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On some identities and Hankel matrices norms involving new defined generalized modified pell numbers

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Abstract — The aim of this paper is to introduce a generalization of Modified Pell numbers. Some identities about this new sequence are obtained and also investigated some relationships with another sequence. Finally, using these sequences the row and column norms of the Hankel matrices are presented.

Subject Classification (2020): 11B37, 11K31.

1. Introduction

In the literature, there are many integer sequences defined by a recurrence relation [1–3]. These sequences have been studied in many areas. One of the mentioned sequences is the Pell sequence. Pell sequence can be defined as [4]:

$$P_{n+1} = 2P_n + P_{n-1} \tag{1.1}$$

for $n \geq 1$ with initial conditions $P_0 = 0$ and $P_1 = 1$.

The Pell-Lucas $\{Q_n\}$ and Modified Pell $\{q_n\}$ sequences are defined by the same recurrence but the initial conditions such as $Q_0 = Q_1 = 2$ and $q_0 = q_1 = 1$ respectively [5].

There are a lot of studies on the generalization of these sequences [6–10]. A generalization of the Pell and Pell-Lucas numbers are defined as follows [11]:

$$P_{k,n} = kP_{k,n-1} + (k-1)P_{k,n-2} \tag{1.2}$$

for $n \geq 2$ and with initial conditions $P_{k,0} = 0$ and $P_{k,1} = 1$.

$Q_{k,n} = kQ_{k,n-1} + (k-1)Q_{k,n-2}$, for $n \geq 2$ with initial conditions $Q_{k,0} = Q_{k,1} = 2$.

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If $k = 2$ in (1.2), we get Pell and Pell-Lucas numbers. $P_{k,n}$ and $Q_{k,n}$ have characteristic equation as follow:

$$r^2 - kr + 1 - k = 0 \tag{1.3}$$

Since $k \geq 2$, the characteristic equation has two roots

$$r_1 = \frac{1}{2}\left(k - \sqrt{k^2 + 4k - 4}\right), r_2 = \frac{1}{2}\left(k + \sqrt{k^2 + 4k - 4}\right). \tag{1.4}$$

$P_{k,n}$ and $Q_{k,n}$ have Explicit formulas for their general terms as follows:

$$P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}, Q_{k,n} = \frac{2\left(1 - r_2\right) r_1^n + 2\left(r_1 - 1\right) r_2^n}{r_1 - r_2}. \tag{1.5}$$

In addition, the expression of the sum of the first terms of the $P_{k,n}$ and $Q_{k,n}$ series is as follows:

$$\sum_{r=0}^{n} P_{k,r} = \frac{1}{2(k-1)}((k-1)P_{k,n} + P_{k,n+1} - 1), \sum_{r=0}^{n} Q_{k,r} = \frac{1}{2(k-1)}((k-1)Q_{k,n} + Q_{k,n+1} + 2k - 4). \tag{1.6}$$

A Hankel matrix [12] is an $n \times n$ symmetric matrix $H_n = \left(h_{ij}\right)$, where $h_{ij} = h_{i+j-1}$, that is a matrix of the form

$$H_n = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots & h_n \\ h_2 & h_3 & h_4 & \cdots & h_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_n & h_{n+1} & h_{n+2} & \cdots & h_{2n-1} \end{bmatrix}.$$

Column norm $\|H_n\|_1$ and row norm $\|H_n\|_\infty$ of Hankel matrix are equivalent and defined as follows:

$$\|H_n\|_1 = \begin{matrix} \max \\ i \leq j \leq 1 \end{matrix} \sum_{i=1}^{n} \left|h_{ij}\right| = \|H_n\|_\infty. \tag{1.7}$$

In this paper, a new generalization of Modified Pell numbers is defined. Then,the generating function, Binet formula, and some identities are obtained. Finally, some special cases of Hankel matrices involving $P_{k,n}$ and $q_{k,n}$ are studied.

2. One-Parameter Generalization of Modified Pell Numbers

In this section, the Modified Pell numbers $q_{k,n}$ is defined. Then, the generating function, the Binet Formula and some identities are obtained.

Definition 2.1. Let $k \geq 2$ be an integer. Then, we define generalized Modified Pell Numbers $q_{k,n}$ as follows:

$$q_{k,n} = kq_{k,n-1} + (k-1)q_{k,n-2} \tag{2.1}$$

for $n \geq 2$ and with initial conditions $q_{k,0} = q_{k,1} = 1$.

The characteristic equation associated to (2.1) is

$$r^2 - kr + 1 - k = 0 \tag{2.2}$$

This equation has two roots

$$r_1 = \frac{1}{2}\left(k - \sqrt{k^2+4k-4}\right), r_2 = \frac{1}{2}\left(k + \sqrt{k^2+4k-4}\right). \tag{2.3}$$

Moreover, the following equations hold true:

$$r_1 + r_2 = k \tag{2.4}$$

$$r_1 r_2 = 1 - k. \tag{2.5}$$

Theorem 2.2. The generating function for the generalized modified pell numbers is

$$H\left(q_{k,n}; x\right) = h_k(x) = \sum_{n=0}^{\infty} q_{k,n} x^n = q_{k,0} + q_{k,1}x + q_{k,2}x^2 + \ldots + q_{k,n}x^n + \ldots. \tag{2.6}$$

Proof.

Using the initial conditions, we get

$$h_k(x) = 1 + x + \sum_{n=2}^{\infty} \left(kq_{k,n-1} + (k-1)\, q_{k,n-2}\right) x^n \tag{2.7}$$

By doing some calculations on the right side of the equation (2.7) , we get

$$\begin{aligned} 1 + x + \sum_{n=2}^{\infty} \left(kq_{k,n-1} + (k-1)\, q_{k,n-2}\right) x^n &= 1 + x + kx \sum_{n=2}^{\infty} q_{k,n-1} x^{n-1} + (k-1)\, x^2 \sum_{n=2}^{\infty} q_{k,n-2} x^{n-2} \\ &= 1 + x + kx \sum_{n=1}^{\infty} q_{k,n} x^n + (k-1)\, x^2 \sum_{n=0}^{\infty} q_{k,n} x^n. \end{aligned}$$

By using equation (2.6), we obtain

$$h_k(x) = 1 + x - kx + kxh_k(x) + (k-1)\, x^2 h_k(x).$$

Hence, we have

$$h_k(x) = \frac{1 + x - kx}{1 - kx - (k-1)\, x^2}$$

Theorem 2.3. (Binet's Formula) The nth Generalized Modified Pell Number is given by

$$q_{k,n} = \frac{(1-r_2)\, r_1^n + (r_1 - 1)\, r_2^n}{r_1 - r_2} \tag{2.8}$$

where r_1,r_2 are given in (2.3).

Proof.

Since the equation (2.2) has two distinct roots, the sequence

$$q_{k,n} = er_1^n + fr_2^n \tag{2.9}$$

is the solution of the equation (2.1). Putting $q_{k,0} = q_{k,1} = 1$ we get $e + f$ and $er_1 + fr_2 = 1$. If we solve this system of linear equations, we obtain $e = \frac{1-r_2}{r_1-r_2}$ and $f = \frac{r_1-1}{r_1-r_2}$.

Using these values and (2.9) we obtain (2.8) as required.

Corollary 2.4. Let k and n be integers and $k \geq 2$ and $n \geq 0$. Then, we get

$$2q_{k,n} = Q_{k,n}.$$

Proof.

By using (1.5), we can prove it.

Corollary 2.5. Let k and n be integers and $k \geq 2$ and $n \geq 0$. Then, we get

$$q_{k,n} = P_{k,n} + (k-1)P_{k,n-1}$$

Proof.

By using (2.8) we get

$$q_{k,n} = \frac{(1-r_2)\, r_1^n + (r_1 - 1)\, r_2^n}{r_1 - r_2} = \frac{r_1^n - r_2^n}{r_1 - r_2} + \frac{r_2^n r_1 - r_1^n r_2}{r_1 - r_2} = \frac{r_1^n - r_2^n}{r_1 - r_2} - r_1 r_2 \frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}.$$

Using (2.5), we have

$$q_{k,n} = P_{k,n} + (k-1)P_{k,n-1}.$$

Corollary 2.6. Let k and n be integers and $k \geq 2$ and $n \geq 0$. Then, we obtain

$$q_{k,n} = P_{k,n+1} - (k-1)P_{k,n}.$$

Proof.

By using Corollary 2.5, we can proof it.

Theorem 2.7. (Catalan's Identitiy) For any positive integer r, we get

$$q_{k,n-r}\, q_{k,n+r} - q_{k,n}^2 = -2(1-k)^{n-r+1} P_{k,r}^2.$$

Proof.

Using (2.8), we have

$$\begin{aligned}
q_{k,n-r}\, q_{k,n+r} - q_{k,n}^2 &= \frac{(1-r_2)r_1^{n-r}+(r_1-1)r_2^{n-r}}{r_1-r_2}\,\frac{(1-r_2)r_1^{n+r}+(r_1-1)r_2^{n+r}}{r_1-r_2} - \left(\frac{(1-r_2)r_1^{n}+(r_1-1)r_2^{n}}{r_1-r_2}\right)^2 \\
&= \frac{(1-r_2)r_1^{n-r}(r_1-1)r_2^{n+r}+(1-r_2)r_1^{n+r}(r_1-1)r_2^{n-r}-2(1-r_2)r_1^{n}(r_1-1)r_2^{n}}{(r_1-r_2)^2} \\
&= \frac{(1-r_2)(r_1-1)(r_1r_2)^n}{(r_1-r_2)^2}(r_1^{-r} r_2^{r} + r_1^{r} r_2^{-r} - 2) \\
&= \frac{(1-r_2)(r_1-1)(r_1r_2)^n}{(r_1-r_2)^2}\,\frac{(r_1^{-2r}+r_2^{2r}-2(r_1r_2)^r)}{(r_1r_2)^r} \\
&= \frac{(1-r_2)(r_1-1)(r_1r_2)^n}{(r_1-r_2)^2}(r_1^{-r} - r_2^{r})^2 \\
&= (r_1r_2)^{n-r}(r_1 - r_1r_2 - 1 + r_2)\left(\frac{r_1^{-r}-r_2^{r}}{r_1-r_2}\right)^2
\end{aligned}.$$

By using (2.5) and (1.5), we get

$$\begin{aligned}
q_{k,n-r}\, q_{k,n+r} - q_{k,n}^2 &= (1-k)^{n-r}(2k-2)P_{k,r}^2 \\
&= -2(1-k)^{n-r+1}P_{k,r}^2.
\end{aligned}$$

Theorem 2.8. Let m, n be positive integers and $m \geq n$. Then, we get

$$q_{k,m}q_{k,n+1} - q_{k,m+1}q_{k,n} = (1-k)^{n+1}P_{k,m-n}.$$

Proof.

By using (2.8), we have

$$\begin{aligned} q_{k,m}q_{k,n+1} - q_{k,m+1}q_{k,n} &= \tfrac{(r_2-1)r_1^m+(1-r_1)r_2^m}{r_2-r_1}\tfrac{(r_2-1)r_1^{n+1}+(1-r_1)r_2^{n+1}}{r_2-r_1} \\ &- \tfrac{(r_2-1)r_1^{m+1}+(1-r_1)r_2^{m+1}}{r_2-r_1}\tfrac{(r_2-1)r_1^{n}+(1-r_1)r_2^{n}}{r_2-r_1} \\ &= (r_2-1)(1-r_1)\tfrac{(r_1r_2)^n(r_1^{m-n}r_2+r_2^{m-n}r_1-r_1^{m-n+1}-r_2^{m-n+1})}{(r_2-r_1)^2} \\ &= (r_2-1)(1-r_1)(r_1r_2)^n\tfrac{r_1^{m-n}-r_2^{m-n}}{r_2-r_1}. \end{aligned}$$

Using (2.5), we get

$$q_{k,m}q_{k,n+1} - q_{k,m+1}q_{k,n} = (1-k)^{n+1}P_{k,m-n}.$$

Theorem 2.9. For all integers $k \geq 2$ and $n \geq 0$ we obtain

$$\sum_{i=0}^{n} q_{k,i} = \frac{1}{2(1-k)}((1-k)q_{k,n} - q_{k,n+1} + 2 - k).$$

Proof.

Note that

$$\begin{aligned} \sum_{i=0}^{n} q_{k,i} &= \tfrac{(1-r_2)}{r_1-r_2}\sum_{i=0}^{n} r_1^n + \tfrac{(r_1-1)}{r_1-r_2}\sum_{i=0}^{n} r_2^n = \tfrac{(1-r_2)}{r_1-r_2}(\tfrac{1-r_1^{n+1}}{1-r_1}) + \tfrac{(r_1-1)}{r_1-r_2}(\tfrac{1-r_2^{n+1}}{1-r_2}) \\ &= \tfrac{(1-r_2)^2(1-r_1^{n+1})-(r_1-1)^2(1-r_2^{n+1})}{(r_1-r_2)(1-r_1)(1-r_2)} = \tfrac{1}{(1-r_1)(1-r_2)}(\tfrac{-(r_1^{n+1})(1-r_2)-r_2^{n+1}(r_1-1)}{(r_1-r_2)} \\ &+ r_1r_2\tfrac{(r_1^n(1-r_2)+r_2^n(r_1-1))}{(r_1-r_2)} + \tfrac{2(r_1-r_2)-(r_1+r_2)(r_1-r_2)}{r_1-r_2}) \end{aligned}$$

.

By using (2.5), (2.8) and by noting that $(1-r_1)(1-r_2) = 2(1-k)$, we get

$$\sum_{i=0}^{n} q_{k,i} = \frac{1}{2(1-k)}(-q_{k,n+1} + (1-k)q_{k,n} + 2 - k).$$

3. Norms of Hankel Matrices Involving $q_{k,n}$ and $P_{k,n}$

In the literature, some important works have been done about norms of some matrices especially, the norms and bounds for the norms of Hankel matrix involving some integer sequences were studied [11],[12]. In this section, we obtain row and column norms of the Hankel matrix using $q_{k,n}$ and $P_{k,n}$.

Theorem 3.1. Let A be a $n \times n$ matrix with $a_{ij} = P_{k,i+j-1}$. Then, we have

$$\|A\|_1 = \|A\|_\infty = \frac{1}{2(k-1)}(q_{k,2n} - q_{k,n}).$$

Proof.

Using $a_{ij} = P_{k,i+j-1}$, we get

$$\|A\|_1 = \max_{i \le j \le 1} \sum_{i=1}^{n} |a_{ij}| = \max_{i \le j \le 1} \{|a_{1j}| + |a_{2j}| + ... + |a_{nj}|\} = P_{k,n}+P_{k,n+1}+...+P_{k,2n-1} = \sum_{i=0}^{2n-1} P_{k,i} - \sum_{i=0}^{n-1} P_{k,i}.$$

Now using (1.6) and Corollary 2.5, we get

$$\|A\|_1 = \frac{1}{2(k-1)}((k-1)P_{k,2n-1} + P_{k,2n-1}) - \frac{1}{2(k-1)}((k-1)P_{k,n-1} + P_{k,n-1}) = \frac{1}{2(k-1)}(q_{k,2n} - q_{k,n}).$$

Theorem 3.2. Let B be a $n \times n$ matrix with $b_{ij} = q_{k,i+j-1}$. Then, we get

$$\|B\|_1 = \|B\|_\infty = \frac{1}{2(1-k)}((k-2)(P_{k,2n+1} - P_{k,n+1}) - k(k-1)(P_{k,2n} - P_{k,n})).$$

Proof.

Using Theorem 2.9, Corollary 2.5 and Corollary 2.6, we obtain

$$\begin{aligned}
\|B\|_1 &= q_{k,n} + q_{k,n+1} + ... + q_{k,2n-1} \\
&= \sum_{i=0}^{2n-1} q_{k,i} - \sum_{i=0}^{n-1} q_{k,i} \\
&= \tfrac{1}{2(1-k)}((1-k)q_{k,2n-1} - q_{k,2n} - (1-k)q_{k,n-1} + q_{k,n}) \\
&= \tfrac{1}{2(1-k)}(-q_{k,2n+1} + (k-1)q_{k,2n} + q_{k,n+1} + (1-k)q_{k,n} \\
&= \tfrac{1}{2(1-k)}((k-2)(-P_{k,2n+1} - (k-1)P_{k,2n} + (k-1)P_{k,2n+1} - (k-1)^2 P_{k,2} \\
&+ P_{k,n+1} + (k-1)P_{k,n} + (1-k)P_{k,n+1} - (k-1)^2 P_{k,n} \\
&= \tfrac{1}{2(1-k)}((k-2)(P_{k,2n+1} - P_{k,n+1}) - k(k-1)(P_{k,2n} - P_{k,n})).
\end{aligned}$$

Author Contributions

The author read and approved the final version of the manuscript.

Conflicts of Interest

The author declares no conflict of interest.

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