We also denote the power set of $\mathcal{P}(X)$ by $\mathcal{P}^2(X)$. In general, a n-*set*-family on X, denoted by $\mathcal{F}_X^{(n)}$ or shortly $\mathcal{F}^{(n)}$, is a subfamily of n-iterated power set operation $P^n(X)$ on X, that is, a subfamily of n-times repeated composition of the power set operation on X with itself. In particular, it is our convention that a 0-set-family $\mathcal{F}^{(0)}$ on X is a subset of X. We denote k-times generalized union of an n-set-family $\mathcal{F}^{(n)}$ with $k \leq n$ by $\bigsqcup^k \mathcal{F}^{(n)}$, that is,

$$\bigsqcup^k \mathcal{F}^{(n)} = \underbrace{\bigcup \cdots \bigcup}_{k \text{ times}} \mathcal{F}^{(n)}.$$

In particular, we use the conventions $\bigsqcup^1 \mathcal{F}^{(n)} = \bigcup \mathcal{F}^{(n)}$ and $\bigsqcup^0 \mathcal{F}^{(n)} = \mathcal{F}^{(n)}$.

The motivation for this paper is to show that there is a special class of graphs, which we will call cumulative graphs, representing any n-set family with $n > 0$, and to obtain its some basic properties.

Corresponding Author Email: kadirhanpolat@agri.edu.tr, orcid.org/0000-0002-3460-2021.

2. Cumulative graphs

We now introduce the concept of cumulative graph, which corresponds to a special class of acyclic digraphs. Notice that in the next definition we bend the rule that each member of a partition of a set is non-empty.

Definition 1. *An acyclic digraph $G = (V, A)$ is called* a cumulative graph *if there exists a partition $\mathcal{V} = \{\emptyset = V_0, V_1, \ldots, V_n\}$ of V, and a partition $\mathcal{A} \cup \mathcal{B}$ of A where $\mathcal{A} = \{\emptyset = A_1, A_2, \ldots, A_n\}$ and $\mathcal{B} = \{\emptyset = B_1, B_2, \ldots, B_n\}$ such that*

(1) *V_i consists of all endpoints in A_i, all tails in B_i and all heads in B_{i+1}, that is, it holds that $V_i = V(G[\cdot, A_i]) \cup V_t(G[\cdot, B_i]) \cup V_h(G[\cdot, B_{i+1}])$ for every $1 \leq i < n$. Also, $V_n = V(G[\cdot, A_i]) \cup V_t(G[\cdot, B_i])$.*
(2) *$uv \in A_i$ and $vw \in A_i$ implies $uw \notin A_i$ for every $1 \leq i \leq n$ (antitransitivity).*
(3) *$uv \in A_i$ and $vw \in B_i$ implies $uw \notin B_i$ for every $1 \leq i \leq n$.*

We denote a cumulative graph by $G = (\mathcal{V}, \mathcal{A}, \mathcal{B})$.

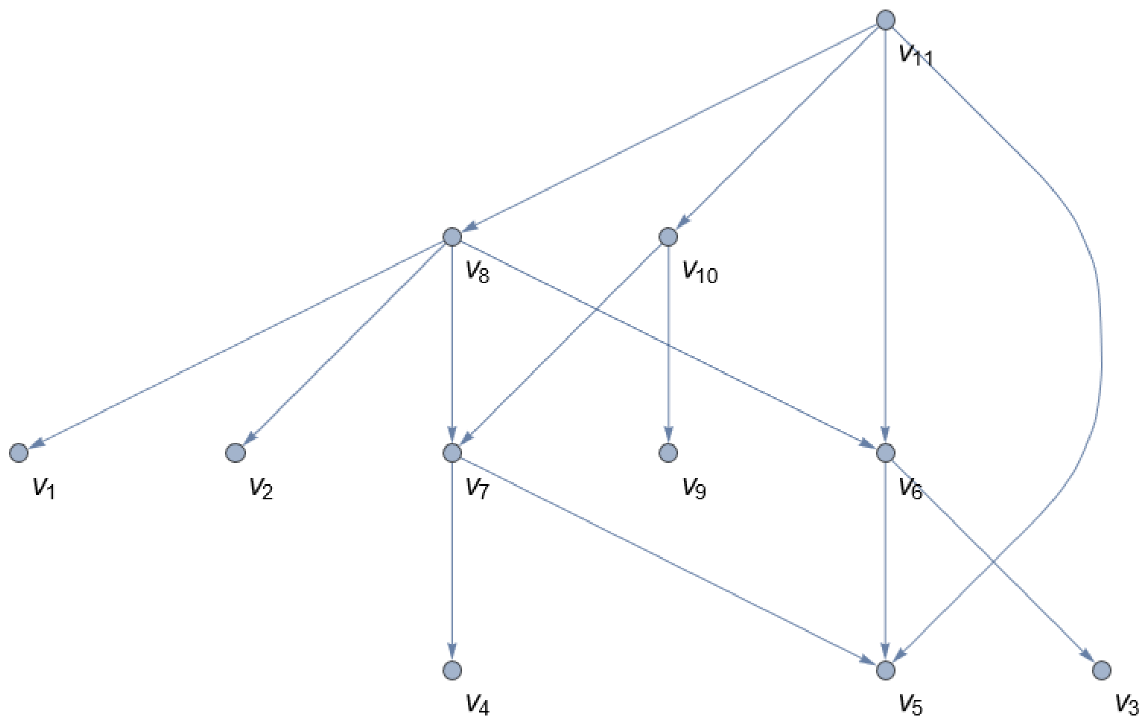

Figure 1. An example of a cumulative graph

Example 1. *Let $G = (V, A)$ be an acyclic digraph with $V = \{v_1, v_2, \ldots, v_{11}\}$ and*

$$A = \{v_{11} \to v_5, v_{11} \to v_6, v_{11} \to v_8, v_{11} \to v_{10}, v_{10} \to v_7, v_{10} \to v_9, v_8 \to v_1, v_8 \to v_2, v_8 \to v_6, v_8 \to v_7, v_7 \to v_4, v_7 \to v_5, v_6 \to v_3, v_6 \to v_5\}.$$

as Figure 1. In order to show that G is a cumulative graph, we set $\mathcal{V} = \{V_0, V_1, V_2, V_3\}$, $\mathcal{A} = \{A_1, A_2, A_3\}$ and $\mathcal{B} = \{B_1, B_2, B_3\}$ where

$$V_0 = \emptyset, V_1 = \{v_1, v_2, v_3, v_4\}, V_2 = \{v_5, v_6, v_7, v_8\}, V_3 = \{v_9, v_{10}, v_{11}\},$$
$$A_1 = \emptyset, A_2 = \{v_8 \to v_6, v_8 \to v_7, v_7 \to v_5, v_6 \to v_5\},$$
$$A_3 = \{v_{11} \to v_{10}, v_{10} \to v_9\}, B_1 = \emptyset,$$
$$B_2 = \{v_8 \to v_1, v_8 \to v_2, v_7 \to v_4, v_6 \to v_3\},$$
$$B_3 = \{v_{11} \to v_5, v_{11} \to v_6, v_{11} \to v_8, v_{10} \to v_7\}.$$

It is easy to check that $\mathcal{V}$ is a partition of V while $\mathcal{A} \cup \mathcal{B}$ is a partition of A. Besides,

$$V(G[\cdot, A_1]) \cup V_t(G[\cdot, B_1]) \cup V_h(G[\cdot, B_2]) = \emptyset \cup \emptyset \cup \{v_1, v_2, v_3, v_4\} = V_1,$$
$$V(G[\cdot, A_2]) \cup V_t(G[\cdot, B_2]) \cup V_h(G[\cdot, B_3]) = \{v_5, v_6, v_7, v_8\} \cup \{v_6, v_7, v_8\} \cup \{v_5, v_6, v_7, v_8\} = V_2,$$
$$V(G[\cdot, A_3]) \cup V_t(G[\cdot, B_3]) = \{v_9, v_{10}, v_{11}\} \cup \{v_{10}, v_{11}\} = V_3$$

which ensure that the first condition is satisfied. It can easily be observed that there exists no arc that does not satisfy the second or third condition. Thus $G = (\mathcal{V}, \mathcal{A}, \mathcal{B})$ is a cumulative graph.

3. A graph representation of a set-family

We claim that a n-set-family with $n \geq 0$ can be represented by a cumulative graph. Starting from an n-set-family $\mathcal{F}^{(n)}$, we perform the steps below to obtain the cumulative graph to represent it.

Step 1: Set $\mathcal{V} := \{V_i | i \in \mathbb{Z}, 0 \leq i \leq n+1\}$ where $V_0 = \emptyset$ and

$$V_i = \{v_j | j \in \mathbb{Z}, \xi_n(i-1) < j \leq \xi_n(i)\}$$

for $1 \leq i \leq n+1$ where

$$\xi_n(k) = \begin{cases} 0 & k = 0 \\ \sum_{i=1}^{k} \left| \bigsqcup^{n-i+1} \mathcal{F}^{(n)} \right| & 0 < k \leq n \end{cases}.$$

Step 2: Define a one-to-one correspondence f_i from V_i to $\bigsqcup^{n-i+1} \mathcal{F}^{(n)}$ for each $1 \leq i \leq n+1$.

Step 3: Set $\mathcal{A} := \{A_i | 1 \leq i \leq n+1\}$ where $A_1 = \emptyset$ and

$$uv \in A_i \Leftrightarrow f_i(v) \text{ is a maximal proper subset of } f_i(u) \text{ in } \bigsqcup^{n-i+1} \mathcal{F}^{(n)}$$

for $2 \leq i \leq n+1$.

Step 4: Set $\mathcal{B} := \{B_i | 1 \leq i \leq n+1\}$ where $B_1 = \emptyset$ and

$$uv \in B_i \Leftrightarrow f_i(u) \text{ is a minimal set containing } f_{i-1}(v) \text{ in } \bigsqcup^{n-i+1} \mathcal{F}^{(n)}$$

for $2 \leq i \leq n+1$.

We give the following example to perform the steps given above for a n-set-family.

Example 2. *Given a 3-set-family*

$$\mathcal{F}^{(3)} = \{\{\{\{b\}, \{c, d\}\}\}, \{\{\{b\}\}, \{\{a\}, \{c, d\}\}\},$$
$$\{\{\{a\}, \{b\}\}, \{\{a\}, \{b\}, \{c, d\}, \{a, b, c, d\}\}\}\}.$$

Let's get the cumulative graph representing it by performing the above four steps.

In the first step, considering $\mathcal{F}^{(3)}$, we get $\xi_n(k)$ as $\xi_3(0) = 0$, $\xi_3(1) = 4$, $\xi_3(2) = 8$, $\xi_3(3) = 13$ and $\xi_3(4) = 16$ since $\bigsqcup^3 \mathcal{F}^{(3)} = \{a, b, c, d\}$, $\bigsqcup^2 \mathcal{F}^{(3)} = \{\{a\}, \{b\}, \{c, d\}, \{a, b, c, d\}\}$,

$$\bigsqcup^1 \mathcal{F}^{(3)} = \{\{\{b\}\}, \{\{a\}, \{b\}\}, \{\{a\}, \{c, d\}\}, \{\{b\}, \{c, d\}\}, \{\{a\}, \{b\}, \{c, d\}, \{a, b, c, d\}\}\}$$

and $\bigsqcup^0 \mathcal{F}^{(3)} = \mathcal{F}^{(3)}$. Hence we get $\mathcal{V} = \{V_0, V_1, V_2, V_3, V_4\}$ where $V_0 = \emptyset$, $V_1 = \{v_1, v_2, v_3, v_4\}$, $V_2 = \{v_5, v_6, v_7, v_8\}$, $V_3 = \{v_9, v_{10}, v_{11}, v_{12}, v_{13}\}$ and $V_4 = \{v_{14}, v_{15}, v_{16}\}$.

In the next step, if the functions f_i from V_i to $\bigsqcup^{4-i} \mathcal{F}^{(3)}$, $1 \leq i \leq 4$ is defined by

$$f_1(v_1) = a, f_1(v_2) = b, f_1(v_3) = c, f_1(v_4) = d,$$
$$f_2(v_5) = \{a\}, f_2(v_6) = \{b\}, f_2(v_7) = \{c, d\}, f_2(v_8) = \{a, b, c, d\},$$
$$f_3(v_9) = \{\{b\}\}, f_3(v_{10}) = \{\{a\}, \{b\}\}, f_3(v_{11}) = \{\{a\}, \{c, d\}\},$$
$$f_3(v_{12}) = \{\{b\}, \{c, d\}\}, f_3(v_{13}) = \{\{a\}, \{b\}, \{c, d\}, \{a, b, c, d\}\},$$
$$f_4(v_{14}) = \{\{\{b\}, \{c, d\}\}\}, f_4(v_{15}) = \{\{\{b\}\}, \{\{a\}, \{c, d\}\}\},$$
$$f_4(v_{16}) = \{\{\{a\}, \{b\}\}, \{\{a\}, \{b\}, \{c, d\}, \{a, b, c, d\}\}\},$$

then it is clear that each of them is a one-to-one correspondence.

In the third step, we first take A_1 as the empty set. Then we get

$$A_2 = \{v_8v_5, v_8v_6, v_8v_7\}$$

since $f_2(v_5)$, $f_2(v_6)$, $f_2(v_7)$ are maximal subsets of $f_2(v_8)$ and one of any other pair of images of members in V_2 under f_2 is not a maximal subset of the other. By similar reasoning, we have

$$A_3 = \{v_{13}v_{10}, v_{13}v_{11}, v_{13}v_{12}, v_{12}v_9, v_{10}v_9\}$$

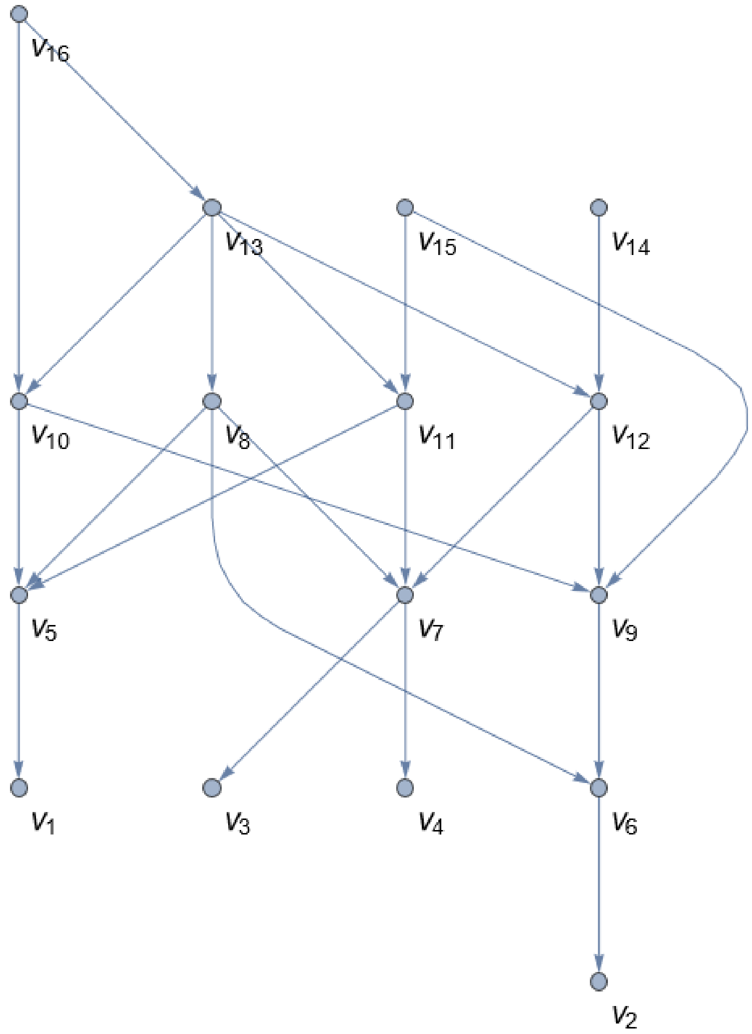

Figure 2. the cumulative graph representing the 3-set-family $\mathcal{F}^{(3)}$ in Example 2

and $A_4 = \emptyset$.

In the last step, we first take $B_1 = \emptyset$. *We obtain* $B_2 = \{v_7v_3, v_7v_4, v_6v_2, v_5v_1\}$ *because* $f_2(v_5)$ *and* $f_2(v_6)$ *is a minimal set containing* $f_1(v_1)$ *and* $f_1(v_2)$*, respectively; also* $f_2(v_7)$ *is a minimal set containing both* $f_1(v_3)$ *and* $f_1(v_4)$. *Following similar arguments, we get*

$$B_3 = \{v_{13}v_8, v_{12}v_7, v_{11}v_5, v_{11}v_7, v_{10}v_5, v_9v_6\}$$

and

$$B_4 = \{v_{16}v_{10}, v_{16}v_{13}, v_{15}v_9, v_{15}v_{11}, v_{14}v_{12}\}.$$

Thus the cumulative graph representing the 3*-set-family* $\mathcal{F}^{(3)}$ *is* $G = (\mathcal{V}, \mathcal{A}, \mathcal{B})$ *with* $\mathcal{V} = \{V_0, V_1, V_2, V_3, V_4\}$, $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ *and* $\mathcal{B} = \{B_1, B_2, B_3, B_4\}$ *as Figure 2 where* V_i*'s,* A_i*'s and* B_i*'s are taken as above.*

4. Conclusion

The cumulative graph introduced in this paper is a special class of graph which represents an arbitrary n-set family, and we have given the steps to be followed to obtain a cumulative graph from an n-set family, with an example implementation.

5. Further work

As a future work, we plan to present the definition of a topological cumulative graph on a set, reconsider some basic concepts of set-theoretic topology on a topological cumulative graph, and obtain some useful results.

References

[1] B. Bollobas. *Modern Graph Theory*. Graduate Texts in Mathematics. Springer New York, 2013.
[2] G. Chartrand and P. Zhang. *A First Course in Graph Theory*. Dover Books on Mathematics. Dover Publications, 2013.
[3] J.L. Gross and J. Yellen. *Handbook of Graph Theory*. Discrete Mathematics and Its Applications. CRC Press, 2003.
[4] K.R. Saoub. *Graph Theory: An Introduction to Proofs, Algorithms, and Applications*. Textbooks in Mathematics. CRC Press, 2021.
[5] R.J. Trudeau. *Introduction to Graph Theory*. Dover Books on Mathematics. Dover Publications, 2013.