

Matching with Restricted Preferences

Mustafa Kadir DOĞAN
doganmk@politics.ankara.edu.tr

Kısıtlı Tercihler ile Eşleşme

Abstract

This paper examines the effects of the introduction of restrictions on the statement of preferences in a two-sided matching model with incomplete information. The model is similar to the process used for college admissions in Turkey. Colleges have unanimous preferences – students with higher ranking in the national examinations are always preferred. We show that the introduction of the restrictions on statement of students' preferences can result in unstable matching between colleges and students.

Key Words : Stable Matching, College Admission System.

JEL Classification Codes : C78, D71.

Özet

Bu makale eksik bilgili çift taraflı eşleşme modelinde tercih bildirmede uygulanan kısıtlamaların etkisini incelemektedir. Bu model Türkiye’de üniversitelere öğrenci yerleştirme sistemine benzerdir. Üniversiteler aynı tercihlere sahiptir – daha yüksek başarı sıralamasına sahip öğrenciler tercih edilmektedir. Tercih bildirmede uygulanan kısıtlamaların üniversiteler ile öğrenciler arasında durağan olmayan eşleşmelere yol açabileceği gösterilmiştir.

Anahtar Sözcükler : Durağan Eşleşme, Üniversitelere Öğrenci Yerleştirme Sistemi.

Acknowledgement

I would like to thank Christophe Chamley, İrfan Cıvcır, Hsueh–Ling Huynh, Zvika Neeman, seminar participants at Boston University and the Seventh International Meeting of the Society for Social Choice at Osaka, Japan.

Beyan

Christophe Chamley, İrfan Cıvcır, Hsueh–Ling Huynh, Zvika Neman ile Boston Üniversitesi'ndeki seminere katılanlara ve Sosyal Tercih Topluluğu'nun Japonya'nın Osaka Kentinde düzenlediği Yedinci Uluslararası Toplantının katılımcılarına teşekkürlerimi sunuyorum.

1. Introduction

In this paper, we examine the effects of the introduction of restrictions on the statement of preferences in a two-sided matching model with incomplete information.¹ The model is similar to the process adopted for college admissions in Turkey.

In Turkey, the process of college admissions is centralized. There is a student placement office that assigns the students to the departments of the colleges. Every year this office administers an examination to every student who wishes to enroll in a college. Each student receives a rank according to his performance on this examination. The higher the performance of a student in the examination, the higher is his rank. After receiving the ranks, each student completes a preference form and submits it to the student placement office. The departments have unanimous preferences – students with higher ranking are always preferred. Finally, the students are assigned to the departments by a pre-announced mechanism by using the students' ranks, preference forms and the preferences of the departments.

Students may have different preferences over the departments. Each student only knows his preferences and has a prior belief about others' preferences. In this process, we are focusing on the point that the student placement office does not allow the students to order his preferences freely. There is a fixed number and the students can submit ordering, at most, this number of departments in the preference form. Since a student cannot declare his preferences for some departments that he may want to enroll, he must choose the departments in the preference form strategically. In Turkey, there are more than 4,000 departments accepting the students but the students are restricted to submit preferences over at most 24 departments. That means the students are exposed to a significant restriction when they are stating their preferences for departments.

If there is no restriction on the statement of preferences, this college admission problem has a unique *stable* matching.² In this matching, the higher-ranked student is assigned to his top choice department, the second highest student is assigned to his top choice department from the available ones, and so on. We show that the restrictions on stating preferences, together with incomplete information of the students about the others' preferences, can result in unstable matching between the departments and the students.

Section two describes the model, section three presents the results, and section four concludes.

¹ An extensive literature survey on two-sided matching models can be found in Roth and Sotomayor (1990,1992).

² The allocation mechanism that is used in Turkish College admission process is analyzed by Ballinski and Sonmez (1999). However, they ignore the restrictions on the statement of preferences.

2. Model

There are n students and m departments. $S = \{s_1, \dots, s_n\}$ denotes the set of students, $D = \{d_1, \dots, d_m\}$ denotes the set of departments, and $C = \{c_{d_1}, \dots, c_{d_m}\}$ denotes the set of capacities of the departments where c_{d_j} is the capacity of d_j .

The students take a test and according to their performance on the test, each student receives a rank. Let s_i denote the student with rank i and $P(d_j)$ denote the preference of d_j over students. The departments have unanimous preferences over the students – they prefer a higher-ranked student to a lower-ranked one. Formally, the preference of d_j is $P(d_j) = \langle s_1, s_2, \dots, s_n \rangle$ for $j = 1, \dots, m$.

Let $u(s_i)$ be the utility of s_i over the departments. We assume that for any s_i , $u(s_i)$ is drawn from a distribution as follows: $u(s_i) = u_k$ with probability $f(u_k)$ for $k = 1, \dots, K$ where $u_k = (u_{k_1}, \dots, u_{k_m})$. A student with utility vector u_k gets u_{k_j} utility if he is placed to d_j . On the other hand, the departments have unanimous preferences over the students. They prefer a higher-ranked student to a lower ranked one.

The nature draws the utilities of the students over departments. Any student observes his utility over departments and his rank, but cannot observe the utilities of the other students. The preferences of the departments and the probability distribution of the students' utilities are common knowledge. Each student submits a preference form in which he can state the ordering of, at most, A departments. Let o_i denote the ordering of s_i (in his preference form) and o_{-i} denote the orderings of other students. $o_i \in O$, where

O is the set of possible orderings.³

There is a matching mechanism G that takes the orderings of the students and creates a matching μ between the students and departments. At matching μ , each student is either assigned to a department or assigned to himself. If a student is assigned to himself, then he is not accepted by any of the departments. $G(o_1, \dots, o_n)$ creates the matching μ by using the following algorithm. Starting with the highest-ranked student, in any iteration one student is assigned to the first available department in his stated preference form. If all the departments in the student's preference form are unavailable, (i.e., they were already assigned to other students), then the student will be assigned to himself. The algorithm continues with the next highest ranked student and concludes after the iteration for the lowest-ranked one.

Let $X_i : \{u(s_i) \rightarrow O\}$ be the strategy set of S_i that has $\|O\|^K = \left[\frac{m!}{(m-A)!} \right]^K$ strategies. Let F be the cumulative distribution function of the density f . We define the matching game with incomplete information as follows.

$$\Gamma = \left(S \cup D, C, \{X_i\}_{i \in \{1, \dots, n\}}, \{u(s_i)\}_{i \in \{1, \dots, n\}}, \{P(d_j)\}_{j \in \{1, \dots, m\}}, F, G \right)$$

Example: 1 Assume there are five students ($n = 5$), three departments ($m = 3$), and the capacity of each department is one. The number of allowed preferences is two ($A = 2$) and a student's utility over departments has following three possibilities ($K = 3$).

³ Note that there are $\frac{m!}{(m-A)!}$ possible orderings.

$$u(s_i) = \begin{cases} u_1 = (3, 2, 1), f(u_1) = 0,6 \\ u_2 = (2, 3, 1), f(u_2) = 0,3 \\ u_3 = (2, 1, 3), f(u_3) = 0,1 \end{cases} \text{ for } \forall s_i$$

For example, if a student's utility is u_1 , he gets three utility if he is assigned to department 1, two utility if he is assigned to department 2, and one utility if he is assigned to department 3. If a student is not assigned to any of the departments, he gets zero utility. Preference of a department over students is $P(d_j) = \langle s_1, s_2, s_3, s_4, s_5 \rangle$ for $\forall d_j$. There are six possible orderings: $O = \{\{1, 2\}, \{1, 3\}, \{2, 1\}, \{2, 3\}, \{3, 1\}, \{3, 2\}\}$

Let $\mu(s_i)$ denote the department to which s_i is assigned under matching μ .

Definition 1 Matching μ is *stable* if there is no student–department pair $\{s_i, d_j\}_{s_i \in S, d_j \in D}$, such that s_i prefers the d_j to $\mu(s_i)$, and d_j prefers the s_i to at least one of the students who is assigned to it under the matching μ . If such a pair exists, then we say that it blocks the matching μ .

3. Results

Gale and Shapley (1962) showed that there always exists at least one stable matching in a two–sided matching problem by introducing the deferred acceptance procedure algorithm.

Proposition 1 There exists a unique stable matching in the college admission problem described above. In this matching, the highest–ranked student is assigned to his top choice, the second highest one is assigned to his top choice among the available departments, and so on.

Proof. The existence of a stable matching is due to Gale and Shapley (1962). Let μ^* denote a stable matching and assume that there is another stable matching μ .

Then, $\mu(s_1) = \mu^*(s_1)$. If this is not true, s_1 and $\mu^*(s_1)$ blocks μ . Also, $\mu(s_2) = \mu^*(s_2)$. Otherwise, s_2 and $\mu^*(s_2)$ blocks μ . By continuing so, $\mu(s_n) = \mu^*(s_n)$. Therefore, $\mu = \mu^*$. ■

Hereafter μ^* denotes the unique stable matching.

Proposition 2 If there is complete information ($u(s_i)$ is common knowledge among students), then G generates the matching μ^* even if $A = 1$.

Proof. If $u(s_i)$ is common knowledge, then any student knows the others' preferences. Assume $A = 1$. The first-ranked student will choose his top choice and be assigned to that department ($\mu^*(s_1)$). The student with rank two infers the first-ranked student's placement, and his best strategy is to choose his top choice from remaining departments, which is $\mu^*(s_2)$. If he chooses another strategy, either he will not be assigned to any of the departments or he prefers $\mu^*(s_2)$ to the department that he will be assigned under that strategy. In general, the student with rank k , ($k \leq n$), knows the preferences and best strategies of other students, so he infers the first $(k-1)$ -ranked students' placements. Consequently, his best strategy is to choose his top choice from available departments, which is $\mu^*(s_k)$. If he chooses another strategy, either he will choose an unavailable department and be assigned to himself, or he will be assigned to a department that he prefers less than $\mu^*(s_k)$. ■

Proposition 3 If there is no restriction on stating preferences (i.e., if $A = m$), then G generates the unique stable matching μ^* .

Proof. If a student states his true preferences, he will be assigned to his most preferred department among the available ones when the algorithm turn comes to him. Consequently, G generates the matching μ^* . Hence, if we show that all students state their true preferences when there is no restriction on stating preferences, then we prove the proposition.

Assume o_i be the true preference ordering of s_i and $G(o_i, o_{-i}) = \mu$ for $\exists o_{-i}$. Suppose that $G(o'_i, o_{-i}) = \mu'$ for $\exists o'_i$ and $\mu \neq \mu'$. We must show that s_i prefers $\mu(s_i)$ to $\mu'(s_i)$. Suppose that s_i prefers $\mu'(s_i)$ to $\mu(s_i)$. Since $G(o'_i, o_{-i}) = \mu'$, $\mu'(s_i)$ is available when the algorithm turn comes to s_i . Hence, under his true preferences, s_i can not be assigned to $\mu(s_i)$, which contradicts $G(o_i, o_{-i}) = \mu$. ■

Definition 2 Equilibrium for the game Γ with incomplete information is $\left\{ \{x_i\}_{i \in \{1, \dots, n\}}, \{\beta_i\}_{i \in \{1, \dots, n\}}, \mu^e \right\}$, where $\{x_i\}$ is the set of the strategies of the students, $\{\beta_i\}$ is the set of beliefs of the students about higher-ranked students' ordering choices, and the matching outcome μ^e which is generated by algorithm G under strategies $\{x_i\}_{i \in \{1, \dots, n\}}$ such that :

1) $x_i \in X_i$ maximizes the expected utility of s_i under the belief β_i , for $\forall s_i \in S$, and

2) beliefs of the students must be derived from strategies according to the Bayes' rule.

Proposition 4 If there is a restriction on the statement of preferences (i.e. if $A < m$), the equilibrium outcome of the game Γ with incomplete information can be unstable.

Proof. We present here an example in which $\mu^e \neq \mu^*$ and another one where $\mu^e = \mu^*$.

In the example 1, equilibrium strategies, beliefs and matching outcome are the following. (We assumed that students select the low-indexed ordering when they are indifferent among the orderings.)

Equilibrium strategies

$$\begin{aligned} x_1 &= \{1, 2\} \text{ if } k = 1, \{2, 1\} \text{ if } k = 2, \{3, 1\} \text{ if } k = 3 \\ x_2 &= \{1, 2\} \text{ if } k = 1, \{2, 1\} \text{ if } k = 2, \{3, 1\} \text{ if } k = 3 \\ x_3 &= \{2, 3\} \text{ if } k = 1, \{2, 3\} \text{ if } k = 2, \{1, 3\} \text{ if } k = 3 \\ x_4 &= \{1, 2\} \text{ if } k = 1, \{2, 1\} \text{ if } k = 2, \{1, 2\} \text{ if } k = 3 \\ x_5 &= \text{any strategy} \end{aligned}$$

Equilibrium beliefs

$$\beta_2 = [0,6 \ 0 \ 0,3 \ 0 \ 0,1 \ 0]$$

$$\beta_3 = \begin{bmatrix} 0,6 & 0 & 0,3 & 0 & 0,1 & 0 \\ 0,6 & 0 & 0,3 & 0 & 0,1 & 0 \end{bmatrix}$$

$$\beta_4 = \begin{bmatrix} 0,6 & 0 & 0,3 & 0 & 0,1 & 0 \\ 0,6 & 0 & 0,3 & 0 & 0,1 & 0 \\ 0 & 0,1 & 0 & 0,9 & 0 & 0 \end{bmatrix}$$

$$\beta_5 = \begin{bmatrix} 0,6 & 0 & 0,3 & 0 & 0,1 & 0 \\ 0,6 & 0 & 0,3 & 0 & 0,1 & 0 \\ 0 & 0,1 & 0 & 0,9 & 0 & 0 \\ 0,7 & 0 & 0,3 & 0 & 0 & 0 \end{bmatrix}$$

The element in row a and column b of the matrix β_i reveals the belief of s_i about the probability that student a chooses the ordering o_b where $o_1 = \{1, 2\}, o_2 = \{1, 3\}, o_3 = \{2, 1\}, o_4 = \{2, 3\}, o_5 = \{3, 1\}, o_6 = \{3, 2\}$. For example, student 4 believes that student 3 chooses the ordering $\{1, 3\}$ with probability 0,1, and the ordering $\{2, 3\}$ with probability 0,9 and the others with probability zero.

Equilibrium outcome

Suppose that nature draws the utilities of the students as follows:

$$u(s_1) = u_2 = (2, 3, 1)$$

$$u(s_2) = u_3 = (2, 1, 3)$$

$$u(s_3) = u_1 = (3, 2, 1)$$

$$u(s_4) = u_1 = (3, 2, 1)$$

$$u(s_5) = u_2 = (2, 3, 1)$$

The unique stable matching μ^* is the following.

$$\mu^* = \left\{ \left(\begin{matrix} s_1 \\ d_2 \end{matrix} \right), \left(\begin{matrix} s_2 \\ d_3 \end{matrix} \right), \left(\begin{matrix} s_3 \\ d_1 \end{matrix} \right), \left(\begin{matrix} s_4 \\ \emptyset \end{matrix} \right), \left(\begin{matrix} s_5 \\ \emptyset \end{matrix} \right) \right\}$$

The equilibrium outcome μ^e is the following.

$$\mu^e = \left\{ \left(\begin{matrix} s_1 \\ d_2 \end{matrix} \right), \left(\begin{matrix} s_2 \\ d_3 \end{matrix} \right), \left(\begin{matrix} s_3 \\ \emptyset \end{matrix} \right), \left(\begin{matrix} s_4 \\ d_1 \end{matrix} \right), \left(\begin{matrix} s_5 \\ \emptyset \end{matrix} \right) \right\}$$

In the unique stable matching μ^* , s_3 is assigned to d_1 and s_4 is assigned to himself. On the other hand, in the equilibrium outcome μ^e , s_4 is assigned to d_1 and

s_3 is assigned to himself. s_3 prefers d_1 instead of being assigned to himself and d_1 prefers s_3 to s_4 . Therefore, $\{s_3, d_1\}$ blocks the equilibrium matching outcome. The equilibrium outcome μ^e is not a stable matching.

Now, suppose that nature draws the utilities of the students as follows.

$$u(s_1) = u_1 = (3, 2, 1)$$

$$u(s_2) = u_2 = (2, 3, 1)$$

$$u(s_3) = u_1 = (3, 2, 1)$$

$$u(s_4) = u_3 = (2, 1, 3)$$

$$u(s_5) = u_2 = (2, 3, 1)$$

Then, the unique stable matching is equal to the equilibrium outcome.

$$\mu^* = \mu^e = \left\{ \begin{pmatrix} s_1 \\ d_2 \end{pmatrix}, \begin{pmatrix} s_2 \\ d_3 \end{pmatrix}, \begin{pmatrix} s_3 \\ d_1 \end{pmatrix}, \begin{pmatrix} s_4 \\ \emptyset \end{pmatrix}, \begin{pmatrix} s_5 \\ \emptyset \end{pmatrix} \right\} \quad \blacksquare$$

4. Conclusion

This paper analyzed the effects of the introduction of restrictions on the statement of preferences in a two-sided matching model with incomplete information. The model is similar to the process used for college admissions in Turkey. In Turkey, the college admission process is centralized and a student placement office assigns the students to the departments of the colleges according to the preferences of the departments and the preference forms submitted by the students. Departments have unanimous preferences – students with higher ranking in the national examination are always preferred. Students are exposed to a restriction on the statement of the preferences; each student can state a preference ordering over a limited number of departments in the preference form. We demonstrated that the restriction on statement of the preferences can result in unstable matching between the departments and the students.

References

- Balinski, M. and T. Sonmez (1999), “A Tale of Two Mechanisms: Student Placement”, *Journal of Economic Theory*, 84: 73–94.
- Gale, D. and L. Shapley (1962), “College Admissions and the Stability of Marriage,” *American Mathematical Monthly*, 69: 9–15.
- Roth, A. E. and M. Sotomayar (1990), *Two–Sided Matching: A Study in Game–Theoretic Modeling and Analysis*, Cambridge, U.K.: Cambridge University Press.
- Roth, A.E. and M. Sotomayor (1992), “Two Sided Matching,” in *Handbook of Game Theory with Economic Applications*, (eds.) Aumann R. and S. Hart, North Holland: Elsevier, 485–541.