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Directional Energy Functionals Through Anholonomic Coordinates

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Abstract

In this paper, a special case of directional energy functional is investigated by computing the directional energy and pseudo-angle of unit vector fields in the ordinary three-dimensional space. This approach is also extended simultaneously to define the critical points of the directional energy functionals of the velocity fields. Then, the restriction of the harmonic maps and the extrema of the directional energy functionals is considered. Finally, we compute directional harmonic and biharmonic equations of the curvature vector fields to generalize total bending or energy of vector fields.

1. Introduction

The energy functional of a given family of unit vector fields is described to be the sectional energy of the mapping of the vector field. This map is generally defined between Riemannian and semi-Riemannian manifolds, where the integration is induced with respect to the definite or indefinite structures of the standard measure. These structures are generally carried by the unit tangent bundles together with the natural connection of the Sasaki metric inherited by the Levi-Civita connection. The consideration of smooth sections of unit vector fields leads also to the volume functional of the corresponding volume immersion. There have been many significant studies about energy and volume so far. For instance, Wiegmink [1,2] focused on the quantitative measure of the total bending and energy of Hopf unit vectors on the sphere $\mathbb{S}^3$ and 2-torus, which double covers the 2-sphere. He developed such an efficient method that it finds out instantly what consequences about minimizing functionals can be determined and what kind of consequences can be anticipated by direct methods. Gluck and Ziller [3] defined other functionals to measure the quantity of deviation of vector fields from parallelism. They also established an elegant approach by interrelating the submanaifolds of the tangent bundle and the volume of unit vectors via the Sasaki metric. Brito [4] computed the absolute minima and critical points of the energy of Hopf unit vector fields in $\mathbb{S}^3$. Wood [5] proved that critical points of the energy of Hopf unit vector fields in the odd-dimensional sphere are both not stable critical points and minima.

The concept of corrected energy and its distribution is improved since volume and energy functionals have specific properties on higher-dimensional spaces. Thus, the existence of absolute minimizing of the traditional volume and energy functionals together with their instable and stable critical points can be investigated. Chacón and Naveira [6] defined the corrected energy by adding the norms of the mean curvature fields and their different weights (orthogonal distribution). Furthermore, Chacón et al. [7] introduced the energy of q-distribution by considering the sectional energy and the Sasaki metric with some applications to the quaternionic Hopf structures.

Altın [8,9,10] improved a very interesting and useful approach to deal with the problem of computing the energy of unit vector fields. She

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showed that one only needs to focus on a space curve together with its associated orthonormal vectors and geometric quantities instead of a manifold to determine the energy of the unit vector fields. Based on this methodology, Altın [8] computed the pseudo-angle and energy of Frenet-Serret orthonormal vector fields in indefinite structures. Altın [9] also obtained the necessary conditions of critical curves and energy minimizer points of velocity vector fields. Finally, Altın [10] proved that the energy of a unit normal vector defined on a Riemannian surface is not dependent on the selection of an orthonormal basis in the tangent space. By using the similar approach, Körpınar and Demirkol [11-13] calculated the energy of many important vectrof fields included from geometric or physical context.

Biharmonic or harmonic maps between Riemannian or semi-Riemannian manifolds are described as critical points of generalized bienergy or energy maps, respectively. It is an obvious fact that being harmonic implies biharmonicity. However, the converse relation does not hold. If the biharmonic map is also a nonharmonic then it is called proper. This difference plays a key role defining several concepts in hydrodynamics and elasticity. It has also been heavily investigated in real space forms, unit sphere, and submanifolds by many mathematicians. For example, Inoguchi [14] and Sasahara [15] defined the fundamental proper biharmonic maps and typical examples in the Sasakian space form. They classified proper biharmonic Legendre surfaces, proper biharmonic Legendre curves, and Hopf cylinders in Sasakian 5-space form and in Sasakian 3-space form, separately. Chen [16-19] studied biharmonicity conditions of the finite type of submanifolds in different space structures via geometry of smooth maps and spectral geometry. Later on, Körpınar and Turhan [20-25] studied energy and total bending of horizontal biharmonic curves, magnetic biharmonic curves, and their flows up to certain surfaces in many space structures.

The manuscript is organized as follows. In Section 2, we briefly review the differential geometry of space curves and anholonomic coordinates in the three-dimensional Euclidean space. Keeping in mind that the geometric representation of unit orthonormal vectors is given by the Frenet-Serret equations in the tangent direction, we remind readers that the geometric representation of unit orthonormal vectors is given by the Gauss Weingarten equations in the normal and binormal directions. In Section 3, we define directional energy and pseudo-angle of unit Frenet-Serret vector fields in the three-dimensional ordinary space. We further compute the energy and pseudo-angle of particular vector fields representing certain planes and the most general form of vector fields. In Section 4, we calculate the critical points of the directional energy functionals of the velocity fields for different cases. For each case, we obtain necessary conditions of minimizing energy of the velocity fields, which leads to obtain some results on the extrema and critical curves. In Section 5, we describe directional harmonic and biharmonic equations asscociated with the directional energy functionals. In Seciton 6, we discuss further potential research topics and summarize our conclusions.

2. Material and Method

2.1. The Geometry of Space Curves and Gauss Weingarten Equations in $\mathbb{E}^3$

Let α be a space curve in $\mathbb{E}^3$ such that three-dimensional coordinate location (x, y, z) indicates a point on α. Furthermore, let $\mathcal{R}$ be a position vector placed at the same reference frame pointing to the location on α.

The Frenet-Serret frame of space curve α is described by the ordered triad of unit orthonormal vectors such that they are mutually perpendicular to each other. It is also called the moving trihedron or moving triple. It includes tangential axis $\mathbf{t}$, principal normal axis $\mathbf{n}$, and binormal axis $\mathbf{b}$. $(\mathbf{t,n,b})$ triad satisfies the following cross product or vector product rule due to cyclic permutations, i.e.

$$\mathbf{t} = \mathbf{n} \times \mathbf{b}, \quad \mathbf{b} = \mathbf{t} \times \mathbf{n}, \quad \mathbf{n} = \mathbf{b} \times \mathbf{t}. \qquad (1)$$

Let (s, n, b) denotes arc distance along with Frenet-Serret vectors $(\mathbf{t,n,b})$, respectively. (s, n, b) establishes a suitable curvilinear coordinate frame if arc distances are restricted appropriately around the origin, for instance $(s = s_0, n = 0, b = 0)$.

In the schematic trihedron, Frenet-Serret vectors of $(\mathbf{t,b})$ span the rectifying plane; Frenet-

Serret vectors of $(\mathbf{t,n})$ span the osculating plane; Frenet-Serret vectors of $(\mathbf{n,b})$ span the normal plane.

Vector calculus components on vector or scalar fields are typically expressed by the fundamental three vector operators. For instance, the divergent operator acts on an arbitrary vector field $\mathcal{Z}$ in the following manner

$$div\mathcal{Z} = \mathbf{t}\cdot\frac{\delta}{\delta s}\mathcal{Z} + \mathbf{n}\cdot\frac{\delta}{\delta s}\mathcal{Z} + \mathbf{b}\cdot\frac{\delta}{\delta s}\mathcal{Z}. \quad (2)$$

The curl operator acts on an arbitrary vector field $\mathcal{Z}$ in the following manner

$$curl\mathcal{Z} = \mathbf{t}\times\frac{\delta}{\delta s}\mathcal{Z} + \mathbf{n}\times\frac{\delta}{\delta s}\mathcal{Z} + \mathbf{b}\times\frac{\delta}{\delta s}\mathcal{Z}. \quad (3)$$

Finally, the gradient operator acts on an arbitrary scalar field $\mathcal{Y}$ in the following manner

$$grad\mathcal{Y} = \frac{\delta\mathcal{Y}}{\delta s}\mathbf{t} + \frac{\delta\mathcal{Y}}{\delta s}\mathbf{n} + \frac{\delta\mathcal{Y}}{\delta s}\mathbf{b}. \quad (4)$$

The Frenet-Serret formulas describe the motion of the ordered triad of unit orthonormal vectors $(\mathbf{t,n,b})$ along with the $s-line$ coordinate curve (vector line of s). In this case, the motion of the triad frame follows a space curve α parametrized by the arc-length s. By definition, the directional derivative of α with respect to the arc-length s is equal to $\mathbf{t}$ i.e.

$$\frac{\delta}{\delta s}\alpha = \mathbf{t}. \quad (5)$$

It is denoted by the following identity

$$\frac{\delta}{\delta s}\alpha = \frac{\partial}{\partial s}\alpha\frac{1}{\sqrt{\frac{\partial}{\partial s}\alpha\cdot\frac{\partial}{\partial s}\alpha}}. \quad (6)$$

In this paper, we always consider the special case in which the curve is supposed to be a unit speed curve i.e.

$$\frac{\delta}{\delta s}\alpha = \frac{\partial}{\partial s}\alpha, \left|\frac{\partial}{\partial s}\alpha\right| = 1. \quad (7)$$

The Frenet-Serret formulas are characterized by taking the directional derivative of the unit orthonormal vectors $(\mathbf{t,n,b})$ with respect to the vector line of s in the following manner

$$\begin{aligned}\frac{\delta}{\delta s}\mathbf{t} &= \kappa\mathbf{n},\\ \frac{\delta}{\delta s}\mathbf{n} &= -\kappa\mathbf{t}+\tau\mathbf{b},\\ \frac{\delta}{\delta s}\mathbf{b} &= -\tau\mathbf{n},\end{aligned}$$

(8)

where κ measures the bending or rate of change of the tangent vector in the $(\mathbf{t,n})$ plane along with the vector line of s. Torsion τ measures the twisting or amount of rotation of the Frenet-Serret triad frame about the $\mathbf{t}$ along with the vector line of s.

The directional derivative of α in the normal and binormal directions are expressed by considering the parametrization with respect to arc-lentgh n and b, respectively. In the normal direction, the directional derivative of α with respect to the arc-length n is equal to $\mathbf{n}$ i.e.

$$\frac{\delta}{\delta n}\alpha = \mathbf{n},$$

(9)

where the unit speed curve parametrization is guaranteed by the following assumption

$$\frac{\delta}{\delta n}\alpha = \frac{\partial}{\partial n}\alpha, \left|\frac{\partial}{\partial n}\alpha\right| = 1. \quad (10)$$

Eqs. (9,10) imply that the tangent vector of the $n-line$ coordinate curve (vector line of n) is $\mathbf{n}$ in the normal direction.

Similarly, in the binormal direction, the directional derivative of α with respect to the arc-length b is equal to $\mathbf{b}$ i.e.

$$\frac{\delta}{\delta b}\alpha = \mathbf{b}, \quad (11)$$

where the unit speed curve parametrization is guarenteed by the following assumption

$$\frac{\delta}{\delta b}\alpha = \frac{\partial}{\partial b}\alpha, \left|\frac{\partial}{\partial b}\alpha\right| = 1. \quad (12)$$

Eqs. (11,12) imply that the tangent vector of the $b-line$ coordinate curve (vector line of b) is $\mathbf{b}$ in the binormal direction. These implications lead to define a new type of variations of the motion of the Frenet-Serret frame by taking the directional derivative of the $(\mathbf{t,n,b})$ along with the $n-line$

coordinate curve and $b-line$ coordinate curve. Accordingly, these variations satisfy that

$$\begin{aligned} \frac{\delta}{\delta n}\mathbf{t} &= \theta_{ns}\mathbf{n}+(\Omega_b+\tau)\mathbf{b}, \\ \frac{\delta}{\delta n}\mathbf{n} &= -\theta_{ns}\mathbf{t}-(div\mathbf{b})\mathbf{b}, \\ \frac{\delta}{\delta n}\mathbf{b} &= -(\Omega_b+\tau)\mathbf{t}+(div\mathbf{b})\mathbf{n}, \end{aligned} \quad (13)$$

and

$$\begin{aligned} \frac{\delta}{\delta b}\mathbf{t} &= -(\Omega_n+\tau)\mathbf{n}+\theta_{bs}\mathbf{b}, \\ \frac{\delta}{\delta b}\mathbf{n} &= (\Omega_n+\tau)\mathbf{t}+(\kappa+div\mathbf{n})\mathbf{b}, \\ \frac{\delta}{\delta b}\mathbf{b} &= -\theta_{bs}\mathbf{t}-(\kappa+div\mathbf{n})\mathbf{n}. \end{aligned} \quad (14)$$

The entire directional differential equation systems of the Frenet-Serret triad vectors given by Eqs. (8,13,14) are called Gauss Weingarten equations [26].

The geometric quantities θ_{ns} and θ_{bs} symbolize the normal deformation of the vector tube in the normal and binormal directions, respectively, in the following manner

$$\theta_{ns}=\mathbf{n}\frac{\delta}{\delta n}\mathbf{t},\ \theta_{ns}=\mathbf{b}\frac{\delta}{\delta n}\mathbf{t}. \quad (15)$$

The divergence of the tangent, normal, and binormal vectors are expressed by the following identities

$$\begin{aligned} div\mathbf{t} &= \theta_{ns}+\theta_{bs}, \\ div\mathbf{n} &= \mathbf{b}\cdot\frac{\delta}{\delta b}\mathbf{n}-\kappa, \\ div\mathbf{b} &= -\mathbf{b}\cdot\frac{\delta}{\delta n}\mathbf{n}. \end{aligned} \quad (16)$$

The curl of the tangent, normal, and binormal vectors are expressed by the following notation

$$curl\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}=\begin{pmatrix}\Omega_s & 0 & \kappa\\ -div\mathbf{b} & \Omega_n & \theta_{ns}\\ \kappa+div\mathbf{n} & -\theta_{bs} & \Omega_b\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}, \quad (17)$$

where $\Omega_s,\Omega_n,\Omega_b$ are called the abnormalities of the $\mathbf{t}-field$, $\mathbf{n}-field$, and $\mathbf{b}-field$ [26]. They are computed by considering the comparison of the two forms for the $curl$ operator. There also exists the following relation among these functions

$$2(\Omega_s-\tau)=\Omega_s+\Omega_n+\Omega_b. \quad (18)$$

3. Results and Discussion

3.1. Directional Energy and Pseudo-angle of Vector Fields in $\mathbb{E}^3$

A generalized volume and energy functionals research is conducted by computing the restricted critical points and constrained variational formulas of unit vector fields. The volume of an immersion from Riemannian manifold to differential one and the energy of a map between Riemannian or semi-Riemannian manifolds are functionals that have been investigated through the variational approach. This technique leads to defining the significant structures of minimal immersion and harmonic map described in a generalized manifold having a vanishing mean curvature and tension. The energy of vector fields is dependent on the constants since the total bending energy can be characterized by the same variational problem. The volume of the immersion is an efficient tool to define the volume functional of a vector field. However, the calculation of the critical points of the constrained volume functional can be completed if each critical vector field is also a critical point of the immersed functional in the entire manifold.

Let $(\mathcal{S},\varpi)$ and $(\mathcal{T},\upsilon)$ be two Riemannian manifolds and ϕ be a differential map between these two manifolds so that the energy functional is defined by the following equality

$$\mathcal{E}(\phi)=\frac{1}{2}\int_{\mathcal{S}_u}\upsilon(d\varpi(e_u),d\varpi(e_u))\mathcal{V}, \quad (19)$$

where $\mathcal{V}$ is a volume in $\mathcal{S}$ and

$$\phi:(\mathcal{S},\varpi)\rightarrow(\mathcal{T},\upsilon). \quad (20)$$

Alternatively, the definition of the energy functional is also denoted by

$$\mathcal{E}(\phi)=\frac{1}{2}\int_{\mathcal{S}}\|d\phi\|^2\mathcal{V}. \quad (21)$$

Let $\mathcal{A}$ be an arbitrary unit vector field defined in $(\mathcal{S},\varpi)$. Then the energy section of $\mathcal{A}:\mathcal{S}\rightarrow\mathcal{B}^*\mathcal{S}$, where $\mathcal{B}^*\mathcal{S}$ is a unit tangent bundle connected with the Sasaki metric $\mathcal{S}_{\mathcal{M}}$, defines the energy of the vector field $\mathcal{A}$. One needs to separate point wise tangent space and the tangent bundle to understand the Sasaki metric on the $\mathcal{BS}$. Further information can be found in the book of Sakai [27]. The standard Levi-Civita connection ∇ can be considered to define a special connection map $\mathcal{L}:\mathcal{B}(\mathcal{B}^*\mathcal{S})\to\mathcal{B}^*\mathcal{S}$ such that it satisfies

$$\sigma\circ\mathcal{L}=\sigma\circ\tilde{\sigma}\,\text{and}\,\sigma\circ\mathcal{L}\ \ \sigma\circ d\sigma, \quad (22)$$

where

$$\begin{aligned}\tilde{\sigma} &= \mathcal{B}(\mathcal{B}^*\mathcal{S})\to\mathcal{B}^*\mathcal{S},\\ \sigma &= \mathcal{B}^*\mathcal{S}\to\mathcal{S}.\end{aligned} \quad (23)$$

Moreover, the connection map $\mathcal{L}$ also verifies

$$\mathcal{L}(d\eta(\psi))=\nabla_{\psi}\psi, \quad (24)$$

where $\psi\in\mathcal{B}_x\mathcal{S}$ and $\eta:\mathcal{S}\to\mathcal{B}^*\mathcal{S}$. Finally, the Sasaki metric $\mathcal{S}_{\mathcal{M}}$ is written by splitting vertical and horizontal orthogonal component in the following way

$$\mathcal{S}_{\mathcal{M}}(\varphi_1,\varphi_2)=d\sigma(\varphi_1)\cdot d\sigma(\varphi_1)+\mathcal{L}(\varphi_1)\cdot\mathcal{L}(\varphi_1), \quad (25)$$

where $\varphi_1,\varphi_2\in\mathcal{B}_{\eta}(\mathcal{B}^*\mathcal{S})$. From Eqs. (19,22-25), one can induce the following identity for the section η as

$$\begin{aligned}\mathcal{S}_{\mathcal{M}}(d\sigma(e_u),d\sigma(e_u)) &= d\sigma(d\eta(e_u))\cdot d\sigma(d\eta(e_u))\\ &+\mathcal{L}(d\sigma(e_u))\cdot\mathcal{L}(d\sigma(e_u))),\end{aligned} \quad (26)$$

where

$$\mathcal{L}(d\sigma(e_u))=\nabla_{e_u}\eta,\ d\sigma\circ d\eta=id_{\mathcal{BS}}. \quad (27)$$

Further details on the corrected energy functional and its distribution can be found in [6,8]. So far, we have given a short overview to introduce the method we use for the rest of the section to compute the directional energy and pseudo-angle of unit vector fields in the ordinary space.

Case 1: In this case, let us assume that α be a space curve in $\mathbb{E}^3$ such that its position vector $\mathcal{R}$ is parametrized by the arc-length n pointing to the location α in the normal direction.

Theorem 3.1.1. The energy of the Frenet-Serret triad vectors $(\mathbf{t,n,b})$ along with the $n-line$ coordinate curve is computed by

$$\begin{aligned}\mathcal{E}_n(\mathbf{t}) &= \frac{1}{2}\int_{\mathcal{R}}(\theta_{ns}^2+(\Omega_b+\tau)^2)dn+\frac{n}{2},\\ \mathcal{E}_n(\mathbf{n}) &= \frac{1}{2}\int_{\mathcal{R}}(\theta_{ns}^2+(div\mathbf{b})^2)dn+\frac{n}{2},\\ \mathcal{E}_n(\mathbf{b}) &= \frac{1}{2}\int_{\mathcal{R}}((\Omega_b+\tau)^2+(div\mathbf{b})^2)dn+\frac{n}{2}.\end{aligned} \quad (28)$$

Proof. From Eqs. (19,27), the splitting space and canonic generalization of the same object can be written due to the Levi-Civita connection map and the so-called Sasaki metric. By considering the clear definition in distribution and sectional energy, we write for unit vector fields $(\mathbf{t,n,b})$ as

$$\mathcal{E}_n(\mathbf{t}) = \frac{1}{2}\int(d\mathbf{t}(\mathbf{n}),d\mathbf{t}(\mathbf{n}))dn, \quad (29)$$

$$\mathcal{E}_n(\mathbf{n}) = \frac{1}{2}\int(d\mathbf{n}(\mathbf{n}),d\mathbf{n}(\mathbf{n}))dn, \quad (30)$$

$$\mathcal{E}_n(\mathbf{b}) = \frac{1}{2}\int(d\mathbf{b}(\mathbf{n}),d\mathbf{b}(\mathbf{n}))dn, \quad (31)$$

where

$$\begin{aligned}\mathcal{S}_{\mathcal{M}}(d\mathbf{t}(\mathbf{n}),d\mathbf{t}(\mathbf{n})) &= d\sigma(\mathbf{t}(\mathbf{n}))\cdot d\sigma(\mathbf{t}(\mathbf{n}))\\ &+\mathcal{L}(\mathbf{t}(\mathbf{n}))\cdot\mathcal{L}(\mathbf{t}(\mathbf{n})),\\ &=\mathbf{n}\cdot\mathbf{n}+(\delta/\delta n)\mathbf{t}\cdot(\delta/\delta n)\mathbf{t},\end{aligned} \quad (32)$$

$$\begin{aligned}\mathcal{S}_{\mathcal{M}}(d\mathbf{n}(\mathbf{n}),d\mathbf{n}(\mathbf{n})) &= d\sigma(\mathbf{n}(\mathbf{n}))\cdot d\sigma(\mathbf{n}(\mathbf{n}))\\ &+\mathcal{L}(\mathbf{n}(\mathbf{n}))\cdot\mathcal{L}(\mathbf{n}(\mathbf{n}))\\ &=\mathbf{n}\cdot\mathbf{n}+(\delta/\delta n)\mathbf{n}\cdot(\delta/\delta n)\mathbf{n},\end{aligned} \quad (33)$$

$$\begin{aligned}\mathcal{S}_{\mathcal{M}}(d\mathbf{b}(\mathbf{n}),d\mathbf{b}(\mathbf{n})) &= d\sigma(\mathbf{b}(\mathbf{n}))\cdot d\sigma(\mathbf{b}(\mathbf{n}))\\ &+\mathcal{L}(\mathbf{b}(\mathbf{n}))\cdot\mathcal{L}(\mathbf{b}(\mathbf{n}))\\ &=\mathbf{n}\cdot\mathbf{n}+(\delta/\delta n)\mathbf{b}\cdot(\delta/\delta n)\mathbf{b}.\end{aligned} \quad (34)$$

Using Eqs. (13,29-34), it is computed respectively that

$$\begin{aligned}\mathcal{E}_n(\mathbf{t}) &= \frac{1}{2}\int_{\mathcal{R}}(\theta_{ns}^2+(\Omega_b+\tau)^2)dn+\frac{n}{2},\\ \mathcal{E}_n(\mathbf{n}) &= \frac{1}{2}\int_{\mathcal{R}}(\theta_{ns}^2+(div\mathbf{b})^2)dn+\frac{n}{2},\\ \mathcal{E}_n(\mathbf{b}) &= \frac{1}{2}\int_{\mathcal{R}}((\Omega_b+\tau)^2+(div\mathbf{b})^2)dn+\frac{n}{2}.\end{aligned}$$

Thus, the proof is completed.

Lemma 3.1.1. The pseudo-angle of the Frenet-Serret triad vectors $(\mathbf{t,n,b})$ along with the $n-line$ coordinate curve is computed by

$$\begin{aligned}\mathcal{P}_n(\mathbf{t}) &= \frac{1}{2}\int_{\mathcal{R}}\sqrt{(\theta_{ns}^2+(\Omega_b+\tau)^2)}dn,\\ \mathcal{P}_n(\mathbf{n}) &= \frac{1}{2}\int_{\mathcal{R}}\sqrt{(\theta_{ns}^2+(div\mathbf{b})^2)}dn,\\ \mathcal{P}_n(\mathbf{b}) &= \frac{1}{2}\int_{\mathcal{R}}\sqrt{((\Omega_b+\tau)^2+(div\mathbf{b})^2)}dn.\end{aligned} \tag{35}$$

Proof. The proof is obvious if one considers the following definitions respectively

$$\mathcal{P}_n(\mathbf{t})=\frac{1}{2}\int_{\mathcal{R}}\|(\delta/\delta n)\mathbf{t}\|dn,\quad \mathcal{P}_n(\mathbf{n})=\frac{1}{2}\int_{\mathcal{R}}\|(\delta/\delta n)\mathbf{n}\|dn,$$
$$\mathcal{P}_n(\mathbf{b})=\frac{1}{2}\int_{\mathcal{R}}\|(\delta/\delta n)\mathbf{b}\|dn. \tag{36}$$

Theorem 3.1.2. i. The energy of the unit vector field $\mathcal{A}^r$ in the rectifying plane along with the $n-line$ coordinate curve is computed by

$$\mathcal{E}_n^{(\mathbf{t,b})}(\mathcal{A}^r)=\frac{1}{2}\int_{\mathcal{R}}((((\delta/\delta n)\lambda-\beta(\Omega_b+\tau))^2+(\lambda\theta_{ns}+\beta div\mathbf{b})^2+((\delta/\delta n)\beta+\lambda(\Omega_b+\tau))^2)dn+\frac{n}{2}, \tag{37}$$

where

$$\mathcal{A}^r=\lambda\mathbf{t}+\beta\mathbf{b},$$

and λ,β are sufficiently smooth functions along with the $n-line$ coordinate curve.

ii. The energy of the unit vector field $\mathcal{A}^o$ in the osculating plane along with the $n-line$ coordinate curve is computed by

$$\mathcal{E}_n^{(\mathbf{t,n})}(\mathcal{A}^o)=\frac{1}{2}\int_{\mathcal{R}}((((\delta/\delta n)\lambda-\beta\theta_{ns})^2+((\delta/\delta n)\beta+\lambda\theta_{ns})^2+(\lambda(\Omega_b+\tau)-\beta div\mathbf{b})^2)dn+\frac{n}{2}, \tag{38}$$

where

$$\mathcal{A}^o=\lambda\mathbf{t}+\beta\mathbf{n},$$

and λ,β are sufficiently smooth functions along with the $n-line$ coordinate curve.

iii. The energy of the unit vector field $\mathcal{A}^n$ in the normal plane along with the $n-line$ coordinate curve is computed by

$$\mathcal{E}_n^{(\mathbf{n,b})}(\mathcal{A}^n)=\frac{1}{2}\int_{\mathcal{R}}((\lambda\theta_{ns}+\beta(\Omega_b+\tau))^2+((\delta/\delta n)\lambda+\beta div\mathbf{b})^2+((\delta/\delta n)\beta-\lambda div\mathbf{b})^2)dn+\frac{n}{2}, \tag{39}$$

where

$$\mathcal{A}^n=\lambda\mathbf{n}+\beta\mathbf{b},$$

and λ,β are sufficiently smooth functions along with the $n-line$ coordinate curve.

Proof. Let us define arbitrary vector fields in the rectifying plane, osculating plane, normal plane, respectively, along with the $n-line$ coordinate curve such that they have the following forms

$$\mathcal{A}^r=\lambda\mathbf{t}+\beta\mathbf{b},\quad \mathcal{A}^o=\lambda\mathbf{t}+\beta\mathbf{n},\quad \mathcal{A}^n=\lambda\mathbf{n}+\beta\mathbf{b}, \tag{40}$$

where λ,β are sufficiently smooth functions along with the $n-line$ coordinate curve. If one uses the similar approach as in the proof of the Theorem 3.1 then it is computed that

$$\begin{aligned}\mathcal{S}_{\mathcal{M}}(d\mathcal{A}^r(\mathbf{n}),d\mathcal{A}^r(\mathbf{n})) &= d\sigma(\mathcal{A}^r(\mathbf{n}))\cdot d\sigma(\mathcal{A}^r(\mathbf{n}))\\ &\quad+\mathcal{L}(\mathcal{A}^r(\mathbf{n}))\cdot\mathcal{L}(\mathcal{A}^r(\mathbf{n}))\\ &= \mathbf{n}\cdot\mathbf{n}+(\delta/\delta n)\mathcal{A}^r\cdot(\delta/\delta n)\mathcal{A}^r,\end{aligned} \tag{41}$$

$$\begin{aligned}\mathcal{S}_{\mathcal{M}}(d\mathcal{A}^o(\mathbf{n}),d\mathcal{A}^o(\mathbf{n})) &= d\sigma(\mathcal{A}^o(\mathbf{n}))\cdot d\sigma(\mathcal{A}^o(\mathbf{n}))\\ &\quad+\mathcal{L}(\mathcal{A}^o(\mathbf{n}))\cdot\mathcal{L}(\mathcal{A}^o(\mathbf{n}))\\ &= \mathbf{n}\cdot\mathbf{n}+(\delta/\delta n)\mathcal{A}^o\cdot(\delta/\delta n)\mathcal{A}^o,\end{aligned} \tag{42}$$

$$\begin{aligned}\mathcal{S}_{\mathcal{M}}(d\mathcal{A}^n(\mathbf{n}),d\mathcal{A}^n(\mathbf{n})) &= d\sigma(\mathcal{A}^n(\mathbf{n}))\cdot d\sigma(\mathcal{A}^n(\mathbf{n}))\\ &\quad+\mathcal{L}(\mathcal{A}^n(\mathbf{n}))\cdot\mathcal{L}(\mathcal{A}^n(\mathbf{n}))\\ &= \mathbf{n}\cdot\mathbf{n}+(\delta/\delta n)\mathcal{A}^n\cdot(\delta/\delta n)\mathcal{A}^n,\end{aligned} \tag{43}$$

Here, if one takes into account Eqs. (13,40-43) the proof can be completed trivially.

Lemma 3.1.2. i. The pseudo-angle of the unit vector field $\mathcal{A}^r$ in the rectifying plane along with the $n-line$ coordinate curve is computed by

$$\mathcal{P}_n^{(\mathbf{t,b})}(\mathcal{A}^r)=\frac{1}{2}\int_{\mathcal{R}}((((\delta/\delta n)\lambda-\beta(\Omega_b+\tau))^2+(\lambda\theta_{ns}+\beta div\mathbf{b})^2+((\delta/\delta n)\beta+\lambda(\Omega_b+\tau))^2)^{1/2}dn, \tag{44}$$

where

$$\mathcal{A}^r=\lambda\mathbf{t}+\beta\mathbf{b},$$

and λ, β are sufficiently smooth functions along with the $n-line$ coordinate curve.

ii. The pseudo-angle of the unit vector field $\mathcal{A}^o$ in the osculating plane along with the $n-line$ coordinate curve is computed by

$$\mathcal{P}_n^{(\mathbf{t},\mathbf{n})}\left(\mathcal{A}^o\right)=\frac{1}{2}\int_{\mathcal{R}}((((\delta/\delta n)\lambda-\beta\theta_{ns})^2+((\delta/\delta n)\beta+\lambda\theta_{ns})^2+(\lambda(\Omega_b+\tau)-\beta div\mathbf{b})^2)^{1/2}dn, \tag{45}$$

where

$$\mathcal{A}^o=\lambda\mathbf{t}+\beta\mathbf{n},$$

and λ, β are sufficiently smooth functions along with the $n-line$ coordinate curve.

iii. The pseudo-angle of the unit vector field $\mathcal{A}^n$ in the normal plane along with the $n-line$ coordinate curve is computed by

$$\mathcal{P}_n^{(\mathbf{n},\mathbf{b})}\left(\mathcal{A}^n\right)=\frac{1}{2}\int_{\mathcal{R}}((\lambda\theta_{ns}+\beta(\Omega_b+\tau))^2+((\delta/\delta n)\lambda+\beta div\mathbf{b})^2+((\delta/\delta n)\beta-\lambda div\mathbf{b})^2)^{1/2}dn, \tag{46}$$

where

$$\mathcal{A}^n=\lambda\mathbf{n}+\beta\mathbf{b},$$

and λ, β are sufficiently smooth functions along with the $n-line$ coordinate curve.

Proof. The proof is obvious if one considers the following definitions respectively

$$\mathcal{P}_n^{(\mathbf{t},\mathbf{b})}\left(\mathcal{A}^r\right)=\frac{1}{2}\int_{\mathcal{R}}\left\|(\delta/\delta n)\mathcal{A}^r\right\|dn,$$
$$\mathcal{P}_n^{(\mathbf{t},\mathbf{n})}\left(\mathcal{A}^o\right)=\frac{1}{2}\int_{\mathcal{R}}\left\|(\delta/\delta n)\mathcal{A}^o\right\|dn, \tag{47}$$
$$\mathcal{P}_n^{(\mathbf{n},\mathbf{b})}\left(\mathcal{A}^n\right)=\frac{1}{2}\int_{\mathcal{R}}\left\|(\delta/\delta n)\mathcal{A}^n\right\|dn,$$

where

$$\mathcal{A}^r=\lambda\mathbf{t}+\beta\mathbf{b},\quad \mathcal{A}^o=\lambda\mathbf{t}+\beta\mathbf{n},\quad \mathcal{A}^n=\lambda\mathbf{n}+\beta\mathbf{b}.$$

Theorem 3.1.3. The generalized energy of the unit vector field $\mathcal{A}$ along with the $n-line$ coordinate curve is computed by

$$\mathcal{E}_n\left(\mathcal{A}\right)=\frac{1}{2}\int_{\mathcal{R}}((((\delta/\delta n)\lambda-\beta\theta_{ns}-\gamma(\Omega_b+\tau))^2+((\delta/\delta n)\beta+\lambda\theta_{ns}+\gamma div\mathbf{b})^2+((\delta/\delta n)\gamma+\lambda(\Omega_b+\tau)-\beta div\mathbf{b})^2)dn+\frac{n}{2}, \tag{48}$$

where

$$\mathcal{A}=\lambda\mathbf{t}+\beta\mathbf{n}+\gamma\mathbf{b},$$

and λ, β, γ are sufficiently smooth functions along with the $n-line$ coordinate curve.

Proof. Let us define the generalized vector field along with the $n-line$ coordinate curve such that it has the following form

$$\mathcal{A}=\lambda\mathbf{t}+\beta\mathbf{n}+\gamma\mathbf{b} \tag{49}$$

where λ, β, γ are sufficiently smooth functions along with the $n-line$ coordinate curve. If one uses the similar approach as in the proof of the Theorem 3.1.1 then it is computed that

$$\begin{aligned}\mathcal{S}_{\mathcal{M}}(d\mathcal{A}(\mathbf{n}),d\mathcal{A}(\mathbf{n}))&=d\sigma(\mathcal{A}(\mathbf{n}))\cdot d\sigma(\mathcal{A}(\mathbf{n}))\\&\quad+\mathcal{L}(\mathcal{A}(\mathbf{n}))\cdot\mathcal{L}(\mathcal{A}(\mathbf{n})),\\&=\mathbf{n}\cdot\mathbf{n}+(\delta/\delta n)\mathcal{A}\cdot(\delta/\delta n)\mathcal{A}.\end{aligned} \tag{50}$$

Here, if one takes into account Eqs. (13,50) the proof can be completed trivially.

Lemma 3.1.3. The generalized pseudo-angle of the unit vector field $\mathcal{A}$ along with the $n-line$ coordinate curve is computed by

$$\mathcal{P}_n\left(\mathcal{A}\right)=\frac{1}{2}\int_{\mathcal{R}}((((\delta/\delta n)\lambda-\beta\theta_{ns}-\gamma(\Omega_b+\tau))^2+((\delta/\delta n)\beta+\lambda\theta_{ns}+\gamma div\mathbf{b})^2+((\delta/\delta n)\gamma+\lambda(\Omega_b+\tau)-\beta div\mathbf{b})^2)^{1/2}dn, \tag{51}$$

where

$$\mathcal{A}=\lambda\mathbf{t}+\beta\mathbf{n}+\gamma\mathbf{b},$$

and λ, β, γ are sufficiently smooth functions along with the $n-line$ coordinate curve.

Proof. The proof is obvious if one considers the following definition

$$\mathcal{P}_n\left(\mathcal{A}\right)=\frac{1}{2}\int_{\mathcal{R}}\left\|(\delta/\delta n)\mathcal{A}\right\|dn, \tag{52}$$

where

$$\mathcal{A}=\lambda\mathbf{t}+\beta\mathbf{n}+\gamma\mathbf{b},$$

and λ, β, γ are sufficiently smooth functions along with the $n-line$ coordinate curve.

Case 2: In this case, let us assume that α be a space curve in $\mathbb{E}^3$ such that its position vector $\mathcal{R}$ is parametrized by the arc-length b pointing to the location α in the binormal direction.

Theorem 3.1.4. The energy of the Frenet-Serret triad vectors $(\mathbf{t},\mathbf{n},\mathbf{b})$ along with the $b-line$ coordinate curve is computed by

$$\mathcal{E}_b(\mathbf{t}) = \frac{1}{2}\int_{\mathcal{R}}(\theta_{bs}^2+(\Omega_n+\tau)^2)db+\frac{b}{2},$$
$$\mathcal{E}_b(\mathbf{n}) = \frac{1}{2}\int_{\mathcal{R}}((\Omega_n+\tau)^2+(\kappa+div\mathbf{n})^2)db+\frac{b}{2},$$
$$\mathcal{E}_b(\mathbf{b}) = \frac{1}{2}\int_{\mathcal{R}}(\theta_{bs}^2+(\kappa+div\mathbf{n})^2)db+\frac{b}{2}. \quad (53)$$

Proof. From Eqs. (19,27), the splitting space and canonic generalization of the same object can be written due to the Levi-Civita connection map and the so-called Sasaki metric. By considering the clear definition in distribution and sectional energy, we write for unit vector fields $(\mathbf{t},\mathbf{n},\mathbf{b})$ as

$$\mathcal{E}_b(\mathbf{t}) = \frac{1}{2}\int(d\mathbf{t}(\mathbf{b}),d\mathbf{t}(\mathbf{b}))db, \quad (54)$$
$$\mathcal{E}_b(\mathbf{n}) = \frac{1}{2}\int(d\mathbf{n}(\mathbf{b}),d\mathbf{n}(\mathbf{b}))db, \quad (55)$$
$$\mathcal{E}_b(\mathbf{b}) = \frac{1}{2}\int(d\mathbf{b}(\mathbf{b}),d\mathbf{b}(\mathbf{b}))db,$$

(56)

where

$$\begin{aligned}\mathcal{S}_{\mathcal{M}}(d\mathbf{t}(\mathbf{b}),d\mathbf{t}(\mathbf{b})) &= d\sigma(\mathbf{t}(\mathbf{b}))\cdot d\sigma(\mathbf{t}(\mathbf{b}))\\ &+\mathcal{L}(\mathbf{t}(\mathbf{b}))\cdot\mathcal{L}(\mathbf{t}(\mathbf{b})), \quad (57)\\ &= \mathbf{b}\cdot\mathbf{b}+(\delta/\delta b)\mathbf{t}\cdot(\delta/\delta b)\mathbf{t},\end{aligned}$$

$$\begin{aligned}\mathcal{S}_{\mathcal{M}}(d\mathbf{n}(\mathbf{b}),d\mathbf{n}(\mathbf{b})) &= d\sigma(\mathbf{n}(\mathbf{b}))\cdot d\sigma(\mathbf{n}(\mathbf{b}))\\ &+\mathcal{L}(\mathbf{n}(\mathbf{b}))\cdot\mathcal{L}(\mathbf{n}(\mathbf{b}))\\ &= \mathbf{b}\cdot\mathbf{b}+(\delta/\delta n)\mathbf{b}\cdot(\delta/\delta n)\mathbf{b},\end{aligned} \quad (58)$$

$$\begin{aligned}\mathcal{S}_{\mathcal{M}}(d\mathbf{b}(\mathbf{b}),d\mathbf{b}(\mathbf{b})) &= d\sigma(\mathbf{b}(\mathbf{b}))\cdot d\sigma(\mathbf{b}(\mathbf{b}))\\ &+\mathcal{L}(\mathbf{b}(\mathbf{b}))\cdot\mathcal{L}(\mathbf{b}(\mathbf{b}))\\ &= \mathbf{b}\cdot\mathbf{b}+(\delta/\delta n)\mathbf{b}\cdot(\delta/\delta n)\mathbf{b}.\end{aligned} \quad (59)$$

Using Eqs. (14,54-59), it is computed respectively that

$$\mathcal{E}_b(\mathbf{t}) = \frac{1}{2}\int_{\mathcal{R}}(\theta_{bs}^2+(\Omega_n+\tau)^2)db+\frac{b}{2},$$
$$\mathcal{E}_b(\mathbf{n}) = \frac{1}{2}\int_{\mathcal{R}}((\Omega_n+\tau)^2+(\kappa+div\mathbf{n})^2)db+\frac{b}{2},$$
$$\mathcal{E}_b(\mathbf{b}) = \frac{1}{2}\int_{\mathcal{R}}(\theta_{bs}^2+(\kappa+div\mathbf{n})^2)db+\frac{b}{2}.$$

Lemma 3.1.4. The pseudo-angle of the Frenet-Serret triad vectors $(\mathbf{t},\mathbf{n},\mathbf{b})$ along with the $b-line$ coordinate curve is computed by

$$\mathcal{P}_b(\mathbf{t}) = \frac{1}{2}\int_{\mathcal{R}}\sqrt{(\theta_{bs}^2+(\Omega_n+\tau)^2)}db,$$
$$\mathcal{P}_b(\mathbf{n}) = \frac{1}{2}\int_{\mathcal{R}}\sqrt{((\Omega_n+\tau)^2+(\kappa+div\mathbf{n})^2)}db,$$
$$\mathcal{P}_b(\mathbf{b}) = \frac{1}{2}\int_{\mathcal{R}}\sqrt{(\theta_{bs}^2+(\kappa+div\mathbf{n})^2)}db. \quad (60)$$

Proof. The proof is obvious if one considers the following definitions respectively

$$\mathcal{P}_b(\mathbf{t}) = \frac{1}{2}\int_{\mathcal{R}}\|(\delta/\delta b)\mathbf{t}\|db, \quad \mathcal{P}_b(\mathbf{n}) = \frac{1}{2}\int_{\mathcal{R}}\|(\delta/\delta b)\mathbf{n}\|db,$$
$$\mathcal{P}_b(\mathbf{b}) = \frac{1}{2}\int_{\mathcal{R}}\|(\delta/\delta b)\mathbf{b}\|db. \quad (61)$$

Theorem 3.1.5. **i**. The energy of the unit vector field $\mathcal{A}^r$ in the rectifying plane along with the $b-line$ coordinate curve is computed by

$$\begin{aligned}\mathcal{E}_b^{(\mathbf{t},\mathbf{b})}(\mathcal{A}^r) &= \frac{1}{2}\int_{\mathcal{R}}(((\delta/\delta b)\lambda-\beta\theta_{bs})^2+(\lambda(\Omega_n+\tau)\\ &+\beta(\kappa+div\mathbf{n}))^2+((\delta/\delta n)\beta+\lambda\theta_{bs})^2)db+\frac{b}{2},\end{aligned} \quad (62)$$

where

$$\mathcal{A}^r = \lambda\mathbf{t}+\beta\mathbf{b},$$

and λ,β are sufficiently smooth functions along with the $b-line$ coordinate curve.

ii. The energy of the unit vector field $\mathcal{A}^o$ in the osculating plane along with the $b-line$ coordinate curve is computed by

$$\begin{aligned}\mathcal{E}_b^{(\mathbf{t},\mathbf{n})}(\mathcal{A}^o) &= \frac{1}{2}\int_{\mathcal{R}}(((\delta/\delta b)\lambda+\beta(\Omega_n+\tau))^2\\ &+((\delta/\delta b)\beta-\lambda(\Omega_n+\tau))^2\\ &+(\lambda\theta_{bs}+\beta(\kappa+div\mathbf{n}))^2)db+\frac{b}{2},\end{aligned} \quad (63)$$

where

$$\mathcal{A}^o = \lambda\mathbf{t}+\beta\mathbf{n},$$

and λ,β are sufficiently smooth functions along with the $b-line$ coordinate curve.

iii. The energy of the unit vector field $\mathcal{A}^n$ in the normal plane along with the $b-line$ coordinate curve is computed by

$$\mathcal{E}_b^{(\mathbf{n},\mathbf{b})}\left(\mathcal{A}^n\right)=\frac{1}{2}\int_{\mathcal{R}}((\lambda(\Omega_n+\tau)-\beta\theta_{bs})^2+((\delta/\delta b)\lambda-\beta(\kappa+div\mathbf{n}))^2+((\delta/\delta b)\beta+\lambda(\kappa+div\mathbf{n}))^2)db+\frac{b}{2}, \quad (64)$$

where

$$\mathcal{A}^n=\lambda\mathbf{n}+\beta\mathbf{b},$$

and λ,β are sufficiently smooth functions along with the $b-line$ coordinate curve.

Proof. Let us define arbitrary vector fields in the rectifying plane, osculating plane, normal plane, respectively, along with the $b-line$ coordinate curve such that they have the following forms

$$\mathcal{A}^r=\lambda\mathbf{t}+\beta\mathbf{b},\quad \mathcal{A}^o=\lambda\mathbf{t}+\beta\mathbf{n},\quad \mathcal{A}^n=\lambda\mathbf{n}+\beta\mathbf{b}, \quad (65)$$

where λ,β are sufficiently smooth functions along with the $b-line$ coordinate curve. If one uses the similar approach as in the proof of the Theorem 3.1.4 then it is computed that

$$\begin{aligned}\mathcal{S}_{\mathcal{M}}(d\mathcal{A}^r(\mathbf{b}),d\mathcal{A}^r(\mathbf{b}))&=d\sigma(\mathcal{A}^r(\mathbf{b}))\cdot d\sigma(\mathcal{A}^r(\mathbf{b}))\\&+\mathcal{L}(\mathcal{A}^r(\mathbf{b}))\cdot\mathcal{L}(\mathcal{A}^r(\mathbf{b}))\\&=\mathbf{b}\cdot\mathbf{b}+(\delta/\delta b)\mathcal{A}^r\cdot(\delta/\delta b)\mathcal{A}^r,\end{aligned} \quad (66)$$

$$\begin{aligned}\mathcal{S}_{\mathcal{M}}(d\mathcal{A}^o(\mathbf{b}),d\mathcal{A}^o(\mathbf{b}))&=d\sigma(\mathcal{A}^o(\mathbf{b}))\cdot d\sigma(\mathcal{A}^o(\mathbf{b}))\\&+\mathcal{L}(\mathcal{A}^o(\mathbf{b}))\cdot\mathcal{L}(\mathcal{A}^o(\mathbf{b}))\\&=\mathbf{b}\cdot\mathbf{b}+(\delta/\delta b)\mathcal{A}^o\cdot(\delta/\delta b)\mathcal{A}^o,\end{aligned} \quad (67)$$

$$\begin{aligned}\mathcal{S}_{\mathcal{M}}(d\mathcal{A}^n(\mathbf{b}),d\mathcal{A}^n(\mathbf{b}))&=d\sigma(\mathcal{A}^n(\mathbf{b}))\cdot d\sigma(\mathcal{A}^n(\mathbf{b}))\\&+\mathcal{L}(\mathcal{A}^n(\mathbf{b}))\cdot\mathcal{L}(\mathcal{A}^n(\mathbf{b}))\\&=\mathbf{b}\cdot\mathbf{b}+(\delta/\delta b)\mathcal{A}^n\cdot(\delta/\delta b)\mathcal{A}^n,\end{aligned} \quad (68)$$

Here, if one takes into account Eqs. (14,65-68), the proof can be completed trivially.

Lemma 3.1.5. i. The pseudo-angle of the unit vector field $\mathcal{A}^r$ in the rectifying plane along with the $b-line$ coordinate curve is computed by

$$\mathcal{P}_b^{(\mathbf{t},\mathbf{b})}\left(\mathcal{A}^r\right)=\frac{1}{2}\int_{\mathcal{R}}((((\delta/\delta b)\lambda-\beta\theta_{bs})^2+(\lambda(\Omega_n+\tau)+\beta(\kappa+div\mathbf{n}))^2+((\delta/\delta n)\beta+\lambda\theta_{bs})^2)^{1/2}db, \quad (69)$$

where

$$\mathcal{A}^r=\lambda\mathbf{t}+\beta\mathbf{b},$$

and λ,β are sufficiently smooth functions along with the $b-line$ coordinate curve.

ii. The pseudo-angle of the unit vector field $\mathcal{A}^o$ in the osculating plane along with the $b-line$ coordinate curve is computed by

$$\mathcal{P}_b^{(\mathbf{t},\mathbf{n})}\left(\mathcal{A}^o\right)=\frac{1}{2}\int_{\mathcal{R}}((((\delta/\delta b)\lambda+\beta(\Omega_n+\tau))^2+((\delta/\delta b)\beta-\lambda(\Omega_n+\tau))^2+(\lambda\theta_{bs}+\beta(\kappa+div\mathbf{n}))^2)^{1/2}db, \quad (70)$$

where

$$\mathcal{A}^o=\lambda\mathbf{t}+\beta\mathbf{n},$$

and λ,β are sufficiently smooth functions along with the $b-line$ coordinate curve.

iii. The pseudo-angle of the unit vector field $\mathcal{A}^n$ in the normal plane along with the $b-line$ coordinate curve is computed by

$$\mathcal{P}_b^{(\mathbf{n},\mathbf{b})}\left(\mathcal{A}^n\right)=\frac{1}{2}\int_{\mathcal{R}}((\lambda(\Omega_n+\tau)-\beta\theta_{bs})^2+((\delta/\delta b)\lambda-\beta(\kappa+div\mathbf{n}))^2+((\delta/\delta b)\beta+\lambda(\kappa+div\mathbf{n}))^2)^{1/2}db, \quad (71)$$

where

$$\mathcal{A}^n=\lambda\mathbf{n}+\beta\mathbf{b},$$

and λ,β are sufficiently smooth functions along with the $b-line$ coordinate curve.

Proof. The proof is obvious if one considers the following definitions respectively

$$\begin{aligned}\mathcal{P}_b^{(\mathbf{t},\mathbf{b})}\left(\mathcal{A}^r\right)&=\frac{1}{2}\int_{\mathcal{R}}\left\|(\delta/\delta b)\mathcal{A}^r\right\|db,\\ \mathcal{P}_b^{(\mathbf{t},\mathbf{n})}\left(\mathcal{A}^o\right)&=\frac{1}{2}\int_{\mathcal{R}}\left\|(\delta/\delta b)\mathcal{A}^o\right\|db,\\ \mathcal{P}_b^{(\mathbf{n},\mathbf{b})}\left(\mathcal{A}^n\right)&=\frac{1}{2}\int_{\mathcal{R}}\left\|(\delta/\delta b)\mathcal{A}^n\right\|db,\end{aligned} \quad (72)$$

where

$$\mathcal{A}^r=\lambda\mathbf{t}+\beta\mathbf{b},\quad \mathcal{A}^o=\lambda\mathbf{t}+\beta\mathbf{n},\quad \mathcal{A}^n=\lambda\mathbf{n}+\beta\mathbf{b}.$$

Theorem 3.1.6. The generalized energy of the unit vector field $\mathcal{A}$ along with the $b-line$ coordinate curve is computed by

$$\mathcal{E}_b(\mathcal{A}) = \frac{1}{2}\int_{\mathcal{R}}(((\delta/\delta b)\lambda + \beta(\Omega_n + \tau) - \gamma\theta_{bs})^2 + ((\delta/\delta b)\beta - \lambda(\Omega_n + \tau) - \gamma(\kappa + div\mathbf{n}))^2 + ((\delta/\delta b)\gamma + \lambda\theta_{bs} + \beta(\kappa + div\mathbf{n}))^2)db + \frac{b}{2}, \tag{73}$$

where

$$\mathcal{A} = \lambda\mathbf{t} + \beta\mathbf{n} + \gamma\mathbf{b},$$

and λ, β, γ are sufficiently smooth functions along with the $b-line$ coordinate curve.

Proof. Let us define the generalized vector field along with the $b-line$ coordinate curve such that it has the following form

$$\mathcal{A} = \lambda\mathbf{t} + \beta\mathbf{n} + \gamma\mathbf{b}, \tag{74}$$

where λ, β, γ are sufficiently smooth functions along with the $b-line$ coordinate curve. If one uses the similar approach as in the proof of the Theorem 3.4 then it is computed that

$$\begin{aligned} \mathcal{S}_{\mathcal{M}}(d\mathcal{A}(\mathbf{b}), d\mathcal{A}(\mathbf{b})) &= d\sigma(\mathcal{A}(\mathbf{b}))\cdot d\sigma(\mathcal{A}(\mathbf{b})) \\ &+ \mathcal{L}(\mathcal{A}(\mathbf{b}))\cdot\mathcal{L}(\mathcal{A}(\mathbf{b})), \\ &= \mathbf{b}\cdot\mathbf{b} + (\delta/\delta b)\mathcal{A}\cdot(\delta/\delta b)\mathcal{A} \end{aligned} \tag{75}$$

Here, if one takes into account Eqs. (14,75) the proof can be completed trivially.

Lemma 3.1.6. The generalized pseudo-angle of the unit vector field $\mathcal{A}$ along with the $b-line$ coordinate curve is computed by

$$\mathcal{P}_b(\mathcal{A}) = \frac{1}{2}\int_{\mathcal{R}}(((\delta/\delta b)\lambda + \beta(\Omega_n + \tau) - \gamma\theta_{bs})^2 + ((\delta/\delta b)\beta - \lambda(\Omega_n + \tau) - \gamma(\kappa + div\mathbf{n}))^2 + ((\delta/\delta b)\gamma + \lambda\theta_{bs} + \beta(\kappa + div\mathbf{n}))^2)^{1/2}db, \tag{76}$$

where

$$\mathcal{A} = \lambda\mathbf{t} + \beta\mathbf{n} + \gamma\mathbf{b},$$

and λ, β, γ are sufficiently smooth functions along with the $b-line$ coordinate curve.

Proof. The proof is obvious if one considers the following definition

$$\mathcal{P}_b(\mathcal{A}) = \frac{1}{2}\int_{\mathcal{R}}\|(\delta/\delta b)\mathcal{A}\|db, \tag{77}$$

where

$$\mathcal{A} = \lambda\mathbf{t} + \beta\mathbf{n} + \gamma\mathbf{b},$$

and λ, β, γ are sufficiently smooth functions along with the $b-line$ coordinate curve.

3.2. Critical Points of the Directional Energy Functionals of the Velocity Fields

In the differential geometry literature, various functionals provide insight into measuring the vector fields described over any semi-Riemannian or Riemannian monifolds. Obtaining the optimal functionals and vector fields is an important task to expand this effort to corresponding distributions. This effort is highly appreciated since it does not require the manifold to be a compact manifold, which yields restriction-free solutions for characterizing many critical points belonging to the mapping from differential manifolds to their submanifolds by projecting their gradients. In the case of the isometry, one also expects to find the correlation between the attitude of the natural variational problem and energy functionals. In this section, we attempt finding the critical points with respect to directional variations through nearby unit normal and unit binormal vector fields, and if so whether it is not unstable with respect to such directional variations. Thus, we aim to deal with a less compelling problem than the harmonic case.

Case 1: In this case, let us assume that α be a space curve in $\mathbb{E}^3$ such that its position vector $\mathcal{R}$ is parametrized by the arc-length n pointing to the location α in the normal direction.

Theorem 3.2.1. The critical point of the minimizing directional energy functional of the velocity vector field $\mathbf{n}$ of the $n-line$ coordinate curve in the normal direction is computed by

$$\int_{\alpha_1}^{\alpha_2}\zeta\theta_{ns}\frac{\delta}{\delta n}\theta_{ns}dn + \int_{\alpha_1}^{\alpha_2}\zeta div\mathbf{b}\frac{\delta}{\delta n}div\mathbf{b}dn = 0, \tag{78}$$

where ζ is a well-defined function along with the $n-line$ coordinate curve in the following form

$$\zeta(n) = (n-\alpha_1)(\alpha_2 - n), \zeta(n) \neq 0\ \forall n \in (\alpha_1, \alpha_2), \tag{79}$$

and

$$\alpha : \mathbb{I} \to \mathbb{E}^3, [\alpha_1, \alpha_2] \subset \mathbb{I}.$$

Proof. From Eqs. (19,28), the minimizing directional energy functional and its directional variation in the normal direction is written as follows

$$\frac{\delta}{\delta\varepsilon}\mathcal{E}_n(\mathbf{n}_\varepsilon)=\frac{1}{2}\int_{\alpha_1}^{\alpha_2}\frac{\delta}{\delta\varepsilon}(\mathbf{n}_\varepsilon\cdot\mathbf{n}_\varepsilon+\frac{\delta}{\delta n}\mathbf{n}_\varepsilon\cdot\frac{\delta}{\delta n}\mathbf{n}_\varepsilon)dn,$$

(80)

where

$$\mathbf{n}_\varepsilon=\mathbf{n}(n,\varepsilon)=(\delta/\delta n)\alpha(n,\varepsilon), \quad (81)$$

$$\alpha(n,\varepsilon)=\begin{pmatrix}\alpha^*(n)+\varepsilon\varpi^*(n),\alpha^{**}(n)\\+\varepsilon\varpi^{**}(n),\alpha^{***}(n)+\varepsilon\varpi^{***}(n)\end{pmatrix}, \quad (82)$$

$$\zeta(n)\mathbf{n}(n)=(\varpi^*(n),\varpi^{**}(n),\varpi^{***}(n)),\ \varpi^i:[\alpha_1,\alpha_2]\to\mathbb{R},$$

(83)

such that

$$\alpha(n,0)=\zeta(\alpha_1)=\zeta(\alpha_2)=0. \quad (84)$$

By using Eqs. (81-84), one computes that

$$\frac{\delta}{\delta\varepsilon}\mathcal{E}_n(\mathbf{n}_\varepsilon)=\int_{\alpha_1}^{\alpha_2}\frac{\delta^2}{\delta\varepsilon\delta n}\mathbf{n}_\varepsilon\cdot\frac{\delta}{\delta n}\mathbf{n}_\varepsilon dn. \quad (85)$$

Further computation yields that

$$\frac{\delta}{\delta\varepsilon}\mathcal{E}_n(\mathbf{n}_0)|_{\varepsilon=0}=\frac{\delta}{\delta\varepsilon}\mathbf{n}_\varepsilon(n,0)\cdot\frac{\delta}{\delta n}\mathbf{n}_\varepsilon(n,0)|_{\alpha_1}^{\alpha_2}-\int_{\alpha_1}^{\alpha_2}\frac{\delta}{\delta\varepsilon}\mathbf{n}_\varepsilon(n,0)\cdot\frac{\delta^2}{\delta n^2}\mathbf{n}_\varepsilon(n,0)dn. \quad (86)$$

Now, we should remind some identities to calculate the above equality. From the partial derivatives with respect to n and $\varepsilon,$ we obtain that

$$\frac{\delta}{\delta\varepsilon}\alpha(n,\varepsilon)=\zeta(n)\mathbf{n}_\varepsilon(n,\varepsilon),\qquad \frac{\delta}{\delta\varepsilon}\alpha(n,0)=\mathbf{n}_\varepsilon(n,0)=\frac{\delta}{\delta n}\mathbf{n} \quad (87)$$

Thus, one can reformulate Eq. (86) by using Eqs. (13,84,87) in the following way

$$\frac{\delta}{\delta\varepsilon}\mathcal{E}_n(\mathbf{n}_0)|_{\varepsilon=0}=\zeta(\theta_{ns}^2+div\mathbf{b}^2)|_{\alpha_1}^{\alpha_2}-3\int_{\alpha_1}^{\alpha_2}\zeta(\theta_{ns}\frac{\delta}{\delta n}\theta_{ns}+div\mathbf{b}\frac{\delta}{\delta n}div\mathbf{b})dn. \quad (88)$$

From Eq. (84), we already know that $\zeta(\alpha_1)=\zeta(\alpha_2)=0.$ Therefore, one can compute that

$$\frac{\delta}{\delta\varepsilon}\mathcal{E}_n(\mathbf{n}_0)|_{\varepsilon=0}=\int_{\alpha_1}^{\alpha_2}\zeta\theta_{ns}\frac{\delta}{\delta n}\theta_{ns}dn+\int_{\alpha_1}^{\alpha_2}\zeta div\mathbf{b}\frac{\delta}{\delta n}div\mathbf{b}dn=0. \quad (89)$$

Finally, the proof is completed and it implies that any path in the normal direction minimizing the directional energy functional $\mathcal{E}_n(\mathbf{n}_\varepsilon)$ must meets Eq. (89).

Case 2: In this case, let us assume that α be a space curve in $\mathbb{E}^3$ such that its position vector $\mathcal{R}$ is parametrized by the arc-length b pointing to the location α in the binormal direction·

Theorem 3.2.2. The critical point of the minimizing directional energy functional of the velocity vector field $\mathbf{b}$ of the $b-line$ coordinate curve in the binormal direction is given by

$$\int_{\alpha_1}^{\alpha_2}\zeta\theta_{bs}\frac{\delta}{\delta b}\theta_{bs}db+\int_{\alpha_1}^{\alpha_2}\zeta(\kappa+div\mathbf{n})\frac{\delta}{\delta b}(\kappa+div\mathbf{n})db=0,$$

(90)

where ζ is a well-defined function along with the $b-line$ coordinate curve in the following form

$$\zeta(b)=(b-\alpha_1)(\alpha_2-b),\zeta(b)\neq 0\ \forall b\in(\alpha_1,\alpha_2), \quad (91)$$

and

$$\alpha:\mathbb{I}\to\mathbb{E}^3,[\alpha_1,\alpha_2]\subset\mathbb{I}.$$

Proof. From Eqs. (19,53), the minimizing directional energy functional and its directional variation in the binormal direction is written as follows

$$\frac{\delta}{\delta\varepsilon}\mathcal{E}_b(\mathbf{b}_\varepsilon)=\frac{1}{2}\int_{\alpha_1}^{\alpha_2}\frac{\delta}{\delta\varepsilon}(\mathbf{b}_\varepsilon\cdot\mathbf{b}_\varepsilon+\frac{\delta}{\delta b}\mathbf{b}_\varepsilon\cdot\frac{\delta}{\delta b}\mathbf{b}_\varepsilon)db,$$

(92)

where

$$\mathbf{b}_\varepsilon=\mathbf{b}(b,\varepsilon)=(\delta/\delta b)\alpha(b,\varepsilon), \quad (93)$$

$$\alpha(b,\varepsilon)=\begin{pmatrix}\alpha^*(b)+\varepsilon\varpi^*(b),\alpha^{**}(b)\\+\varepsilon\varpi^{**}(b),\alpha^{***}(b)+\varepsilon\varpi^{***}(b)\end{pmatrix}, \quad (94)$$

$$\zeta(b)\mathbf{b}(b)=(\varpi^*(b),\varpi^{**}(b),\varpi^{***}(b)),\ \varpi^i:[\alpha_1,\alpha_2]\to\mathbb{R},$$

(95)

such that

$$\alpha(b,0)=\zeta(\alpha_1)=\zeta(\alpha_2)=0. \quad (96)$$

By using Eqs. (93,96), one computes that

$$\frac{\delta}{\delta\varepsilon}\mathcal{E}_b(\mathbf{b}_\varepsilon)=\int_{\alpha_1}^{\alpha_2}\frac{\delta^2}{\delta\varepsilon\delta b}\mathbf{b}_\varepsilon\cdot\frac{\delta}{\delta b}\mathbf{b}_\varepsilon db. \quad (97)$$

Further computation yields that

$$\frac{\delta}{\delta\varepsilon}\mathcal{E}_b(\mathbf{b}_0)|_{\varepsilon=0}=\frac{\delta}{\delta\varepsilon}\mathbf{b}_\varepsilon(b,0)\cdot\frac{\delta}{\delta b}\mathbf{b}_\varepsilon(b,0)|_{\alpha_1}^{\alpha_2}-\int_{\alpha_1}^{\alpha_2}\frac{\delta}{\delta\varepsilon}\mathbf{b}_\varepsilon(b,0)\cdot\frac{\delta^2}{\delta b^2}\mathbf{b}_\varepsilon(b,0)db. \quad (98)$$

Now, we should remind some identities to calculate the above equality. From the partial derivatives with respect to b and $\varepsilon,$ we obtain that

$$\frac{\delta}{\delta\varepsilon}\alpha(b,\varepsilon)=\zeta(b)\mathbf{b}_{\varepsilon}(b,\varepsilon),$$
$$\frac{\delta}{\delta\varepsilon}\alpha(b,0)=\mathbf{b}_{\varepsilon}(b,0)=\frac{\delta}{\delta b}\mathbf{b}, \tag{99}$$

Thus, one can reformulate Eq. (98) by using Eqs. (14,96,99) in the following way

$$\frac{\delta}{\delta\varepsilon}\mathcal{E}_b(\mathbf{b}_0)|_{\varepsilon=0}=\zeta\left(\theta_{bs}^2+(\kappa+div\mathbf{n})^2\right)|_{\alpha_1}^{\alpha_2}$$
$$-3\int_{\alpha_1}^{\alpha_2}\zeta(\theta_{bs}\frac{\delta}{\delta b}\theta_{bs}+(\kappa+div\mathbf{n})\frac{\delta}{\delta b}(\kappa+div\mathbf{n}))db. \tag{100}$$

From Eq. (96), we already know that $\zeta(\alpha_1)=\zeta(\alpha_2)=0$. Therefore, one can compute that

$$\frac{\delta}{\delta\varepsilon}\mathcal{E}_b(\mathbf{b}_0)|_{\varepsilon=0}=\int_{\alpha_1}^{\alpha_2}\zeta\theta_{bs}\frac{\delta}{\delta b}\theta_{bs}db+\int_{\alpha_1}^{\alpha_2}\zeta(\kappa$$
$$+div\mathbf{n})\frac{\delta}{\delta b}(\kappa+div\mathbf{n})db=0. \tag{101}$$

Finally, the proof is completed and it implies that any path in the binormal direction minimizing the directional energy functional $\mathcal{E}_b(\mathbf{b}_\varepsilon)$ must meets Eq. (101).

3.3. Harmonic Maps and the Extrema of the Directional Energy Functionals

If the differential of the divergence of a map between semi-Rieamannian or Riemannian manifolds vanish then this map is called harmonic. Critical points of this map are computed by investigating the corresponding energy functional. This differential is also written in terms of the tension field associated with the energy functional given by the Euler-Lagrange operator. Harmonic maps are seen in several different contexts. For instance, these maps are used to characterize the geodesics, or they are considered to compute minimal surface energy via the Dirichlet-Douglas integral, or they are taken into account to define holomorphic maps through the compatible metrics in certain manifolds, etc. The most striking problem for harmonic maps is solved by the improvement of variational theory. Even though there is no generalized solution family or theory ensuring the exact and entire solutions to these maps the second-order semilinear Euler-Lagrange type of elliptic equation systems provide some advantages. Accordingly, it helps obtaining some approximate solutions and outlining some results. In this section, we describe the directional harmonic and biharmonic equations of curvature vector fields in the three-dimensional ordinary space. Thus, we aim to obtain further generalization on the directional biharmonic maps and bienergy between Riemannian manifolds.

Case 1: In this case, let us assume that α be a space curve in $\mathbb{E}^3$ such that its position vector $\mathcal{R}$ is parametrized by the arc-length n pointing to the location α in the normal direction.

The partial derivative of the Frenet-Serret triad in the normal direction implies that the curvature vector field associated with the space curve is expressed by

$$\mathbb{C}=\frac{\delta}{\delta n}\mathbf{n}=-\theta_{ns}\mathbf{t}-(div\mathbf{b})\mathbf{b}. \tag{102}$$

Let Δ denotes the Laplacian operator acting on the tangent space of all smooth sections. Thus, in the normal direction, Δ is defined explicitly by

$$\Delta=-\frac{\delta}{\delta n}\frac{\delta}{\delta n}. \tag{103}$$

Then, the normal biharmonicity condition of the curvature vector field $\mathbb{C}$ is given by

$$\frac{\delta}{\delta n}\frac{\delta}{\delta n}\frac{\delta}{\delta n}\alpha=0, \tag{104}$$

where

$$\Delta\mathbb{C}=\pi\mathbb{C},\pi\in\mathbb{R}. \tag{105}$$

Taking the partial derivative $\mathbb{C}$ with respect to n, and considering Eq. (13) it is computed that

$$\Delta\mathbb{C}\ =\ (-\frac{\delta^2}{\delta n^2}\theta_{ns}+2(\Omega_b+\tau)\frac{\delta}{\delta n}div\mathbf{b}$$
$$+div\mathbf{b}\frac{\delta}{\delta n}(\Omega_b+\tau)+\theta_{ns}(\theta_{ns}^2+(div\mathbf{b})^2$$
$$+(\Omega_b+\tau)^2))\mathbf{t}-3(\theta_{ns}\frac{\delta}{\delta n}\theta_{ns}+div\mathbf{b}\frac{\delta}{\delta n}div\mathbf{b})\mathbf{n}$$
$$+(-\frac{\delta^2}{\delta n^2}div\mathbf{b}-2(\Omega_b+\tau)\frac{\delta}{\delta n}\theta_{ns}-\theta_{ns}\frac{\delta}{\delta n}(\Omega_b$$
$$+\tau)+div\mathbf{b}(\theta_{ns}^2+(div\mathbf{b})^2+(\Omega_b+\tau)^2))\mathbf{b}. \tag{106}$$

From Eqs. (102,105,106), the normal biharmonicity condition of the curvature vector field $\mathbb{C}$ is satisfied if and only if the following identities hold

$$0=-\frac{\delta^2}{\delta n^2}\theta_{ns}+2(\Omega_b+\tau)\frac{\delta}{\delta n}div\mathbf{b}+div\mathbf{b}\frac{\delta}{\delta n}(\Omega_b+\tau)+\theta_{ns}(\ +\pi+(\Omega_b+\tau)^2), \tag{107}$$

$$0=-\frac{\delta^2}{\delta n^2}div\mathbf{b}-2(\Omega_b+\tau)\frac{\delta}{\delta n}\theta_{ns}-\theta_{ns}\frac{\delta}{\delta n}(\Omega_b+\tau)+div\mathbf{b}(\ +\pi+(\Omega_b+\tau)^2), \tag{108}$$

where $\mathcal{K}$ is a constant value such that

$$\mathcal{K}=\theta_{ns}^2+div\mathbf{b}^2.$$

Case 2: In this case, let us assume that α be a space curve in $\mathbb{E}^3$ such that its position vector $\mathcal{R}$ is parametrized by the arc-length b pointing to the location α in the binormal direction.

The partial derivative of the Frenet-Serret triad in the binormal direction implies that the curvature vector field associated with the space curve is expressed by

$$\mathbb{C}=\frac{\delta}{\delta b}\mathbf{b}=-\theta_{bs}\mathbf{t}-(\kappa+div\mathbf{n})\mathbf{n}. \tag{109}$$

Let Δ denotes the Laplacian operator acting on the tangent space of all smooth sections. Thus, in the binormal direction, Δ is defined explicitly by

$$\Delta=-\frac{\delta}{\delta b}\frac{\delta}{\delta b}. \tag{110}$$

Then, the binormal biharmonicity condition of the curvature vector field $\mathbb{C}$ is given by

$$\frac{\delta}{\delta b}\frac{\delta}{\delta b}\frac{\delta}{\delta b}\alpha=0, \tag{111}$$

where

$$\Delta\mathbb{C}=\pi\mathbb{C},\pi\in\mathbb{R}. \tag{112}$$

Taking the partial derivative $\mathbb{C}$ with respect to b, and considering Eq. (14) it is computed that

$$\Delta\mathbb{C} = (-\frac{\delta^2}{\delta b^2}\theta_{bs}-2(\Omega_n+\tau)\frac{\delta}{\delta b}(\kappa+div\mathbf{n})-(\kappa+div\mathbf{n})\frac{\delta}{\delta b}(\Omega_n+\tau)+\theta_{bs}(\theta_{bs}^2+(\kappa+div\mathbf{n})^2+(\Omega_n+\tau)^2))\mathbf{t}+(-\frac{\delta^2}{\delta b^2}(\kappa+div\mathbf{n})+2(\Omega_n+\tau)\frac{\delta}{\delta b}\theta_{bs}+\theta_{bs}\frac{\delta}{\delta b}(\Omega_n+\tau)+(\kappa+div\mathbf{n})(\theta_{bs}^2+(\kappa+div\mathbf{n})^2+(\Omega_n+\tau)^2))\mathbf{n}-3(\theta_{bs}\frac{\delta}{\delta b}\theta_{bs}+(\kappa+div\mathbf{n})\frac{\delta}{\delta b}(\kappa+div\mathbf{n}))\mathbf{b}. \tag{113}$$

From Eqs. (109,112,113), the binormal biharmonicity condition of the curvature vector field $\mathbb{C}$ is satisfied if and only if the following identities hold

$$0=-\frac{\delta^2}{\delta b^2}\theta_{bs}-2(\Omega_n+\tau)\frac{\delta}{\delta b}(\kappa+div\mathbf{n})-(\kappa+div\mathbf{n})\frac{\delta}{\delta b}(\Omega_n+\tau)+\theta_{bs}(\ +\pi+(\Omega_n+\tau)^2), \tag{114}$$

$$0=-\frac{\delta^2}{\delta b^2}(\kappa+div\mathbf{n})+2(\Omega_n+\tau)\frac{\delta}{\delta b}\theta_{bs}+\theta_{bs}\frac{\delta}{\delta b}(\Omega_n+\tau)+(\kappa+div\mathbf{n})(\ +\pi+(\Omega_n+\tau)^2), \tag{115}$$

where $\mathcal{K}$ is a constant value such that

$$\mathcal{K}=\theta_{bs}^2+(\kappa+div\mathbf{n})^2.$$

4. Conclusion and Suggestions

It is well-known that energy functionals are significant in the improvement of the variational theory in mathematics. They are the key ingredient of the calculation of energy and volume of unit vector fields in different geometric and physical space structures. They are also highly effectively used in determining the absolute minima or maxima of critical curves, obtaining critical points of extrema curves, measuring bending or total bending quantities, defining curvature energy of extremal curves, etc. In this paper, we investigate the energy of Frenet-Serret unit vector fields under infinitesimal variation of the arc-length parameter determined in the normal and binormal directions.

It is called directional energy along with the paper. We define the corresponding directional energy functionals associated with the energy of the unit tangent, normal, and binormal vectors in the normal and binormal directions. We also give explicit identities for the computation of the energy minimizers and the critical points of the unit velocity vector fields in the normal and binormal directions. Finally, we calculate the necessary and sufficient conditions of directional harmonic and biharmonic equations of the curvature vector field in the ordinary space to present insight into the directional biharmonic maps and bienergy between Riemannian manifolds. In the future, we will solve these partial differential equations of the Euler-Lagrange type to examine the effect of changing the arc-length parameter from s to n and b. This also leads to understand the behavior of the geometric quantities of the anholonomic coordinates when their characterizations are shaped due to the influence of certain energy or volume functionals.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

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