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Solvability of a Three-Dimensional System of Nonlinear Difference Equations

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Abstract
In this paper, we solve the following three-dimensional system of difference equations

$$x_n = \frac{y_{n-4}z_{n-5}}{y_{n-1}\left(a_n + b_n z_{n-2}x_{n-3}y_{n-4}z_{n-5}\right)},$$
$$y_n = \frac{z_{n-4}x_{n-5}}{z_{n-1}\left(\alpha_n + \beta_n x_{n-2}y_{n-3}z_{n-4}x_{n-5}\right)},$$
$$z_n = \frac{x_{n-4}y_{n-5}}{x_{n-1}\left(A_n + B_n y_{n-2}z_{n-3}x_{n-4}y_{n-5}\right)},\ n \in \mathbb{N}_0,$$

where the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$, $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$ and the initial values x_{-j}, y_{-j}, $j = \overline{1,5}$, are real numbers. In addition, the constant coefficients of the mentioned system is solved in closed form. Finally, we also describe the forbidden set of solutions of the system of difference equations.

Keywords: *System of difference equations; Closed-form; Forbidden set.*

AMS Subject Classification (2020): *Primary: 39A10 ; Secondary: 39A20; 39A23.*

1. Introduction

Difference equations emerge from generation functions, numerical solutions of differential equations or mathematical models of physical events. Therefore, difference equations or systems of difference equations are important for many researchers. Because they use them in economics, physics, biology, engineering. Especially, mathematicians are interested in system of difference equations or difference equations [1–8, 10–18, 20–22, 25–39]. For example, the difference equation

$$x_{n+1} = \frac{x_{n-3}x_{n-4}}{x_n\left(\pm 1 \pm x_{n-1}x_{n-2}x_{n-3}x_{n-4}\right)},\ n \in \mathbb{N}_0, \tag{1.1}$$

was studied in [9]. Elsayed have shown that this difference equation can be solved in closed form by using the method of induction.

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In addition, Stević found the general solution of following extension of difference equations (1.1)

$$x_{n+1}=\frac{x_{n-3}x_{n-4}}{x_n\left(a+bx_{n-1}x_{n-2}x_{n-3}x_{n-4}\right)},\ n\in\mathbb{N}_0, \tag{1.2}$$

where the parameters a, b and the initial values x_{-j}, $j=\overline{0,4}$, are complex numbers in [24].

The authors of [19] found formulas for exact solutions of the following equations

$$x_{n+1}=\frac{x_{n-3}x_{n-4}}{x_n\left(a_n+b_nx_{n-1}x_{n-2}x_{n-3}x_{n-4}\right)},\ n\in\mathbb{N}_0, \tag{1.3}$$

where a_n and b_n are real sequences.

Moreover, in [40], the following system of difference equations

$$x_n=\frac{x_{n-4}y_{n-5}}{y_{n-1}\left(a_n+b_nx_{n-2}y_{n-3}x_{n-4}y_{n-5}\right)},\ y_n=\frac{y_{n-4}x_{n-5}}{x_{n-1}\left(\alpha_n+\beta_ny_{n-2}x_{n-3}y_{n-4}x_{n-5}\right)},\ n\in\mathbb{N}_0, \tag{1.4}$$

was solved by Yazlik and Kara where the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$ and the initial values x_{-i},y_{-i}, $i=\overline{1,5}$, are real numbers. Further, we investigated asymptotic behavior and periodicity of solutions of system (1.4) when all sequences are constant.

In this paper, we study the following system of difference equations

$$\begin{aligned}x_n&=\frac{y_{n-4}z_{n-5}}{y_{n-1}\left(a_n+b_nz_{n-2}x_{n-3}y_{n-4}z_{n-5}\right)},\\ y_n&=\frac{z_{n-4}x_{n-5}}{z_{n-1}\left(\alpha_n+\beta_nx_{n-2}y_{n-3}z_{n-4}x_{n-5}\right)},\\ z_n&=\frac{x_{n-4}y_{n-5}}{x_{n-1}\left(A_n+B_ny_{n-2}z_{n-3}x_{n-4}y_{n-5}\right)},\ n\in\mathbb{N}_0,\end{aligned} \tag{1.5}$$

where the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$, $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$ and the initial values x_{-j},y_{-j}, $j=\overline{1,5}$, are real numbers. System (1.5) is a generalization of equation (1.1), equation (1.2), equation (1.3) and system (1.4). Our aim in this paper is to show that system (1.5) is solvable in closed form by using the method of transformation. In addition, the forbidden set of initial values for solutions of system (1.5) is described. Then, for the case when all the coefficients are constant, solutions of system (1.5) are obtained.

Lemma 1.1. *[23] Let $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ be two sequences of real numbers and the sequences y_{km+i}, $i=\overline{0,k-1}$, be solutions of the equations*

$$y_{km+i}=a_{km+i}y_{k(m-1)+i}+b_{km+i},\ m\in\mathbb{N}_0. \tag{1.6}$$

Then, for each fixed $i=\overline{0,k-1}$ and $m\geq-1$, equation (1.6) has the general solution

$$y_{km+i}=y_{i-k}\prod_{j=0}^{m}a_{kj+i}+\sum_{s=0}^{m}b_{ks+i}\prod_{j=s+1}^{m}a_{kj+i}.$$

Further, if $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are constant and $i=\overline{0,k-1}$, $m\geq-1$, then

$$y_{km+i}=\begin{cases}a^{m+1}y_{i-k}+b\frac{1-a^{m+1}}{1-a}, & \text{if } a\neq1,\\ y_{i-k}+b\left(m+1\right), & \text{if } a=1.\end{cases}$$

2. Closed-Form Solutions of System (1.5)

In this section, we show that the system (1.5) is solvable in closed form. We will deal only with well-defined solutions to system (1.5). Hence, we assume that

$$x_n\neq0,\ y_n\neq0,\ z_n\neq0,\ n\geq-5,$$

and

$$a_n+b_nz_{n-2}x_{n-3}y_{n-4}z_{n-5}\neq0,\ \ \alpha_n+\beta_nx_{n-2}y_{n-3}z_{n-4}x_{n-5}\neq0,\ \ A_n+B_ny_{n-2}z_{n-3}x_{n-4}y_{n-5}\neq0,\ n\in\mathbb{N}_0.$$

Let $\{(x_n, y_n, z_n)\}_{n\geq -5}$ be solutions of system (1.5). If at least one of the initial values x_{-k}, y_{-k}, z_{-k}, $k = \overline{1,5}$ is equal to zero, then the solutions of system (1.5) is not defined. For instance, if $x_{-5} = 0$, then $y_0 = 0$ and so x_1 is not defined. Similarly, if $y_{-5} = 0$ (or $z_{-5} = 0$) then $z_0 = 0$ (or $x_0 = 0$) and so y_1 (or z_1) is not defined. For $k = \overline{1,4}$, the other cases are similar.
On the other hand, if $x_{n_1} = 0$ ($n_1 \in \mathbb{N}_0$), $x_n \neq 0$, for every $n < n_1$. Then according to the first equation in (1.5) we get that $y_{n_1-4} = 0$ or $z_{n_1-5} = 0$. If $y_{n_1-4} = 0$, then according to the second equation in (1.5) we get that $z_{n_1-8} = 0$. If $z_{n_1-5} = 0$, then according to the third equation in (1.5) we get that $y_{n_1-10} = 0$. Repeating this procedure, we have a $i_1 \in \{1,2,3,4,5\}$ such that $y_{-i_1} = 0$ or $z_{-i_1} = 0$. Similarly, if $y_{n_2} = 0$ ($n_2 \in \mathbb{N}_0$), $y_n \neq 0$, for every $n < n_2$. Then according to the second equation in (1.5) we get that $z_{n_2-4} = 0$ or $x_{n_2-5} = 0$. If $z_{n_2-4} = 0$, then according to the third equation in (1.5) we get that $x_{n_2-8} = 0$. If $x_{n_2-5} = 0$, then according to the first equation in (1.5) we get that $z_{n_2-10} = 0$. Repeating this procedure, we have a $i_2 \in \{1,2,3,4,5\}$ such that $z_{-i_2} = 0$ or $x_{-i_2} = 0$. If $z_{n_3} = 0$ ($n_3 \in \mathbb{N}_0$), $z_n \neq 0$, for every $n < n_3$. Then according to the third equation in (1.5) we get that $x_{n_3-4} = 0$ or $y_{n_3-5} = 0$. If $x_{n_3-4} = 0$, then according to the first equation in (1.5) we get that $y_{n_3-8} = 0$. If $y_{n_3-5} = 0$, then according to the second equation in (1.5) we get that $x_{n_3-10} = 0$. Repeating this procedure, we have a $i_3 \in \{1,2,3,4,5\}$ such that $x_{-i_3} = 0$ or $y_{-i_3} = 0$. Repeating this procedure we find a $i \in \{1,2,3,4,5\}$ such that $x_{-i} = 0$ or $y_{-i} = 0$ or $z_{-i} = 0$. As we have proved above, such solutions are not defined. Hence, of some interest is the case when

$$x_n \neq 0,\ y_n \neq 0,\ z_n \neq 0,\ n \geq -5.$$

Note that the system (1.5) can be written in the form

$$\begin{aligned} x_n y_{n-1} z_{n-2} x_{n-3} &= \frac{z_{n-2} x_{n-3} y_{n-4} z_{n-5}}{(a_n + b_n z_{n-2} x_{n-3} y_{n-4} z_{n-5})}, \\ y_n z_{n-1} x_{n-2} y_{n-3} &= \frac{x_{n-2} y_{n-3} z_{n-4} x_{n-5}}{(\alpha_n + \beta_n x_{n-2} y_{n-3} z_{n-4} x_{n-5})}, \\ z_n x_{n-1} y_{n-2} z_{n-3} &= \frac{y_{n-2} z_{n-3} x_{n-4} y_{n-5}}{(A_n + B_n y_{n-2} z_{n-3} x_{n-4} y_{n-5})},\ n \in \mathbb{N}_0. \end{aligned} \tag{2.1}$$

Employing the change of variables

$$u_n = \frac{1}{x_n y_{n-1} z_{n-2} x_{n-3}},\ v_n = \frac{1}{y_n z_{n-1} x_{n-2} y_{n-3}},\ w_n = \frac{1}{z_n x_{n-1} y_{n-2} z_{n-3}},\ n \geq -2, \tag{2.2}$$

system (1.5) is transformed into the following system of linear difference equations

$$u_n = a_n w_{n-2} + b_n,\ v_n = \alpha_n u_{n-2} + \beta_n,\ w_n = A_n v_{n-2} + B_n,\ n \in \mathbb{N}_0, \tag{2.3}$$

from system (2.3), we get

$$u_{n+6} = a_{n+6} A_{n+4} \alpha_{n+2} u_n + a_{n+6} A_{n+4} \beta_{n+2} + a_{n+6} B_{n+4} + b_{n+6},\ n \geq -2, \tag{2.4}$$

$$v_{n+6} = \alpha_{n+6} a_{n+4} A_{n+2} v_n + \alpha_{n+6} a_{n+4} B_{n+2} + \alpha_{n+6} b_{n+4} + \beta_{n+6},\ n \geq -2, \tag{2.5}$$

$$w_{n+6} = A_{n+6} \alpha_{n+4} a_{n+2} w_n + A_{n+6} \alpha_{n+4} b_{n+2} + A_{n+6} \beta_{n+4} + B_{n+6},\ n \geq -2, \tag{2.6}$$

which are nonhomogeneous linear sixth-order difference equations with variable coefficient. If we apply the decomposition of indexes $n \to 6n + j$, for some $n \in \mathbb{N}_0$ and $j = \overline{-2,3}$ to (2.4) and (2.6), then they become

$$u_{6(n+1)+j} = a_{6n+j+6} A_{6n+j+4} \alpha_{6n+j+2} u_{6n+j} + a_{6n+j+6} A_{6n+j+4} \beta_{6n+j+2} + a_{6n+j+6} B_{6n+j+4} + b_{6n+j+6}, \tag{2.7}$$

$$v_{6(n+1)+j} = \alpha_{6n+j+6} a_{6n+j+4} A_{6n+j+2} v_{6n+j} + \alpha_{6n+j+6} a_{6n+j+4} B_{6n+j+2} + \alpha_{6n+j+6} b_{6n+j+4} + \beta_{6n+j+6}, \tag{2.8}$$

$$w_{6(n+1)+j} = A_{6n+j+6} \alpha_{6n+j+4} a_{6n+j+2} w_{6n+j} + A_{6n+j+6} \alpha_{6n+j+4} b_{6n+j+2} + A_{6n+j+6} \beta_{6n+j+4} + B_{6n+j+6}, \tag{2.9}$$

for $n \in \mathbb{N}_0$, which are first-order 6-equations. Let $u_n^{(j)} = u_{6n+j}$, $v_n^{(j)} = v_{6n+j}$, $w_n^{(j)} = w_{6n+j}$ for $n \in \mathbb{N}_0$ and $j = \overline{-2,3}$ and

$$\begin{aligned} \gamma_n^{(j)} &= a_{6n+j+6} A_{6n+j+4} \alpha_{6n+j+2}, \\ \delta_n^{(j)} &= a_{6n+j+6} A_{6n+j+4} \beta_{6n+j+2} + a_{6n+j+6} B_{6n+j+4} + b_{6n+j+6}, \end{aligned} \tag{2.10}$$

$$\widehat{\gamma}_n^{(j)} = \alpha_{6n+j+6} a_{6n+j+4} A_{6n+j+2},$$
$$\widehat{\delta}_n^{(j)} = \alpha_{6n+j+6} a_{6n+j+4} B_{6n+j+2} + \alpha_{6n+j+6} b_{6n+j+4} + \beta_{6n+j+6}, \tag{2.11}$$

$$\widetilde{\gamma}_n^{(j)} = A_{6n+j+6} \alpha_{6n+j+4} a_{6n+j+2},$$
$$\widetilde{\delta}_n^{(j)} = A_{6n+j+6} \alpha_{6n+j+4} b_{6n+j+2} + A_{6n+j+6} \beta_{6n+j+4} + B_{6n+j+6}. \tag{2.12}$$

Then equations in (2.7)-(2.9) can be written in the form

$$u_{n+1}^{(j)} = \gamma_n^{(j)} u_n^{(j)} + \delta_n^{(j)},\ n \in \mathbb{N}_0, \tag{2.13}$$

$$v_{n+1}^{(j)} = \widehat{\gamma}_n^{(j)} v_n^{(j)} + \widehat{\delta}_n^{(j)},\ n \in \mathbb{N}_0, \tag{2.14}$$

$$w_{n+1}^{(j)} = \widetilde{\gamma}_n^{(j)} w_n^{(j)} + \widetilde{\delta}_n^{(j)},\ n \in \mathbb{N}_0, \tag{2.15}$$

for $j = \overline{-2,3}$.
From (2.13)-(2.15) and Lemma 1.1, we have

$$u_n^{(j)} = \left(\prod_{k=0}^{n-1} \gamma_k^{(j)}\right) u_0^{(j)} + \sum_{i=0}^{n-1} \left(\prod_{k=i+1}^{n-1} \gamma_k^{(j)}\right) \delta_i^{(j)}, \tag{2.16}$$

$$v_n^{(j)} = \left(\prod_{k=0}^{n-1} \widehat{\gamma}_k^{(j)}\right) v_0^{(j)} + \sum_{i=0}^{n-1} \left(\prod_{k=i+1}^{n-1} \widehat{\gamma}_k^{(j)}\right) \widehat{\delta}_i^{(j)}, \tag{2.17}$$

$$w_n^{(j)} = \left(\prod_{k=0}^{n-1} \widetilde{\gamma}_k^{(j)}\right) w_0^{(j)} + \sum_{i=0}^{n-1} \left(\prod_{k=i+1}^{n-1} \widetilde{\gamma}_k^{(j)}\right) \widetilde{\delta}_i^{(j)}, \tag{2.18}$$

for $n \in \mathbb{N}_0$, $j = \overline{-2,3}$. Using (2.10)-(2.12) in equations (2.16)-(2.18), we obtain

$$\begin{aligned} u_{6n+j} = &\left(\prod_{k=0}^{n-1} \left(a_{6k+j+6} A_{6k+j+4} \alpha_{6k+j+2}\right)\right) u_j \\ &+ \sum_{i=0}^{n-1} \left(\prod_{k=i+1}^{n-1} \left(a_{6k+j+6} A_{6k+j+4} \alpha_{6k+j+2}\right)\right) \left(a_{6i+j+6} A_{6i+j+4} \beta_{6i+j+2} + a_{6i+j+6} B_{6i+j+4} + b_{6i+j+6}\right), \end{aligned} \tag{2.19}$$

$$\begin{aligned} v_{6n+j} = &\left(\prod_{k=0}^{n-1} \left(\alpha_{6k+j+6} a_{6k+j+4} A_{6k+j+2}\right)\right) v_j \\ &+ \sum_{i=0}^{n-1} \left(\prod_{k=i+1}^{n-1} \left(\alpha_{6k+j+6} a_{6k+j+4} A_{6k+j+2}\right)\right) \left(\alpha_{6i+j+6} a_{6i+j+4} B_{6i+j+2} + \alpha_{6i+j+6} b_{6i+j+4} + \beta_{6i+j+6}\right), \end{aligned} \tag{2.20}$$

$$\begin{aligned} w_{6n+j} = &\left(\prod_{k=0}^{n-1} \left(A_{6k+j+6} \alpha_{6k+j+4} a_{6k+j+2}\right)\right) w_j \\ &+ \sum_{i=0}^{n-1} \left(\prod_{k=i+1}^{n-1} \left(A_{6k+j+6} \alpha_{6k+j+4} a_{6k+j+2}\right)\right) \left(A_{6i+j+6} \alpha_{6i+j+4} b_{6i+j+2} + A_{6i+j+6} \beta_{6i+j+4} + B_{6i+j+6}\right), \end{aligned} \tag{2.21}$$

for $n \in \mathbb{N}_0$, $j = \overline{-2,3}$.
When the coefficients are constants i.e., $a_n = a$, $b_n = b$, $\alpha_n = \alpha$, $\beta_n = \beta$, $A_n = A$ and $B_n = B$, formulas (2.19)-(2.21) becomes

$$u_{6n+j} = \begin{cases} \left(a\alpha A\right)^n u_j + \frac{1-(a\alpha A)^n}{1-a\alpha A} \left(aA\beta + aB + b\right), & a\alpha A \neq 1, \\ u_j + \left(aA\beta + aB + b\right) n, & a\alpha A = 1, \end{cases} \quad n \in \mathbb{N}_0, \tag{2.22}$$

$$v_{6n+j} = \begin{cases} (a\alpha A)^n v_j + \frac{1-(a\alpha A)^n}{1-a\alpha A}(\alpha a B + \alpha b + \beta), & a\alpha A \neq 1, \\ v_j + (\alpha a B + \alpha b + \beta) n, & a\alpha A = 1, \end{cases} \quad n \in \mathbb{N}_0, \tag{2.23}$$

$$w_{6n+j} = \begin{cases} (a\alpha A)^n w_j + \frac{1-(a\alpha A)^n}{1-a\alpha A}(A\alpha b + A\beta + B), & a\alpha A \neq 1, \\ w_j + (A\alpha b + A\beta + B) n, & a\alpha A = 1, \end{cases} \quad n \in \mathbb{N}_0, \tag{2.24}$$

for $j = \overline{-2,3}$. From equalities in (2.2), we get

$$x_n = \frac{1}{u_n y_{n-1} z_{n-2} x_{n-3}} = \frac{v_{n-1}}{u_n} y_{n-4} = \frac{v_{n-1} w_{n-5}}{u_n v_{n-4}} z_{n-8} = \frac{v_{n-1} w_{n-5} u_{n-9}}{u_n v_{n-4} w_{n-8}} x_{n-12}, \tag{2.25}$$

$$y_n = \frac{1}{v_n z_{n-1} x_{n-2} y_{n-3}} = \frac{w_{n-1}}{v_n} z_{n-4} = \frac{w_{n-1} u_{n-5}}{v_n w_{n-4}} x_{n-8} = \frac{w_{n-1} u_{n-5} v_{n-9}}{v_n w_{n-4} u_{n-8}} y_{n-12}, \tag{2.26}$$

$$z_n = \frac{1}{w_n x_{n-1} y_{n-2} z_{n-3}} = \frac{u_{n-1}}{w_n} x_{n-4} = \frac{u_{n-1} v_{n-5}}{w_n u_{n-4}} y_{n-8} = \frac{u_{n-1} v_{n-5} w_{n-9}}{w_n u_{n-4} v_{n-8}} z_{n-12}, \tag{2.27}$$

for $n \geq 7$, from which it follows that

$$x_{12m+6l+r} = x_{6l+r-12} \prod_{s=0}^{m} \frac{v_{6\left(2s+l+1+\lfloor\frac{r-5}{6}\rfloor\right)+r-7-6\lfloor\frac{r-5}{6}\rfloor}}{u_{6\left(2s+l+1+\lfloor\frac{r-4}{6}\rfloor\right)+r-6-6\lfloor\frac{r-4}{6}\rfloor}} \frac{w_{6\left(2s+l+\lfloor\frac{r-3}{6}\rfloor\right)+r-5-6\lfloor\frac{r-3}{6}\rfloor}}{v_{6\left(2s+l+\lfloor\frac{r-2}{6}\rfloor\right)+r-4-6\lfloor\frac{r-2}{6}\rfloor}} \times \frac{u_{6\left(2s+l+\lfloor\frac{r-7}{6}\rfloor\right)+r-9-6\lfloor\frac{r-7}{6}\rfloor}}{w_{6\left(2s+l+\lfloor\frac{r-6}{6}\rfloor\right)+r-8-6\lfloor\frac{r-6}{6}\rfloor}}, \tag{2.28}$$

$$y_{12m+6l+r} = y_{6l+r-12} \prod_{s=0}^{m} \frac{w_{6\left(2s+l+1+\lfloor\frac{r-5}{6}\rfloor\right)+r-7-6\lfloor\frac{r-5}{6}\rfloor}}{v_{6\left(2s+l+1+\lfloor\frac{r-4}{6}\rfloor\right)+r-6-6\lfloor\frac{r-4}{6}\rfloor}} \frac{u_{6\left(2s+l+\lfloor\frac{r-3}{6}\rfloor\right)+r-5-6\lfloor\frac{r-3}{6}\rfloor}}{w_{6\left(2s+l+\lfloor\frac{r-2}{6}\rfloor\right)+r-4-6\lfloor\frac{r-2}{6}\rfloor}} \times \frac{v_{6\left(2s+l+\lfloor\frac{r-7}{6}\rfloor\right)+r-9-6\lfloor\frac{r-7}{6}\rfloor}}{u_{6\left(2s+l+\lfloor\frac{r-6}{6}\rfloor\right)+r-8-6\lfloor\frac{r-6}{6}\rfloor}}, \tag{2.29}$$

and

$$z_{12m+6l+r} = z_{6l+r-12} \prod_{s=0}^{m} \frac{u_{6\left(2s+l+1+\lfloor\frac{r-5}{6}\rfloor\right)+r-7-6\lfloor\frac{r-5}{6}\rfloor}}{w_{6\left(2s+l+1+\lfloor\frac{r-4}{6}\rfloor\right)+r-6-6\lfloor\frac{r-4}{6}\rfloor}} \frac{v_{6\left(2s+l+\lfloor\frac{r-3}{6}\rfloor\right)+r-5-6\lfloor\frac{r-3}{6}\rfloor}}{u_{6\left(2s+l+\lfloor\frac{r-2}{6}\rfloor\right)+r-4-6\lfloor\frac{r-2}{6}\rfloor}} \times \frac{w_{6\left(2s+l+\lfloor\frac{r-7}{6}\rfloor\right)+r-9-6\lfloor\frac{r-7}{6}\rfloor}}{v_{6\left(2s+l+\lfloor\frac{r-6}{6}\rfloor\right)+r-8-6\lfloor\frac{r-6}{6}\rfloor}}, \tag{2.30}$$

for every $m \in \mathbb{N}_0$, $l \in \{1, 2\}$ and $r = \overline{1,6}$. Employing (2.22)-(2.24) in (2.28)-(2.30), we have

$$x_{12m+6l+r} = x_{6l+r-12} \prod_{s=0}^{m} \frac{D_{s,l,r}}{C_{s,l,r}} \frac{E_{s,l,r}}{G_{s,l,r}} \frac{F_{s,l,r}}{H_{s,l,r}}, \tag{2.31}$$

$$y_{12m+6l+r} = y_{6l+r-12} \prod_{s=0}^{m} \frac{\widehat{D}_{s,l,r}}{\widehat{C}_{s,l,r}} \frac{\widehat{E}_{s,l,r}}{\widehat{G}_{s,l,r}} \frac{\widehat{F}_{s,l,r}}{\widehat{H}_{s,l,r}}, \tag{2.32}$$

$$z_{12m+6l+r} = z_{6l+r-12} \prod_{s=0}^{m} \frac{\widetilde{D}_{s,l,r}}{\widetilde{C}_{s,l,r}} \frac{\widetilde{E}_{s,l,r}}{\widetilde{G}_{s,l,r}} \frac{\widetilde{F}_{s,l,r}}{\widetilde{H}_{s,l,r}}, \tag{2.33}$$

for every $m \in \mathbb{N}_0$, $l \in \{1,2\}$ and $r = \overline{1,6}$, where

$$C_{s,l,r} = \left(\prod_{k=0}^{2s+l+\lfloor \frac{r-4}{6} \rfloor} \left(a_{6k+r-6\lfloor \frac{r-4}{6} \rfloor} A_{6k+r-2-6\lfloor \frac{r-4}{6} \rfloor} \alpha_{6k+r-4-6\lfloor \frac{r-4}{6} \rfloor} \right) \right) u_{r-6-6\lfloor \frac{r-4}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l+\lfloor \frac{r-4}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l+\lfloor \frac{r-4}{6} \rfloor} \left(a_{6k+r-6\lfloor \frac{r-4}{6} \rfloor} A_{6k+r-2-6\lfloor \frac{r-4}{6} \rfloor} \alpha_{6k+r-4-6\lfloor \frac{r-4}{6} \rfloor} \right) \right)$$
$$\times \left(a_{6i+r-6\lfloor \frac{r-4}{6} \rfloor} A_{6i+r-2-6\lfloor \frac{r-4}{6} \rfloor} \beta_{6i+r-4-6\lfloor \frac{r-4}{6} \rfloor} + a_{6i+r-6\lfloor \frac{r-4}{6} \rfloor} B_{6i+r-2-6\lfloor \frac{r-4}{6} \rfloor} + b_{6i+r-6\lfloor \frac{r-4}{6} \rfloor} \right),$$

$$D_{s,l,r} = \left(\prod_{k=0}^{2s+l+\lfloor \frac{r-5}{6} \rfloor} \left(\alpha_{6k+r-1-6\lfloor \frac{r-5}{6} \rfloor} a_{6k+r-3-6\lfloor \frac{r-5}{6} \rfloor} A_{6k+r-5-6\lfloor \frac{r-5}{6} \rfloor} \right) \right) v_{r-7-6\lfloor \frac{r-5}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l+\lfloor \frac{r-5}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l+\lfloor \frac{r-5}{6} \rfloor} \left(\alpha_{6k+r-1-6\lfloor \frac{r-5}{6} \rfloor} a_{6k+r-3-6\lfloor \frac{r-5}{6} \rfloor} A_{6k+r-5-6\lfloor \frac{r-5}{6} \rfloor} \right) \right)$$
$$\times \left(\alpha_{6i+r-1-6\lfloor \frac{r-5}{6} \rfloor} a_{6i+r-3-6\lfloor \frac{r-5}{6} \rfloor} B_{6i+r-5-6\lfloor \frac{r-5}{6} \rfloor} + \alpha_{6i+r-1-6\lfloor \frac{r-5}{6} \rfloor} b_{6i+r-3-6\lfloor \frac{r-5}{6} \rfloor} + \beta_{6i+r-1-6\lfloor \frac{r-5}{6} \rfloor} \right),$$

$$E_{s,l,r} = \left(\prod_{k=0}^{2s+l-1+\lfloor \frac{r-3}{6} \rfloor} \left(A_{6k+r+1-6\lfloor \frac{r-3}{6} \rfloor} \alpha_{6k+r-1-6\lfloor \frac{r-3}{6} \rfloor} a_{6k+r-3-6\lfloor \frac{r-3}{6} \rfloor} \right) \right) w_{r-5-6\lfloor \frac{r-3}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l-1+\lfloor \frac{r-3}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l-1+\lfloor \frac{r-3}{6} \rfloor} \left(A_{6k+r+1-6\lfloor \frac{r-3}{6} \rfloor} \alpha_{6k+r-1-6\lfloor \frac{r-3}{6} \rfloor} a_{6k+r-3-6\lfloor \frac{r-3}{6} \rfloor} \right) \right)$$
$$\times \left(A_{6i+r+1-6\lfloor \frac{r-3}{6} \rfloor} \alpha_{6i+r-1-6\lfloor \frac{r-3}{6} \rfloor} b_{6i+r-3-6\lfloor \frac{r-3}{6} \rfloor} + A_{6i+r+1-6\lfloor \frac{r-3}{6} \rfloor} \beta_{6i+r-1-6\lfloor \frac{r-3}{6} \rfloor} + B_{6i+r+1-6\lfloor \frac{r-3}{6} \rfloor} \right),$$

$$F_{s,l,r} = \left(\prod_{k=0}^{2s+l-1+\lfloor \frac{r-7}{6} \rfloor} \left(a_{6k+r-3-6\lfloor \frac{r-7}{6} \rfloor} A_{6k+r-5-6\lfloor \frac{r-7}{6} \rfloor} \alpha_{6k+r-7-6\lfloor \frac{r-7}{6} \rfloor} \right) \right) u_{r-9-6\lfloor \frac{r-7}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l-1+\lfloor \frac{r-7}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l-1+\lfloor \frac{r-7}{6} \rfloor} \left(a_{6k+r-3-6\lfloor \frac{r-7}{6} \rfloor} A_{6k+r-5-6\lfloor \frac{r-7}{6} \rfloor} \alpha_{6k+r-7-6\lfloor \frac{r-7}{6} \rfloor} \right) \right)$$
$$\times \left(a_{6i+r-3-6\lfloor \frac{r-7}{6} \rfloor} A_{6i+r-5-6\lfloor \frac{r-7}{6} \rfloor} \beta_{6i+r-7-6\lfloor \frac{r-7}{6} \rfloor} + a_{6i+r-3-6\lfloor \frac{r-7}{6} \rfloor} B_{6i+r-5-6\lfloor \frac{r-7}{6} \rfloor} + b_{6i+r-3-6\lfloor \frac{r-7}{6} \rfloor} \right),$$

$$G_{s,l,r} = \left(\prod_{k=0}^{2s+l-1+\lfloor \frac{r-2}{6} \rfloor} \left(\alpha_{6k+r+2-6\lfloor \frac{r-2}{6} \rfloor} a_{6k+r-6\lfloor \frac{r-2}{6} \rfloor} A_{6k+r-2-6\lfloor \frac{r-2}{6} \rfloor} \right) \right) v_{r-4-6\lfloor \frac{r-2}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l-1+\lfloor \frac{r-2}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l-1+\lfloor \frac{r-2}{6} \rfloor} \left(\alpha_{6k+r+2-6\lfloor \frac{r-2}{6} \rfloor} a_{6k+r-6\lfloor \frac{r-2}{6} \rfloor} A_{6k+r-2-6\lfloor \frac{r-2}{6} \rfloor} \right) \right)$$
$$\times \left(\alpha_{6i+r+2-6\lfloor \frac{r-2}{6} \rfloor} a_{6i+r-6\lfloor \frac{r-2}{6} \rfloor} B_{6i+r-2-6\lfloor \frac{r-2}{6} \rfloor} + \alpha_{6i+r+2-6\lfloor \frac{r-2}{6} \rfloor} b_{6i+r-6\lfloor \frac{r-2}{6} \rfloor} + \beta_{6i+r+2-6\lfloor \frac{r-2}{6} \rfloor} \right),$$

$$H_{s,l,r} = \left(\prod_{k=0}^{2s+l-1+\lfloor \frac{r-6}{6} \rfloor} \left(A_{6k+r-2-6\lfloor \frac{r-6}{6} \rfloor} \alpha_{6k+r-4-6\lfloor \frac{r-6}{6} \rfloor} a_{6k+r-6-6\lfloor \frac{r-6}{6} \rfloor} \right) \right) w_{r-8-6\lfloor \frac{r-6}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l-1+\lfloor \frac{r-6}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l-1+\lfloor \frac{r-6}{6} \rfloor} \left(A_{6k+r-2-6\lfloor \frac{r-6}{6} \rfloor} \alpha_{6k+r-4-6\lfloor \frac{r-6}{6} \rfloor} a_{6k+r-6-6\lfloor \frac{r-6}{6} \rfloor} \right) \right)$$
$$\times \left(A_{6i+r-2-6\lfloor \frac{r-6}{6} \rfloor} \alpha_{6i+r-4-6\lfloor \frac{r-6}{6} \rfloor} b_{6i+r-6-6\lfloor \frac{r-6}{6} \rfloor} + A_{6i+r-2-6\lfloor \frac{r-6}{6} \rfloor} \beta_{6i+r-4-6\lfloor \frac{r-6}{6} \rfloor} + B_{6i+r-2-6\lfloor \frac{r-6}{6} \rfloor} \right),$$

$$\widehat{C}_{s,l,r} = \left(\prod_{k=0}^{2s+l+\lfloor \frac{r-4}{6} \rfloor} \left(\alpha_{6k+r-6\lfloor \frac{r-4}{6} \rfloor} a_{6k+r-2-6\lfloor \frac{r-4}{6} \rfloor} A_{6k+r-4-6\lfloor \frac{r-4}{6} \rfloor} \right) \right) v_{r-6-6\lfloor \frac{r-4}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l+\lfloor \frac{r-4}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l+\lfloor \frac{r-4}{6} \rfloor} \left(\alpha_{6k+r-6\lfloor \frac{r-4}{6} \rfloor} a_{6k+r-2-6\lfloor \frac{r-4}{6} \rfloor} A_{6k+r-4-6\lfloor \frac{r-4}{6} \rfloor} \right) \right)$$
$$\times \left(\alpha_{6i+r-6\lfloor \frac{r-4}{6} \rfloor} a_{6i+r-2-6\lfloor \frac{r-4}{6} \rfloor} B_{6i+r-4-6\lfloor \frac{r-4}{6} \rfloor} + \alpha_{6i+r-6\lfloor \frac{r-4}{6} \rfloor} b_{6i+r-2-6\lfloor \frac{r-4}{6} \rfloor} + \beta_{6i+r-6\lfloor \frac{r-4}{6} \rfloor} \right),$$

$$\widehat{D}_{s,l,r} = \left(\prod_{k=0}^{2s+l+\lfloor \frac{r-5}{6} \rfloor} \left(A_{6k+r-1-6\lfloor \frac{r-5}{6} \rfloor} \alpha_{6k+r-3-6\lfloor \frac{r-5}{6} \rfloor} a_{6k+r-5-6\lfloor \frac{r-5}{6} \rfloor} \right) \right) w_{r-7-6\lfloor \frac{r-5}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l+\lfloor \frac{r-5}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l+\lfloor \frac{r-5}{6} \rfloor} \left(A_{6k+r-1-6\lfloor \frac{r-5}{6} \rfloor} \alpha_{6k+r-3-6\lfloor \frac{r-5}{6} \rfloor} a_{6k+r-5-6\lfloor \frac{r-5}{6} \rfloor} \right) \right)$$
$$\times \left(A_{6i+r-1-6\lfloor \frac{r-5}{6} \rfloor} \alpha_{6i+r-3-6\lfloor \frac{r-5}{6} \rfloor} b_{6i+r-5-6\lfloor \frac{r-5}{6} \rfloor} + A_{6i+r-1-6\lfloor \frac{r-5}{6} \rfloor} \beta_{6i+r-3-6\lfloor \frac{r-5}{6} \rfloor} + B_{6i+r-1-6\lfloor \frac{r-5}{6} \rfloor} \right),$$

$$\widehat{E}_{s,l,r} = \left(\prod_{k=0}^{2s+l-1+\lfloor \frac{r-3}{6} \rfloor} \left(a_{6k+r+1-6\lfloor \frac{r-3}{6} \rfloor} A_{6k+r-1-6\lfloor \frac{r-3}{6} \rfloor} \alpha_{6k+r-3-6\lfloor \frac{r-3}{6} \rfloor} \right) \right) u_{r-5-6\lfloor \frac{r-3}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l-1+\lfloor \frac{r-3}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l-1+\lfloor \frac{r-3}{6} \rfloor} \left(a_{6k+r+1-6\lfloor \frac{r-3}{6} \rfloor} A_{6k+r-1-6\lfloor \frac{r-3}{6} \rfloor} \alpha_{6k+r-3-6\lfloor \frac{r-3}{6} \rfloor} \right) \right)$$
$$\times \left(a_{6i+r+1-6\lfloor \frac{r-3}{6} \rfloor} A_{6i+r-1-6\lfloor \frac{r-3}{6} \rfloor} \beta_{6i+r-3-6\lfloor \frac{r-3}{6} \rfloor} + a_{6i+r+1-6\lfloor \frac{r-3}{6} \rfloor} B_{6i+r-1-6\lfloor \frac{r-3}{6} \rfloor} + b_{6i+r+1-6\lfloor \frac{r-3}{6} \rfloor} \right),$$

$$\widehat{F}_{s,l,r} = \left(\prod_{k=0}^{2s+l-1+\lfloor \frac{r-7}{6} \rfloor} \left(\alpha_{6k+r-3-6\lfloor \frac{r-7}{6} \rfloor} a_{6k+r-5-6\lfloor \frac{r-7}{6} \rfloor} A_{6k+r-7-6\lfloor \frac{r-7}{6} \rfloor} \right) \right) v_{r-9-6\lfloor \frac{r-7}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l-1+\lfloor \frac{r-7}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l-1+\lfloor \frac{r-7}{6} \rfloor} \left(\alpha_{6k+r-3-6\lfloor \frac{r-7}{6} \rfloor} a_{6k+r-5-6\lfloor \frac{r-7}{6} \rfloor} A_{6k+r-7-6\lfloor \frac{r-7}{6} \rfloor} \right) \right)$$
$$\times \left(\alpha_{6i+r-3-6\lfloor \frac{r-7}{6} \rfloor} a_{6i+r-5-6\lfloor \frac{r-7}{6} \rfloor} B_{6i+r-7-6\lfloor \frac{r-7}{6} \rfloor} + \alpha_{6i+r-3-6\lfloor \frac{r-7}{6} \rfloor} b_{6i+r-5-6\lfloor \frac{r-7}{6} \rfloor} + \beta_{6i+r-3-6\lfloor \frac{r-7}{6} \rfloor} \right),$$

$$\widehat{G}_{s,l,r}=\left(\prod_{k=0}^{2s+l-1+\lfloor\frac{r-2}{6}\rfloor}\left(A_{6k+r+2-6\lfloor\frac{r-2}{6}\rfloor}\alpha_{6k+r-6\lfloor\frac{r-2}{6}\rfloor}a_{6k+r-2-6\lfloor\frac{r-2}{6}\rfloor}\right)\right)w_{r-4-6\lfloor\frac{r-2}{6}\rfloor}$$
$$+\sum_{i=0}^{2s+l-1+\lfloor\frac{r-2}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l-1+\lfloor\frac{r-2}{6}\rfloor}\left(A_{6k+r+2-6\lfloor\frac{r-2}{6}\rfloor}\alpha_{6k+r-6\lfloor\frac{r-2}{6}\rfloor}a_{6k+r-2-6\lfloor\frac{r-2}{6}\rfloor}\right)\right)$$
$$\times\left(A_{6i+r+2-6\lfloor\frac{r-2}{6}\rfloor}\alpha_{6i+r-6\lfloor\frac{r-2}{6}\rfloor}b_{6i+r-2-6\lfloor\frac{r-2}{6}\rfloor}+A_{6i+r+2-6\lfloor\frac{r-2}{6}\rfloor}\beta_{6i+r-6\lfloor\frac{r-2}{6}\rfloor}+B_{6i+r+2-6\lfloor\frac{r-2}{6}\rfloor}\right),$$

$$\widehat{H}_{s,l,r}=\left(\prod_{k=0}^{2s+l-1+\lfloor\frac{r-6}{6}\rfloor}\left(a_{6k+r-2-6\lfloor\frac{r-6}{6}\rfloor}A_{6k+r-4-6\lfloor\frac{r-6}{6}\rfloor}\alpha_{6k+r-6-6\lfloor\frac{r-6}{6}\rfloor}\right)\right)u_{r-8-6\lfloor\frac{r-6}{6}\rfloor}$$
$$+\sum_{i=0}^{2s+l-1+\lfloor\frac{r-6}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l-1+\lfloor\frac{r-6}{6}\rfloor}\left(a_{6k+r-2-6\lfloor\frac{r-6}{6}\rfloor}A_{6k+r-4-6\lfloor\frac{r-6}{6}\rfloor}\alpha_{6k+r-6-6\lfloor\frac{r-6}{6}\rfloor}\right)\right)$$
$$\times\left(a_{6i+r-2-6\lfloor\frac{r-6}{6}\rfloor}A_{6i+r-4-6\lfloor\frac{r-6}{6}\rfloor}\beta_{6i+r-6-6\lfloor\frac{r-6}{6}\rfloor}+a_{6i+r-2-6\lfloor\frac{r-6}{6}\rfloor}B_{6i+r-4-6\lfloor\frac{r-6}{6}\rfloor}+b_{6i+r-2-6\lfloor\frac{r-6}{6}\rfloor}\right),$$

$$\widetilde{C}_{s,l,r}=\left(\prod_{k=0}^{2s+l+\lfloor\frac{r-4}{6}\rfloor}\left(A_{6k+r-6\lfloor\frac{r-4}{6}\rfloor}\alpha_{6k+r-2-6\lfloor\frac{r-4}{6}\rfloor}a_{6k+r-4-6\lfloor\frac{r-4}{6}\rfloor}\right)\right)w_{r-6-6\lfloor\frac{r-4}{6}\rfloor}$$
$$+\sum_{i=0}^{2s+l+\lfloor\frac{r-4}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l+\lfloor\frac{r-4}{6}\rfloor}\left(A_{6k+r-6\lfloor\frac{r-4}{6}\rfloor}\alpha_{6k+r-2-6\lfloor\frac{r-4}{6}\rfloor}a_{6k+r-4-6\lfloor\frac{r-4}{6}\rfloor}\right)\right)$$
$$\times\left(A_{6i+r-6\lfloor\frac{r-4}{6}\rfloor}\alpha_{6i+r-2-6\lfloor\frac{r-4}{6}\rfloor}b_{6i+r-4-6\lfloor\frac{r-4}{6}\rfloor}+A_{6i+r-6\lfloor\frac{r-4}{6}\rfloor}\beta_{6i+r-2-6\lfloor\frac{r-4}{6}\rfloor}+B_{6i+r-6\lfloor\frac{r-4}{6}\rfloor}\right),$$

$$\widetilde{D}_{s,l,r}=\left(\prod_{k=0}^{2s+l+\lfloor\frac{r-5}{6}\rfloor}\left(a_{6k+r-1-6\lfloor\frac{r-5}{6}\rfloor}A_{6k+r-3-6\lfloor\frac{r-5}{6}\rfloor}\alpha_{6k+r-5-6\lfloor\frac{r-5}{6}\rfloor}\right)\right)u_{r-7-6\lfloor\frac{r-5}{6}\rfloor}$$
$$+\sum_{i=0}^{2s+l+\lfloor\frac{r-5}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l+\lfloor\frac{r-5}{6}\rfloor}\left(a_{6k+r-1-6\lfloor\frac{r-5}{6}\rfloor}A_{6k+r-3-6\lfloor\frac{r-5}{6}\rfloor}\alpha_{6k+r-5-6\lfloor\frac{r-5}{6}\rfloor}\right)\right)$$
$$\times\left(a_{6i+r-1-6\lfloor\frac{r-5}{6}\rfloor}A_{6i+r-3-6\lfloor\frac{r-5}{6}\rfloor}\beta_{6i+r-5-6\lfloor\frac{r-5}{6}\rfloor}+a_{6i+r-1-6\lfloor\frac{r-5}{6}\rfloor}B_{6i+r-3-6\lfloor\frac{r-5}{6}\rfloor}+b_{6i+r-1-6\lfloor\frac{r-5}{6}\rfloor}\right),$$

$$\widetilde{E}_{s,l,r}=\left(\prod_{k=0}^{2s+l-1+\lfloor\frac{r-3}{6}\rfloor}\left(\alpha_{6k+r+1-6\lfloor\frac{r-3}{6}\rfloor}a_{6k+r-1-6\lfloor\frac{r-3}{6}\rfloor}A_{6k+r-3-6\lfloor\frac{r-3}{6}\rfloor}\right)\right)v_{r-5-6\lfloor\frac{r-3}{6}\rfloor}$$
$$+\sum_{i=0}^{2s+l-1+\lfloor\frac{r-3}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l-1+\lfloor\frac{r-3}{6}\rfloor}\left(\alpha_{6k+r+1-6\lfloor\frac{r-3}{6}\rfloor}a_{6k+r-1-6\lfloor\frac{r-3}{6}\rfloor}A_{6k+r-3-6\lfloor\frac{r-3}{6}\rfloor}\right)\right)$$
$$\times\left(\alpha_{6i+r+1-6\lfloor\frac{r-3}{6}\rfloor}a_{6i+r-1-6\lfloor\frac{r-3}{6}\rfloor}B_{6i+r-3-6\lfloor\frac{r-3}{6}\rfloor}+\alpha_{6i+r+1-6\lfloor\frac{r-3}{6}\rfloor}b_{6i+r-1-6\lfloor\frac{r-3}{6}\rfloor}+\beta_{6i+r+1-6\lfloor\frac{r-3}{6}\rfloor}\right),$$

$$\widetilde{F}_{s,l,r} = \left(\prod_{k=0}^{2s+l-1+\lfloor\frac{r-7}{6}\rfloor}\left(A_{6k+r-3-6\lfloor\frac{r-7}{6}\rfloor}\alpha_{6k+r-5-6\lfloor\frac{r-7}{6}\rfloor}a_{6k+r-7-6\lfloor\frac{r-7}{6}\rfloor}\right)\right)w_{r-9-6\lfloor\frac{r-7}{6}\rfloor}$$
$$+\sum_{i=0}^{2s+l-1+\lfloor\frac{r-7}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l-1+\lfloor\frac{r-7}{6}\rfloor}\left(A_{6k+r-3-6\lfloor\frac{r-7}{6}\rfloor}\alpha_{6k+r-5-6\lfloor\frac{r-7}{6}\rfloor}a_{6k+r-7-6\lfloor\frac{r-7}{6}\rfloor}\right)\right)$$
$$\times\left(A_{6i+r-3-6\lfloor\frac{r-7}{6}\rfloor}\alpha_{6i+r-5-6\lfloor\frac{r-7}{6}\rfloor}b_{6i+r-7-6\lfloor\frac{r-7}{6}\rfloor}+A_{6i+r-3-6\lfloor\frac{r-7}{6}\rfloor}\beta_{6i+r-5-6\lfloor\frac{r-7}{6}\rfloor}+B_{6i+r-3-6\lfloor\frac{r-7}{6}\rfloor}\right),$$

$$\widetilde{G}_{s,l,r} = \left(\prod_{k=0}^{2s+l-1+\lfloor\frac{r-2}{6}\rfloor}\left(a_{6k+r+2-6\lfloor\frac{r-2}{6}\rfloor}A_{6k+r-6\lfloor\frac{r-2}{6}\rfloor}\alpha_{6k+r-2-6\lfloor\frac{r-2}{6}\rfloor}\right)\right)u_{r-4-6\lfloor\frac{r-2}{6}\rfloor}$$
$$+\sum_{i=0}^{2s+l-1+\lfloor\frac{r-2}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l-1+\lfloor\frac{r-2}{6}\rfloor}\left(a_{6k+r+2-6\lfloor\frac{r-2}{6}\rfloor}A_{6k+r-6\lfloor\frac{r-2}{6}\rfloor}\alpha_{6k+r-2-6\lfloor\frac{r-2}{6}\rfloor}\right)\right)$$
$$\times\left(a_{6i+r+2-6\lfloor\frac{r-2}{6}\rfloor}A_{6i+r-6\lfloor\frac{r-2}{6}\rfloor}\beta_{6i+r-2-6\lfloor\frac{r-2}{6}\rfloor}+a_{6i+r+2-6\lfloor\frac{r-2}{6}\rfloor}B_{6i+r-6\lfloor\frac{r-2}{6}\rfloor}+b_{6i+r+2-6\lfloor\frac{r-2}{6}\rfloor}\right),$$

$$\widetilde{H}_{s,l,r} = \left(\prod_{k=0}^{2s+l-1+\lfloor\frac{r-6}{6}\rfloor}\left(\alpha_{6k+r-2-6\lfloor\frac{r-6}{6}\rfloor}a_{6k+r-4-6\lfloor\frac{r-6}{6}\rfloor}A_{6k+r-6-6\lfloor\frac{r-6}{6}\rfloor}\right)\right)v_{r-8-6\lfloor\frac{r-6}{6}\rfloor}$$
$$+\sum_{i=0}^{2s+l-1+\lfloor\frac{r-6}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l-1+\lfloor\frac{r-6}{6}\rfloor}\left(\alpha_{6k+r-2-6\lfloor\frac{r-6}{6}\rfloor}a_{6k+r-4-6\lfloor\frac{r-6}{6}\rfloor}A_{6k+r-6-6\lfloor\frac{r-6}{6}\rfloor}\right)\right)$$
$$\times\left(\alpha_{6i+r-2-6\lfloor\frac{r-6}{6}\rfloor}a_{6i+r-4-6\lfloor\frac{r-6}{6}\rfloor}B_{6i+r-6-6\lfloor\frac{r-6}{6}\rfloor}+\alpha_{6i+r-2-6\lfloor\frac{r-6}{6}\rfloor}b_{6i+r-4-6\lfloor\frac{r-6}{6}\rfloor}+\beta_{6i+r-2-6\lfloor\frac{r-6}{6}\rfloor}\right).$$

The previous computations prove the next theorem.

Theorem 2.1. *Suppose that $\{(x_n, y_n, z_n)\}_{n\geq -5}$ is a well-defined solution of system (1.5). Then, the general solutions of system (1.5) are given by equations in (2.31)-(2.33).*

By the following theorem, we characterize the forbidden set of the initial values for system (1.5).

Theorem 2.2. *Assume that $a_n \neq 0$, $b_n \neq 0$, $\alpha_n \neq 0$, $\beta_n \neq 0$, $A_n \neq 0$, $B_n \neq 0$, for every $n \in \mathbb{N}_0$. Then the forbidden set of the initial values for system (1.5) is given by the set*

$$\mathcal{F} = \bigcup_{m\in\mathbb{N}_0}\bigcup_{i=0}^{1}\Big\{(x_{-5},x_{-4},\ldots,x_{-1},y_{-5},y_{-4},\ldots,y_{-1},z_{-5},z_{-4},\ldots,z_{-1})\in\mathbb{R}^{15}:$$
$$z_{i-2}x_{i-3}y_{i-4}z_{i-5}=\frac{1}{c_m},\, x_{i-2}y_{i-3}z_{i-4}x_{i-5}=\frac{1}{d_m},\ y_{i-2}z_{i-3}x_{i-4}y_{i-5}=\frac{1}{e_m}\text{where}$$
$$c_m := -\sum_{j=0}^{m}\left(\frac{B_{6j+i+2}+A_{6j+i+2}b_{6j+i}+A_{6j+i+2}\alpha_{6j+i}b_{6j+i-2}}{A_{6j+i+2}\alpha_{6j+i}a_{6j+i-2}}\right)\prod_{l=0}^{j-1}\frac{1}{A_{6l+i+2}\alpha_{6l+i}a_{6l+i-2}}\neq 0,$$
$$d_m := -\sum_{j=0}^{m}\left(\frac{b_{6j+i+2}+a_{6j+i+2}\beta_{6j+i}+a_{6j+i+2}A_{6j+i}\beta_{6j+i-2}}{a_{6j+i+2}A_{6j+i}\alpha_{6j+i-2}}\right)\prod_{l=0}^{j-1}\frac{1}{a_{6l+i+2}A_{6l+i}\alpha_{6l+i-2}}\neq 0,,$$
$$e_m := -\sum_{j=0}^{m}\left(\frac{\beta_{6j+i+2}+\alpha_{6j+i+2}B_{6j+i}+\alpha_{6j+i+2}a_{6j+i}B_{6j+i-2}}{\alpha_{6j+i+2}a_{6j+i}A_{6j+i-2}}\right)\prod_{l=0}^{j-1}\frac{1}{\alpha_{6l+i+2}a_{6l+i}A_{6l+i-2}}\neq 0$$
$$\bigcup\bigcup_{j=1}^{5}\Big\{(x_{-5},x_{-4},\ldots,x_{-1},y_{-5},y_{-4},\ldots,y_{-1},z_{-5},z_{-4},\ldots,z_{-1})\in\mathbb{R}^{15}: x_{-j}=0,\ y_{-j}=0,\ z_{-j}=0\Big\}. \quad (2.34)$$

Proof. At the beginning of Section 2, we have obtained that the set

$$\bigcup_{j=1}^{5}\left\{(x_{-5},x_{-4},\dots,x_{-1},y_{-5},y_{-4},\dots,y_{-1},z_{-5},z_{-4},\dots,z_{-1})\in\mathbb{R}^{15}: x_{-j}=0,\ y_{-j}=0,\ z_{-j}=0\right\}$$

belongs to the forbidden set of the initial values for system (1.5). Now, we suppose that $x_n\neq 0$, $y_n\neq 0$ and $z_n\neq 0$. Note that the system (1.5) is not defined, when the conditions $a_n+b_nz_{n-2}x_{n-3}y_{n-4}z_{n-5}=0$, $\alpha_n+\beta_nx_{n-2}y_{n-3}z_{n-4}x_{n-5}=0$ or $A_n+B_ny_{n-2}z_{n-3}x_{n-4}y_{n-5}=0$, that is, $z_{n-2}x_{n-3}y_{n-4}z_{n-5}=-\frac{a_n}{b_n}$, $x_{n-2}y_{n-3}z_{n-4}x_{n-5}=-\frac{\alpha_n}{\beta_n}$ or $y_{n-2}z_{n-3}x_{n-4}y_{n-5}=-\frac{A_n}{B_n}$, for some $n\in\mathbb{N}_0$, are satisfied(Here we consider that $b_n\neq 0$, $\beta_n\neq 0$ and $B_n\neq 0$ for every $n\in\mathbb{N}_0$). From this and equations in (2.2), we get

$$u_{6m+i}=-\frac{\beta_{6m+i+2}}{\alpha_{6m+i+2}},\ v_{6m+i}=-\frac{B_{6m+i+2}}{A_{3m+i+2}},\ w_{6m+i}=-\frac{b_{6m+i+2}}{a_{6m+i+2}},\tag{2.35}$$

for some $m\in\mathbb{N}_0$ and $i=\overline{-2,3}$. Hence, we can determine the forbidden set of the initial values for system (1.5) by using the substitution $u_n=\frac{1}{x_ny_{n-1}z_{n-2}x_{n-3}}$, $v_n=\frac{1}{y_nz_{n-1}x_{n-2}y_{n-3}}$, $w_n=\frac{1}{z_nx_{n-1}y_{n-2}z_{n-3}}$. Now, we consider the functions

$$\begin{aligned} f_{6m+i+2}(t) &:= a_{6m+i+2}t+b_{6m+i+2},\\ g_{6m+i+2}(t) &:= \alpha_{6m+i+2}t+\beta_{6m+i+2},\\ h_{6m+i+2}(t) &:= A_{6m+i+2}t+B_{6m+i+2}, \end{aligned}\tag{2.36}$$

for $m\in\mathbb{N}_0$, $i=\overline{-2,3}$, which correspond to the system (2.3). From (2.35) and (2.36), we can write

$$u_{6m+i}=f_{6m+i}\circ h_{6m+i-2}\circ g_{6m+i-4}\cdots\circ f_i\circ h_{i-2}\circ g_{i-4}(u_{i-6}),\tag{2.37}$$

$$v_{6m+i}=g_{6m+i}\circ f_{6m+i-2}\circ h_{6m+i-4}\cdots\circ g_i\circ f_{i-2}\circ h_{i-4}(v_{i-6}),\tag{2.38}$$

$$w_{6m+i}=h_{6m+i}\circ g_{6m+i-2}\circ f_{6m+i-4}\cdots\circ h_i\circ g_{i-2}\circ f_{i-4}(w_{i-6}),\tag{2.39}$$

where $m\in\mathbb{N}_0$, and $i=\overline{4,9}$. By using (2.35) and implicit forms (2.37)-(2.39) and considering $f^{-1}_{6m+i+2}(0)=-\frac{b_{6m+i+2}}{a_{6m+i+2}}$, $g^{-1}_{6m+i+2}(0)=-\frac{\beta_{6m+i+2}}{\alpha_{6m+i+2}}$, $h^{-1}_{6m+i+2}(0)=-\frac{B_{6m+i+2}}{A_{6m+i+2}}$, for $m\geq -1$ and $i=\overline{4,9}$, we get

$$u_{i-6}=g^{-1}_{i-4}\circ h^{-1}_{i-2}\circ f^{-1}_{i}\circ\cdots\circ g^{-1}_{6m+i-4}\circ h^{-1}_{6m+i-2}\circ f^{-1}_{6m+i}(0),\tag{2.40}$$

$$v_{i-6}=h^{-1}_{i-4}\circ f^{-1}_{i-2}\circ g^{-1}_{i}\circ\cdots\circ h^{-1}_{6m+i-4}\circ f^{-1}_{6m+i-2}\circ g^{-1}_{6m+i}(0),\tag{2.41}$$

$$w_{i-6}=f^{-1}_{i-4}\circ g^{-1}_{i-2}\circ h^{-1}_{i}\circ\cdots\circ f^{-1}_{6m+i-4}\circ g^{-1}_{6m+i-2}\circ h^{-1}_{6m+i}(0),\tag{2.42}$$

where $f^{-1}_{6m+i+2}(t)=\frac{t-b_{6m+i+2}}{a_{6m+i+2}}$, $g^{-1}_{6m+i+2}(t)=\frac{t-\beta_{6m+i+2}}{\alpha_{6m+i+2}}$, $h^{-1}_{6m+i+2}(t)=\frac{t-B_{6m+i+2}}{A_{6m+i+2}}$, $m\geq -1$, $i=\overline{4,9}$. From (2.40)-(2.42), we get

$$u_{i-6} = -\sum_{j=0}^{m}\left(\frac{b_{6j+i}+a_{6j+i}B_{6j+i-2}+a_{6j+i}A_{6j+i-2}\beta_{6j+i-4}}{a_{6j+i}A_{6j+i-2}\alpha_{6j+i-4}}\right)\prod_{l=0}^{j-1}\frac{1}{a_{6l+i}A_{6l+i-2}\alpha_{6l+i-4}},$$

$$v_{i-6} = -\sum_{j=0}^{m}\left(\frac{\beta_{6j+i}+\alpha_{6j+i}b_{6j+i-2}+\alpha_{6j+i}a_{6j+i-2}B_{6j+i-4}}{\alpha_{6j+i}a_{6j+i-2}A_{6j+i-4}}\right)\prod_{l=0}^{j-1}\frac{1}{\alpha_{6l+i}a_{6l+i-2}A_{6l+i-4}},$$

$$w_{i-6} = -\sum_{j=0}^{m}\left(\frac{B_{6j+i}+A_{6j+i}\beta_{6j+i-2}+A_{6j+i}\alpha_{6j+i-2}b_{6j+i-4}}{A_{6j+i}\alpha_{6j+i-2}a_{6j+i-4}}\right)\prod_{l=0}^{j-1}\frac{1}{A_{6l+i}\alpha_{6l+i-2}a_{6l+i-4}},$$

for some $m\in\mathbb{N}_0$ and $i=\overline{4,9}$. This means that if one of the conditions in (2.40)-(2.42) holds, then m-th iteration or $(m+1)$-th iteration in system (1.5) can not be calculated. □

3. Solutions of System (1.5) with Constant Coefficients

In this section, we give the forms of solutions of system (1.5) when all the coefficients are constant. We assume that $a_n = a$, $b_n = b$, $\alpha_n = \alpha$, $\beta_n = \beta$, $A_n = A$ and $B_n = B$ for every $n \in \mathbb{N}_0$. In this case, system (1.5) is written as in the form

$$\begin{aligned} x_n &= \frac{y_{n-4}z_{n-5}}{y_{n-1}\left(a + bz_{n-2}x_{n-3}y_{n-4}z_{n-5}\right)}, \\ y_n &= \frac{z_{n-4}x_{n-5}}{z_{n-1}\left(\alpha + \beta x_{n-2}y_{n-3}z_{n-4}x_{n-5}\right)}, \\ z_n &= \frac{x_{n-4}y_{n-5}}{x_{n-1}\left(A + By_{n-2}z_{n-3}x_{n-4}y_{n-5}\right)}, \ n \in \mathbb{N}_0. \end{aligned} \tag{3.1}$$

In (2.28)-(2.30), if we replace the formulas given in (2.22)-(2.24), then the solution of system (3.1) is given by

$$\begin{aligned} x_{12m+6l+r} = x_{6l+r-12} \prod_{s=0}^{m} & \frac{(a\alpha A)^{2s+l+1+\lfloor \frac{r-5}{6} \rfloor} v_{r-7-6\lfloor \frac{r-5}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+1+\lfloor \frac{r-5}{6} \rfloor}\right)(\alpha a B + \alpha b + \beta)}{(a\alpha A)^{2s+l+1+\lfloor \frac{r-4}{6} \rfloor} u_{r-6-6\lfloor \frac{r-4}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+1+\lfloor \frac{r-4}{6} \rfloor}\right)(aA\beta + aB + b)} \\ & \times \frac{(a\alpha A)^{2s+l+\lfloor \frac{r-3}{6} \rfloor} w_{r-5-6\lfloor \frac{r-3}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor \frac{r-3}{6} \rfloor}\right)(A\alpha b + A\beta + B)}{(a\alpha A)^{2s+l+\lfloor \frac{r-2}{6} \rfloor} v_{r-4-6\lfloor \frac{r-2}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor \frac{r-2}{6} \rfloor}\right)(\alpha a B + \alpha b + \beta)} \\ & \times \frac{(a\alpha A)^{2s+l+\lfloor \frac{r-7}{6} \rfloor} u_{r-9-6\lfloor \frac{r-7}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor \frac{r-7}{6} \rfloor}\right)(aA\beta + aB + b)}{(a\alpha A)^{2s+l+\lfloor \frac{r-6}{6} \rfloor} w_{r-8-6\lfloor \frac{r-6}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor \frac{r-6}{6} \rfloor}\right)(A\alpha b + A\beta + B)}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} y_{12m+6l+r} = y_{6l+r-12} \prod_{s=0}^{m} & \frac{(a\alpha A)^{2s+l+1+\lfloor \frac{r-5}{6} \rfloor} w_{r-7-6\lfloor \frac{r-5}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+1+\lfloor \frac{r-5}{6} \rfloor}\right)(A\alpha b + A\beta + B)}{(a\alpha A)^{2s+l+1+\lfloor \frac{r-4}{6} \rfloor} v_{r-6-6\lfloor \frac{r-4}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+1+\lfloor \frac{r-4}{6} \rfloor}\right)(\alpha a B + \alpha b + \beta)} \\ & \times \frac{(a\alpha A)^{2s+l+\lfloor \frac{r-3}{6} \rfloor} u_{r-5-6\lfloor \frac{r-3}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor \frac{r-3}{6} \rfloor}\right)(aA\beta + aB + b)}{(a\alpha A)^{2s+l+\lfloor \frac{r-2}{6} \rfloor} w_{r-4-6\lfloor \frac{r-2}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor \frac{r-2}{6} \rfloor}\right)(A\alpha b + A\beta + B)} \\ & \times \frac{(a\alpha A)^{2s+l+\lfloor \frac{r-7}{6} \rfloor} v_{r-9-6\lfloor \frac{r-7}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor \frac{r-7}{6} \rfloor}\right)(\alpha a B + \alpha b + \beta)}{(a\alpha A)^{2s+l+\lfloor \frac{r-6}{6} \rfloor} u_{r-8-6\lfloor \frac{r-6}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor \frac{r-6}{6} \rfloor}\right)(aA\beta + aB + b)}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} z_{12m+6l+r} = z_{6l+r-12} \prod_{s=0}^{m} & \frac{(a\alpha A)^{2s+l+1+\lfloor \frac{r-5}{6} \rfloor} u_{r-7-6\lfloor \frac{r-5}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+1+\lfloor \frac{r-5}{6} \rfloor}\right)(aA\beta + aB + b)}{(a\alpha A)^{2s+l+1+\lfloor \frac{r-4}{6} \rfloor} w_{r-6-6\lfloor \frac{r-4}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+1+\lfloor \frac{r-4}{6} \rfloor}\right)(A\alpha b + A\beta + B)} \\ & \times \frac{(a\alpha A)^{2s+l+\lfloor \frac{r-3}{6} \rfloor} v_{r-5-6\lfloor \frac{r-3}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor \frac{r-3}{6} \rfloor}\right)(\alpha a B + \alpha b + \beta)}{(a\alpha A)^{2s+l+\lfloor \frac{r-2}{6} \rfloor} u_{r-4-6\lfloor \frac{r-2}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor \frac{r-2}{6} \rfloor}\right)(aA\beta + aB + b)} \\ & \times \frac{(a\alpha A)^{2s+l+\lfloor \frac{r-7}{6} \rfloor} w_{r-9-6\lfloor \frac{r-7}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor \frac{r-7}{6} \rfloor}\right)(A\alpha b + A\beta + B)}{(a\alpha A)^{2s+l+\lfloor \frac{r-6}{6} \rfloor} v_{r-8-6\lfloor \frac{r-6}{6} \rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor \frac{r-6}{6} \rfloor}\right)(\alpha a B + \alpha b + \beta)}, \end{aligned} \tag{3.4}$$

if $a\alpha A \neq 1$,

$$x_{12m+6l+r}=x_{6l+r-12}\prod_{s=0}^{m}\frac{v_{r-7-6\lfloor\frac{r-5}{6}\rfloor}+(\alpha aB+\alpha b+\beta)\left(2s+l+1+\lfloor\frac{r-5}{6}\rfloor\right)}{u_{r-6-6\lfloor\frac{r-4}{6}\rfloor}+(aA\beta+aB+b)\left(2s+l+1+\lfloor\frac{r-4}{6}\rfloor\right)}$$
$$\times\frac{w_{r-5-6\lfloor\frac{r-3}{6}\rfloor}+(A\alpha b+A\beta+B)\left(2s+l+\lfloor\frac{r-3}{6}\rfloor\right)}{v_{r-4-6\lfloor\frac{r-2}{6}\rfloor}+(\alpha aB+\alpha b+\beta)\left(2s+l+\lfloor\frac{r-2}{6}\rfloor\right)}$$
$$\times\frac{u_{r-9-6\lfloor\frac{r-7}{6}\rfloor}+(aA\beta+aB+b)\left(2s+l+\lfloor\frac{r-7}{6}\rfloor\right)}{w_{r-8-6\lfloor\frac{r-6}{6}\rfloor}+(A\alpha b+A\beta+B)\left(2s+l+\lfloor\frac{r-6}{6}\rfloor\right)},\tag{3.5}$$

$$y_{12m+6l+r}=y_{6l+r-12}\prod_{s=0}^{m}\frac{w_{r-7-6\lfloor\frac{r-5}{6}\rfloor}+(A\alpha b+A\beta+B)\left(2s+l+1+\lfloor\frac{r-5}{6}\rfloor\right)}{v_{r-6-6\lfloor\frac{r-4}{6}\rfloor}+(\alpha aB+\alpha b+\beta)\left(2s+l+1+\lfloor\frac{r-4}{6}\rfloor\right)}$$
$$\times\frac{u_{r-5-6\lfloor\frac{r-3}{6}\rfloor}+(aA\beta+aB+b)\left(2s+l+\lfloor\frac{r-3}{6}\rfloor\right)}{w_{r-4-6\lfloor\frac{r-2}{6}\rfloor}+(A\alpha b+A\beta+B)\left(2s+l+\lfloor\frac{r-2}{6}\rfloor\right)}$$
$$\times\frac{v_{r-9-6\lfloor\frac{r-7}{6}\rfloor}+(\alpha aB+\alpha b+\beta)\left(2s+l+\lfloor\frac{r-7}{6}\rfloor\right)}{u_{r-8-6\lfloor\frac{r-6}{6}\rfloor}+(aA\beta+aB+b)\left(2s+l+\lfloor\frac{r-6}{6}\rfloor\right)},\tag{3.6}$$

$$z_{12m+6l+r}=z_{6l+r-12}\prod_{s=0}^{m}\frac{u_{r-7-6\lfloor\frac{r-5}{6}\rfloor}+(aA\beta+aB+b)\left(2s+l+1+\lfloor\frac{r-5}{6}\rfloor\right)}{w_{r-6-6\lfloor\frac{r-4}{6}\rfloor}+(A\alpha b+A\beta+B)\left(2s+l+1+\lfloor\frac{r-4}{6}\rfloor\right)}$$
$$\times\frac{v_{r-5-6\lfloor\frac{r-3}{6}\rfloor}+(\alpha aB+\alpha b+\beta)\left(2s+l+\lfloor\frac{r-3}{6}\rfloor\right)}{u_{r-4-6\lfloor\frac{r-2}{6}\rfloor}+(aA\beta+aB+b)\left(2s+l+\lfloor\frac{r-2}{6}\rfloor\right)}$$
$$\times\frac{w_{r-9-6\lfloor\frac{r-7}{6}\rfloor}+(A\alpha b+A\beta+B)\left(2s+l+\lfloor\frac{r-7}{6}\rfloor\right)}{v_{r-8-6\lfloor\frac{r-6}{6}\rfloor}+(\alpha aB+\alpha b+\beta)\left(2s+l+\lfloor\frac{r-6}{6}\rfloor\right)},\tag{3.7}$$

if $a\alpha A=1$, for every $m\in\mathbb{N}_0$, $l\in\{1,2\}$ and $r=\overline{1,6}$.

4. Conclusion

In this paper, we have studied the following system of difference equations

$$x_n=\frac{y_{n-4}z_{n-5}}{y_{n-1}\left(a_n+b_nz_{n-2}x_{n-3}y_{n-4}z_{n-5}\right)},$$
$$y_n=\frac{z_{n-4}x_{n-5}}{z_{n-1}\left(\alpha_n+\beta_nx_{n-2}y_{n-3}z_{n-4}x_{n-5}\right)},$$
$$z_n=\frac{x_{n-4}y_{n-5}}{x_{n-1}\left(A_n+B_ny_{n-2}z_{n-3}x_{n-4}y_{n-5}\right)},\ n\in\mathbb{N}_0,$$

where the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$, $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$ and the initial values x_{-j},y_{-j}, $j=\overline{1,5}$, are real numbers.

Firstly, we have solved above system in closed form by using suitable transformation. In addition, we also characterize the forbidden set of solutions of the system of difference equations. Finally, we have obtained solutions of aforementioned system with constant coefficients.

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