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Cofinitely (Weak) G-Supplemented Lattices

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ABSTRACT

In this work, cofinitely (weak) g-supplemented lattices are defined and some properties of these lattices are investigated. It is shown that quotient sublattices of cofinitely (weak) g-supplemented lattices are cofinitely (weak) g-supplemented. If $\{a_i/0\}_{i\in I}$ is a collection of cofinitely (weak) g-supplemented sublattices of L and $1 = \bigvee_{i\in I} a_i$, then L is also cofinitely (weak) g-supplemented. It is proved that without loss of generality weak g-supplements of cofinite elements in compactly generated lattices are compact. An example showing that this is not true for lattices which are not cofinitely generated is given. A condition is given under which a compactly generated cofinitely weak g-supplemented lattice is cofinitely g-supplemented.

Keywords: *Modular lattices, G-small elements, G-supplemented lattices*

Eşsonlu (Zayıf) G-Tümlenmiş Kafesler

ÖZ

Bu çalışmada eşsonlu (zayıf) g-tümlenmiş kafesler tanımlandı ve bu kafeslerin bazı özellikleri incelendi. Eşsonlu (zayıf) g-tümlenmiş kafeslerin bölüm alt kafeslerinin de eşsonlu (zayıf) g-tümlenmiş olduğu gösterildi. Herhangi sayıda eşsonlu (zayıf) g-tümlenmiş kafeslerin supremumu da eşsonlu (zayıf) g-tümlenmiştir. Kompakt üretilmiş kafeslerde eşsonlu elemanların zayıf g-tümleyenlerinin kompakt elemanlar olarak kabul edilebileceği kanıtlandı. Bu özelliğin kompakt üretilmiş olmayan kafesler için doğru olmadığına bir örnek verildi. Eşsonlu zayıf g-tümlenmiş kompakt üretilmiş kafeslerin eşsonlu g-tümlenmiş olması için gerekli bir koşul verildi.

Anahtar Kelimeler: Modüler kafesler, G-küçük elemanlar, G-tümlenmiş kafesler

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I. INTRODUCTION

Throughout this paper, L will be an arbitrary complete modular lattice with smallest element 0 and greatest element 1. A *quotient sublattice* b/a is the set of elements $x \in L$ such that $a \leq x \leq b$ (see [3]). If $a \vee b \neq 1$ for every $b \neq 1$ in a lattice L, then a is called a *small* element of L. It is denoted by $a \ll L$. If $a \leq b$ and $a \neq b$, then it is written as $a < b$. Small elements have the following properties:

Lemma 1.1. ([3, Lemma 7.2, Lemma 7.3 and Lemma 12.4]) *Let L be a lattice and $a, b \in L$ such that $a < b$.*

1. *If $a \ll b/0$, then $a \vee c \ll (b \vee c)/c$ for every $c \in L$.*
2. *$b \ll L$ if and only if $a \ll L$ and $b \ll 1/a$.*
3. *Let $c' \ll c/0$ and $d' \ll d/0$. Then $c' \vee d' \ll (c \vee d)/0$.*

An element a of L is said to be *essential* if $a \wedge b \neq 0$ whenever $b \neq 0$ in L. It is denoted by $a \trianglelefteq L$ (see [4]). An element $a \in L$ is called a *generalized small* (or shortly *g-small*) element if $b = 1$ for every $b \trianglelefteq L$ with $a \vee b = 1$ and it is denoted by $a \ll_g L$. Clearly every small element of a lattice L is g-small, but the converse of this is not true in general (see [8, Example 1]). We have the following properties of g-small elements:

Lemma 1.2. [8, Lemma 1] *The following properties hold for elements a, b, c, d of a lattice L.*

1. *If $a \leq b$ and $b \ll_g L$, then $a \ll_g L$.*
2. *If $a \ll_g b/0$, then $a \ll_g t/0$ for every element $t \in L$ with $b \leq t$.*
3. *If $a \ll_g L$, then $a \vee b \ll_g 1/b$.*
4. *If $a \ll_g b/0$ and $c \ll_g d/0$, then $a \vee c \ll_g (b \vee d)/0$.*

An element a of L is called a *supplement* of an element b if $a \vee b = 1$ and a is minimal with respect to this property. Equivalently, an element a is a supplement of b in L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$ (see [3, Proposition 12.1]). Reducing the last condition to $a \wedge b \ll L$ we obtain the definition of *weak supplements*. A lattice L is called a *supplemented* (respectively, a *weakly supplemented*) lattice if for every element a of L there is a supplement (respectively, a weak supplement) in L. Let b be an element of a lattice L. If $1 = a \vee b$ and $1 = a \vee t$ with $t \trianglelefteq b/0$ implies that $t = b$, then b is said to be *g-supplement* of a in L. If every element of L has a g-supplement in L, then L is called a *g-supplemented* lattice (see [8]). Recently, g-supplemented lattices are introduced and studied by Ökten in [8].

If the quotient sublattice $1/a$ is compact, then a is called a *cofinite* element in L. Here $1/a$ is compact means that $1 = \bigvee_{i \in F} x_i$ for some finite subset F of I, when $1 = \bigvee_{i \in I} x_i$ for some elements $x_i \geq a$. If there is a (weak) supplement for every cofinite element of a lattice L in L, then L is said to be *cofinitely (weak) supplemented.* Cofinitely weak supplemented lattices and cofinitely supplemented lattices are studied in [1] and [2].

In this paper, we introduce and study cofinitely (weak) g-supplemented lattices. These lattices are generalizations of cofinitely (weak) g-supplemented modules which are studied in [5] and [7]. Namely, if there is a (weak) g-supplement for every cofinite element of L in L, then L is said to be *cofinitely (weak) g-supplemented*. We give examples of lattices showing that not all generalizations are true. Some results proved for lattices provide new results for modules.

In Section 2, cofinitely g-supplemented lattices are studied. It is shown in Proposition 2.4 that quotient sublattices of cofinitely g-supplemented lattices are also cofinitely g-supplemented. It is proved in Proposition 2.6 that every cofinite element of $1/rad_g(L)$ is a complement in a cofinitely g-supplemented lattice L. It is proved in Theorem 2.9 that if $\{a_i/0\}_{i \in I}$ is a collection of cofinitely g-supplemented sublattices of L and $1 = \bigvee_{i \in I} a_i$, then L is also cofinitely g-supplemented. It is shown that if l is a g-supplement of a g-maximal element m of a compactly generated lattice$L$ having zero socle, then $l/0$ is a g-local lattice (see Proposition 2.11).

In Section 3, cofinitely weak g-supplemented lattices are discussed. It is shown that without loss of generality weak g-supplements of cofinite elements in compactly generated lattices are compact (see Proposition 3.2). An example showing that Proposition 3.2 need not be true for lattices which are not compactly generated is given in Example 3.3. It is proved that quotient sublattices of cofinitely weak g-supplemented lattices are cofinitely weak g-supplemented (see Proposition 3.4). It is given in Theorem 3.7 that if $\{a_i/0\}_{i\in I}$ is a collection of cofinitely weak g-supplemented sublattices of L and $1 = \bigvee_{i\in I} a_i$, then L is a cofinitely weak g-supplemented lattice. We give a condition under which a compactly generated cofinitely weak g-supplemented lattice is cofinitely g-supplemented (see Theorem 3.9).

II. COFINITELY G-SUPPLEMENTED LATTICES

In this section, cofinitely g-supplemented lattices are defined and some properties of them are given.

Definition 2.1. A lattice L is called a *cofinitely g-supplemented* lattice if each cofinite element of L has a g-supplement in L.

We will use following lemmas in the sequel of the paper.

Lemma 2.2. [8, Lemma 2] *An element b is a g-supplement of an element a in L if and only if $a \vee b = 1$ and $a \wedge b \ll_g b/0$.*

Lemma 2.3. [8, Lemma 5] *Let L be a lattice and $a, b, c \in L$ with $c \leq a$. If b is a g-supplement of a in L, then $b \vee c$ is a g-supplement of a in $1/c$.*
Proposition 2.4. *If L is a cofinitely g-supplemented lattice, then the quotient sublattice $1/a$ is also cofinitely g-supplemented for every element a of L.*

Proof. To show that $1/a$ is a cofinitely g-supplemented quotient sublattice for an arbitrary element a of L take a cofinite element b of $1/a$. Then $1/b$ is a compact sublattice of $1/a$ and therefore $1/b$ is a compact quotient sublattice of L. So b is also cofinite in L. Since L is cofinitely g-supplemented, b has a g-supplement x in L. Therefore, $a \vee x$ is a g-supplement of b in $1/a$ by Lemma 2.3.

Recall that the *radical* of L is the meet of all maximal elements ($\neq 1$) in L (see [3]). The radical of L is denoted by $rad(L)$. Let m be a maximal element ($\neq 1$) of a lattice L. If $m \trianglelefteq L$, then m is called a *g-maximal* element of L. The meet of all g-maximal elements of L is said to be the *g-radical* of L. The g-radical of L is denoted by $rad_g(L)$. If L has no g-maximal element, then $rad_g(L) = 1$ (see [8]).

Lemma 2.5. *The following properties hold for a lattice L.*

1. $rad(L) \leq rad_g(L)$ *by* [8, Corollary 6].
2. *If* $a \ll_g L$, *then* $a \leq rad_g(L)$ *by* [8, Lemma 6].
3. $rad_g(a/0) \leq rad_g(L)$ *for an element a of L by* [8, Lemma 7].

Recall that if every element a of L is a complement of an element in L, i.e. $a \vee b = 1$ and $a \wedge b = 0$ for some b in L, then L is called a *complemented* lattice (see [3]).

Proposition 2.6. *If L is a cofinitely g-supplemented lattice, then every cofinite element of the quotient sublattice $1/\, rad_g(L)$ is a complement.*

Proof. Let a be a cofinite element of $1/\ rad_g(L)$. Then a is a cofinite element of L and therefore a has a g-supplement b in L. That is, $a \vee b = 1$ and $a \wedge \mathrm{b} \ll_g \mathrm{b}\ /\ 0$. So $a \wedge b\ \leq rad_g(b/0) \leq rad_g(L)$, by Lemma 2.5. Hence $1 = a \vee b \vee rad_g(L)$ and $a \wedge \left(b \vee rad_g(L)\right) = (a \wedge b) \vee rad_g(L) = rad_g(L)$. Thus, the element $b \vee rad_g(L)$ is a complement of the element a in the quotient sublattice $1\ /\ rad_g(L)$ of L.

Lemma 2.7. *Let $m/0$ be cofinitely g-supplemented lattice for an elementm of a lattice L and u be a cofinite element of L. If $m \vee u$ has a g-supplement in L, then u has a g-supplement in L.*

Proof. Let a be a g-supplement of $m \vee u$ in L, i.e. $(m \vee u) \vee a = 1$ and $(m \vee u) \wedge a \ll_g a/0$.Since u is a cofinite element of L, $a \vee u$ is also a cofinite element of L. $m/[m \wedge (a \vee u)] = [m \vee (a \vee u)]/(a \vee u) = 1/(a \vee u)$. Therefore, $m \wedge (a \vee u)$ is a cofinite element of $m/0$. Since $m/0$ is cofinitely g-supplemented, $m \wedge (a \vee u)$ has a g-supplement b in $m/0$. That is, $[m \wedge (a \vee u)] \vee b = m$ and $[m \wedge (a \vee u)] \wedge b \ll_g b/0$. Now we have $1 = (m \vee u) \vee a = [m \wedge (a \vee u)] \vee b \vee a \vee u = b \vee a \vee u$. Then by [3, Lemma 12.3] and Lemma 1.2 (4),$u \wedge (a \vee b) \leq [a \wedge (u \vee b)] \vee [b \wedge (u \vee a)] \leq [a \wedge (m \vee u)] \vee [b \wedge m \wedge (a \vee u)] \ll_g (a \vee b)/0$. Thus $(a \vee b)$ is a g-supplement of u in L.

Lemma 2.8. *Let m_i be elements of a lattice L for $i = 1, \ldots, n$ with $m_i\ /\ 0$ cofinitely g-supplemented for each $i\ =\ 1, \ldots, n$ and u be cofinite in L. If $u \vee m_1 \vee \ldots \vee m_n$ has a g-supplement in L, then there is a g-supplement of u in L.*

Proof. Clear from Lemma 2.7.

Next it is proved that an arbitrary join of cofinitely g-supplemented sublattices is also cofinitely g-supplemented.

Theorem 2.9. *If $\{a_i/0\}_{\ i \in I}$ is a collection of cofinitely g-supplemented sublattices of L and $1 = \bigvee_{i \in I} a_i$, then L is also cofinitely g-supplemented.*

Proof. Take a cofinite element u of L. $1 = \bigvee_{i \in I} a_i = u \vee (\bigvee_{i \in I} a_i) = \bigvee_{i \in I}\ (u \vee a_i)$. Since $1/u$ is compact, there exists a finite subset F of I such that $1 = \bigvee_{i \in F} a_i = u \vee (\bigvee_{i \in F} a_i) = \bigvee_{i \in \mathrm{F}}\ (u \vee a_i)$. Thus, there is a g-supplement of u in L by Lemma 2.8.

We can define generalized local or briefly g-local lattices as a generalization of generalized local modules which is defined in [6].

Definition 2.10. If a lattice L has the largest essential element different from 1 or L has no nonzero essential element, then L is called a *generalized local* (or shortly *g-local*) lattice.

Recall that an element a of L is said to be *compact*, if for every subset $\{x_i \mid i \in I\}$ of L with $a \leq \bigvee_{i \in I} x_i$ there exists a finite subset F of I such that $a \leq \bigvee_{i \in F} x_i$. If greatest element 1 of L is compact, then L called a *compact lattice*. If every element of L is a join of compact elements, then L is called a *compactly generated* (or an *algebraic*) lattice (see [9]). If $a < b$ and $a \leq c < b$ implies $c = a$, then we say that a is *covered by* b (or b *covers* a). An element a of L is said to be *atom* if 0 is covered by a (see [10]). The *socle* of L is the join of all the atoms in L. It is denoted by $soc(L)$ ([3]). In a lattice L if l is a supplement of a maximal element m, then the sublattice $l/0$ is local (see [2, Lemma 2.3]). This is not true for g-supplements of a maximal element m of a lattice in general. But we have the following result.

Proposition 2.11. *Let L be a compactly generated lattice with a zero socle and m be a g-maximal element of L. If l is a g-supplement of m, then $l/0$ is a g-local lattice. Moreover $l \wedge m$ is the largest essential element ($\neq l$) of $l/0$.*

Proof. l is a g-supplement of m if and only if $l \vee m = 1$ and $l \wedge m \ll_g l/0$ by Lemma 2.2. Since $m \trianglelefteq L$, $l \wedge m \trianglelefteq l/0$ by [3, Ex. 4.5]. Since $l / 0$ is a compactly generated lattice with zero socle, there are proper essential elements of $l/0$ by [3, Ex.5.11]. Let $a \in l/0$ such that $a \trianglelefteq l/0$ and $a \neq l$. If $a \leq m$, then $a \leq l \wedge m$. If $a \nleq m$ ($a \nleq l \wedge m$), then since m is maximal, $a \vee m = 1$. Now $l = l \wedge 1 = l \wedge (a \vee m) = a \vee (l \wedge m)$. Since $l \wedge m \ll_g l/0$, $a = l$. This is a contradiction. Thus $l \wedge m$ is the largest essential element ($\neq l$) of $l/0$.

III. COFINITELY WEAK G-SUPPLEMENTED LATTICES

In this section, cofinitely (weak) g-supplemented lattices are defined and some properties of cofinitely (weak) g-supplemented lattices are obtained.

Definition 3.1. If every cofinite element of a lattice L has a weak g-supplement in L, then L is said to be *cofinitely weak g-supplemented.*

Without loss of generality, weak g-supplements of cofinite elements can be regarded as compact elements for compactly generated lattices.

Proposition 3.2. *If b is a weak g-supplement of a cofinite element a in a compactly generated lattice L, then there is a weak g-supplement c of a in L such that $c \leq b$ and c is compact.*

Proof. Since L is compactly generated, $b = \bigvee_{i \in I} c_i$ where each c_i is compact. Then $1 = a \vee b = a \vee (\bigvee_{i \in I} c_i) = \bigvee_{i \in I} (a \vee c_i)$. Since $1 / a$ is compact, $1 = \bigvee_{i \in F} (a \vee c_i) = a \vee (\bigvee_{i \in F} c_i)$ for some finite subset F of I. Also $c = \bigvee_{i \in I} c_i$ is compact by [3, Proposition 2.1]. Since $c \leq b$, $a \wedge c \leq a \wedge b \ll_g L$ and therefore $a \wedge c \ll_g L$, by Lemma 1.2 (1). Thus c is a weak g-supplement of a in L.

The following example shows that Proposition 3.2 need not be true for lattices that are not compactly generated.

Example 3.3. Let $L = \{(x, 0) \mid x \in [0,1]\} \cup \{(0, y) \mid y \in [0,1]\} \subseteq R^2$ and define the order $\leq$ on L as follows. $(a, b) \leq (c, d)$ if either $b = d = 0$ and $a \leq c$; or $a = c = 0$ and $b \leq d$; or $b = c = 0$ and $a \leq d$. One can easily verify that L is a complete modular lattice with the largest element$(0,1)$ and the smallest element $(0,0)$. Since the quotient sublattice $(0,1)/(1,0)$ is simple, it is compact. So $(1,0)$ is a cofinite element of L. Let a be a real number with $0 < a < 1$. Clearly, $(0, a) \vee (1,0) = (0,1)$ and $(0, a) \wedge (1,0) = (a, 0)$ is small and therefore g-small in L, so $(0, a)$ is a weak g-supplement of $(1,0)$ in L. On the other hand, there is no compact element in L except for $(0,0)$, therefore there is no compact weak g-supplement (b, c) of $(1,0)$ with $(b, c) \leq (0, a)$.

Next result shows that every quotient sublattice of a cofinitely weak g-supplemented lattice is also cofinitely weak g-supplemented.

Proposition 3.4. *If L is a cofinitely weak g-supplemented lattice, then $1/a$ is also a cofinitely weak g-supplemented sublattice for every element a of L.*

Proof. Take a cofinite element b from$1/a$. Since $1/b$ is a compact sublattice of $1/a$, $1/b$ is a compact quotient sublattice of L, that is b is cofinite in L. Since L is a cofinitely weak g-supplemented lattice, b has a weak g-supplement x in L, i.e. $x \vee b = 1$ and $x \wedge b \ll_g L$. Since $x \wedge b \ll_g L$, $(x \vee a) \wedge b = (x \wedge b) \vee a \ll_g 1/a$ by Lemma 1.2 (3). So $x \vee a$ is a weak g-supplement of b in $1/a$.

Now to show that any arbitrary join of cofinitely weak g-supplemented principal ideals is again cofinitely weak g-supplemented we need following lemmas whose proofs can be obtained by slight modification of Lemma 2.7 and Lemma 2.8.

Lemma 3.5. *Let $m/0$ be a cofinitely weak g-supplemented sublattice of L for an element m of L and u be a cofinite in L. If $m \vee u$ has a weak g-supplement in L, then there is a weak g-supplement of u in L.*

Lemma 3.6. *Let m_i be elements of a lattice L for $i = 1,\dots,n$ with $m_i / 0$ cofinitely weak g-supplemented for each $i = 1,\dots,n$ and u be a cofinite element of L. If there is a weak g-supplement of $u \vee m_1 \vee \dots \vee m_n$ in L, then there is a weak g-supplement of u in L.*

Theorem 3.7. *If $\{a_i/0\}_{i\in I}$ is a collection of cofinitely weak g-supplemented sublattices of L and $1 = \bigvee_{i\in I} a_i$, then L is cofinitely weak g-supplemented.*

Proof. Clear by Theorem 2.9.

We know that cofinitely g-supplemented lattices are cofinitely weak g-supplemented. Next, we give a condition under which the converse of this is true for compactly generated lattices. But first we need following lemma.

Lemma 3.8. *If a cofinite element a has a weak g-supplement b in a compactly generated lattice L and for every compact element c with $c \le b$, $rad_g(c/0) = c \wedge rad_g(L)$, then a has a compact g-supplement in L.*

Proof. Since a is cofinite, $1/a$ is compact. So, by Proposition 3.2, a has a compact weak g-supplement c with $c \le b$, i.e. $1 = a \vee c$ and $a \wedge c \ll_g L$. Then $a \wedge c \le rad_g(L)$ by Lemma 2.5 (2). So $a \wedge c \le c \wedge rad_g(L) = rad_g(c/0)$. Since c is compact, $rad(c/0) \ll c/0$ by [9, Proposition 9 (iii)] and therefore $rad(c/0) \ll_g c/0$. Since $a \wedge c \le rad_g(c/0)$, $a \wedge c \ll_g c/0$ by Lemma 1.2 (1). Hence c is a compact g-supplement of a in L.

Using Lemma 3.8 we obtain the following results.

Theorem 3.9. *Let $rad_g(c/0) = c \wedge rad_g(L)$ for every compact element c of a compactly generated lattice L. Then L is cofinitely weak g-supplemented if and only if L is cofinitely g-supplemented.*

Proof. (⇒) Take a cofinite element a of L. Since L is cofinitely weak g-supplemented there is a weak g-supplement b of a in L. Therefore, there is a g-supplement of a in L by Lemma 3.8. Thus L is cofinitely g-supplemented.
(⇐) Clear since every g-supplemented lattice is weakly g-supplemented.

Corollary 3.10. *Let $rad_g(c/0) = c \wedge rad_g(L)$ for every compact element c of a compact lattice L. Then L is weakly g-supplemented if and only if L is g-supplemented. Furthermore, in this case every compact element of L is a g-supplement.*

Proof. First part is clear by Theorem 3.9 since every element in a compact lattice is cofinite. If c is a compact element in L, then there is a weak g-supplement b of c in L. That is $c \vee b = 1$ and $c \wedge b = c \wedge rad_g(L) = rad_g(c/0) \ll_g c/0$. This means that the element c, which is compact, is a g-supplement in L.

Let R be an associative ring with identity and M be a unitary right R-module. Then the *radical* of M is defined to be the intersection of all maximal submodules of M and it is denoted by Rad M. If M has no maximal submodule, then Rad $M = M$. Now we have the following analogues of Theorem 3.9 and Corollary 3.10 which are new for modules.

Theorem 3.11. *Let M be a right R-module such that for every finitely generated submodule N of M,* Rad $N = N \cap$ Rad M. *Then M is is cofinitely weak g-supplemented if and only if M is cofinitely g-supplemented.*

Corollary 3.12. *Let M be a finitely generated right R-module such that for every finitely generated submodule N of M,* Rad $N = N \cap$ Rad M. *Then M is weakly g-supplemented if and only if M is g-supplemented. Furthermore, in this case every finitely generated submodule of M is a g-supplement.*

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