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Research Article

Characterizations of the Ruled Surfaces with Modified Orthogonal Frame

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Abstract

In our paper, a new ruled surface is created using a modified orthogonal frame. Then, the distribution parameter of this surface, the striction line, the normal vector field and the fundamental form of the surface are calculated and differential geometric properties are examined. In the last part of the study, examples related to this new ruled surface are given.

Keywords: Ruled surfaces, Serret-Frenet frame, Modified orthogonal frame.

Regle Yüzeylerin Modifiye Çatı ile Karakterizasyonları

Öz

Çalışmamızda, modifiye çatı kullanılarak yeni bir regle yüzey oluşturulmuştur. Daha sonra bu yüzeyin dağılma parametresi, striksiyon çizgisi, yüzeyin normal vektör alanı ve esas formları hesaplanıp diferansiyel geometrik özellikleri incelenmektedir. Çalışmanın son kısmında ise bu yeni regle yüzey ile ilgili örnekler verilmiştir.

Anahtar Kelimeler: Regle yüzeyleri, Serret-Frenet çatısı, Modifiye çatı.

1. Introduction

Ruled surfaces firstly were found and studied by Monge (1850). In the later years, Guggenheimer (1963) and Hoschek (1971) studied ruled surfaces by developing different perspectives in geometry. Ruled surfaces are the surfaces obtained by a continuous movement of a straight line on a curve in the space which is called a base curve. The straight lines lean against the direction. It has been several studies done on the ruled surfaces in the differential geometry. The ruled surfaces have still continued to be one of the most attractive subjects in differential geometry. Since the structure that ruled surfaces have, they can be studied in a variety of fields such as architecture, spatial mechanics, computer-aided design, etc. Thus, the importance of these surfaces is increasing more and more every passing day [1, 2]. On the other side, in the differential geometry, the Serret-Frenet formulas defined the respective geometric and kinematic properties of a particle that moves through a continuous and differentiable curve without any relation with any movement in the 3-dimensional Euclidean space. The respective properties of the curve are independent of any other movement. The frame is known as {T, N,

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B} is made up of tangent vector, principal normal vector, and binormal vector, respectively. Also, it defines the derivatives of these vectors in terms of each other. This frame got its name from the two French mathematicians who discovered it independently from each other. Jean Frederic Frenet (1847) and, Joseph Alfred Serret (1851) discovered it. The Serret-Frenet frame composes the keystone of the other frame studies [3]. On the other hand, the Serret-Frenet frame is not insufficient at the points where the curvature of the curve is zero. In other words, if the second derivative of the curve is equal to zero, then the Serret-Frenet frame is not identified. Because of this reason, until today, various alternative frames have been constituted for analyzing the properties of the curves. One of these frames is the study made by Sasai(1984). He worked on an orthogonal frame and obtained the modified orthogonal frame corresponding to the Serret-Frenet frame. The frame elements in Sasai's study were obtained by multiplying each element by κ, the coefficient of curvature of the Serret-Frenet vectors, respectively [4]. And then, Karacan and Bükçü have developed the study of Sasai and obtained the newly modified orthogonal frame through the coefficient of torsion τ by Serret-Frenet vectors [5, 6, 7, 8]. Additionally, Eren and Kosal examined the tubular surfaces and special ruled surfaces using the modified orthogonal frame [9, 10]. In this study, the definitions of the new ruled surfaces which are made up of the tangent vector and the base curve whose direction is composed of the coefficients of curvature and torsion of the modified orthogonal frame have been given. Then, the geometrical properties of these surfaces have been analyzed. At the end of the study, various examples of the new ruled surfaces have been given.

2. Preliminaries

In this section, it will be first given the certain properties of the ruled surface and the defined curve, and then, after giving the outlines of the Serret-Frenet frame, the relation between its modified orthogonal frame will be demonstrated. Then, the general characteristics of the modified orthogonal frame and some of the geometric properties of the curves will be presented.

Definition 2.1. Assume that the curve $\alpha(s): I \subset \mathbb{R} \to E^3$ is a regular curve that is parametrized by the arc length parameter s. When T is the unit tangent vector field, N is the principal normal field, and B is a binormal vector field, the Serret-Frenet frame of the curve λ is defined by $T = \alpha'(s),\ N = \frac{T'(s)}{\|T'(s)\|},\ B = T \times N$ the derivative formulas of this frame are given by the equations $T' = \kappa N,\ N' = -\kappa T + \tau B,\ B' = -\tau N$. $\kappa = \|T'\|$ and $\tau = -\langle B', N \rangle$ denote the curvature and the torsion of the curve, respectively. The curvature denotes the amount of deviation of the curve from the tangent line. The torsion represents the amount of deviation from the osculating plane. Here, that the curvature is zero shows that α is a straight line, and that the torsion is zero shows that α is a plane curve [3].

Definition 2.2. (The Modified Orthogonal Frame) Let the curve $\alpha(s): I \subset \mathbb{R} \to E^3$ be a regular curve in Euclidean 3-space. Thus, the modified orthogonal frame prepared with using the curvature κ in the direction of the curve α is defined as $e_1 = T, e_2 = \kappa N, e_3 = \kappa B$, the derivative equations of the frame are

$$\begin{pmatrix} e_1'(s) \\ e_2'(s) \\ e_3'(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \frac{\kappa'}{\kappa} \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix}.$$

If the value κ is the same as in the Serret-Frenet frame, then $\tau = \frac{\det[\lambda', \lambda'', \lambda''']}{\kappa^2}$. The inner products in this frame are defined as $\langle e_1, e_1 \rangle = 1, \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = \kappa^2$, $\langle e_1, e_2 \rangle = \langle e_1, e_3 \rangle = \langle e_2, e_3 \rangle = 0$. Let $\alpha(s)$ be a regular curve that is parametrized by the arch length parameter. Therefore, the modified orthogonal frame prepared with the curvature κ in the direction of the curve $\alpha(s)$ is defined as $e_1 = T, e_2 = \tau N, e_3 = \tau B$ and the derivative equations of the frame are

$$\begin{pmatrix} e_1'(s) \\ e_2'(s) \\ e_3'(s) \end{pmatrix} = \begin{pmatrix} 0 & \frac{\kappa}{\tau} & 0 \\ -\kappa\tau & \frac{\tau'}{\tau} & \tau \\ 0 & -\tau & \frac{\tau'}{\tau} \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix}.$$

If the value κ is the same as in the Serret-Frenet frame, then $\tau = \frac{\det[\lambda', \lambda'', \lambda''']}{\kappa^2}$. The inner products in this frame are defined as $\langle e_1, e_1 \rangle = 1, \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = \tau^2$, $\langle e_1, e_2 \rangle = \langle e_1, e_3 \rangle = \langle e_2, e_3 \rangle = 0$ in [11].

Definition 2.5. (Ruled Surface) Let $M \subset E^3$ be a regular surface. If there is a line of E^3 in M at the point $\forall P \in M$, then M is called a ruled surface. The line passing through the point $P \in M$ and staying in M is called the direction of the surface M. Since $\lambda(s)$ is a based curve and $X(s)$ is a direction, then a ruled surface is defined by the equation $\gamma(s,v) = \lambda(s) + vX(s)$ [12, 3].

Definition 2.6. (Distribution Parameter) Let $\gamma(s,v) = \lambda(s) + vX(s)$ be a ruled surface and $\gamma : I \times R \to M$, the distribution parameter of this surface is defined by $P_X = \frac{\det[\lambda', X, X']}{\|X'\|^2}$.

According to the coordinate variances, the distribution parameter (dral) for ruled surfaces is the simplest differential invariant [3].

Definition 2.7. Let $\gamma(s,v)$ be a ruled surface in the 3-dimensional Euclidean space. $\overline{\lambda}(s)$ is the position vector of the striction line; thus the striction line is defined by $\overline{\lambda}(s)=\lambda(s)-\frac{\langle X',T\rangle}{\|X'\|^2}X(s)$, [3].

3. Main Theorem and Proof

3.1. Ruled Surfaces with Modified orthogonal frame Formed by κ

In this section, a new ruled surface whose direction has been defined by the elements of the modified orthogonal frame is defined and the differential properties such as distribution parameter, striction line and mean curvature are given.

Let a ruled surface

$$\gamma: I\times R\rightarrow R^3,\ \ (s,v)\rightarrow\gamma(s,v)=\lambda(s)+vX(s) \tag{1}$$

be given with the based curve $\lambda(s)=e_1(s)$ in the Euclidean 3-space. When the direction vector $X(s)=Sp\{e_1,e_2,e_3\}$ is chosen in a way that it will be formed by the elements of the modified orthogonal frame where $\forall x_1,x_2,x_3\in R$, the direction vector $X(s)$ is

$$X(s)=x_1e_1+x_2e_2+x_3e_3,\quad (x_1^2+x_2^2+x_3^2=1). \tag{2}$$

When the first derivative of the direction vector is taken, the formulation

$$X'=-x_2\kappa^2e_1+\left(x_1+x_2\frac{\kappa'}{\kappa}-x_3\tau\right)e_2+\left(x_2\tau+x_3\frac{\kappa'}{\kappa}\right)e_3 \tag{3}$$

is obtained. From the Definition 2.2, the tangent of the curve is

$$\lambda'(s)=e_1'=e_2, \tag{4}$$

and when the equations (2), (3), and (4) are substituted in the definition of the distribution parameter in Definition 2.6, the distribution parameter of the curve $\lambda(s)$ is found by the equation

$$P_x=\frac{-x_2x_3\kappa^2-x_1x_2\tau-x_1x_3\frac{\kappa'}{\kappa}}{x_2^2\kappa^4+\left(x_1+x_2\frac{\kappa'}{\kappa}-x_3\tau\right)^2+\left(x_2\tau+x_3\frac{\kappa'}{\kappa}\right)^2}. \tag{5}$$

The position vector of the striction line $\overline{\lambda}$ in Definition 2.7. is obtained as

$$\overline{\lambda} = \lambda + \frac{x_2\kappa^2\left(x_1e_1 + x_2e_2 + x_3e_3\right)}{x_2\kappa^4 + \left(x_1 + x_2\dfrac{\kappa'}{\kappa} - x_3\tau\right)^2 + \left(x_2\tau + x_3\dfrac{\kappa'}{\kappa}\right)^2}. \tag{6}$$

The first fundamental form of the ruled surface can be given as follows:

$$I = Eds^2 + 2Fdsdv + Gdv^2 \tag{7}$$

$$E = \langle\gamma_s, \gamma_s\rangle,\ F = \langle\gamma_s, \gamma_v\rangle,\ G = \langle\gamma_v, \gamma_v\rangle, \tag{8}$$

$$\gamma_s = \lambda' - vx_2\kappa^2 e_1 + \left(1 + vx_1 + vx_2\frac{\kappa'}{\kappa} - vx_3\tau\right)e_2 + \left(vx_2\tau + vx_3\frac{\kappa'}{\kappa}\right)e_3 \tag{9}$$

$$\gamma_v = x_1e_1 + x_2e_2 + x_3e_3. \tag{10}$$

Thus, when the equations obtained in equations (9) and (10) are substituted in equation (8), the coefficients

$$\begin{aligned} E &= v^2x_2^2\kappa^4 + \left(1 + vx_1 + vx_2\frac{\kappa'}{\kappa} - vx_3\tau\right)^2\kappa^2 + \left(vx_2\tau + vx_3\frac{\kappa'}{\kappa}\right)^2\kappa^2 \\ F &= \kappa^2\left(x_2 + vx_1x_2 + vx_2^2\frac{\kappa'}{\kappa} + vx_3^2\frac{\kappa'}{\kappa}\right), \quad G = 1 \end{aligned} \tag{11}$$

are obtained. When equation (11) is substituted in equation (7), the first fundamental form of the surface is found as

$$\begin{aligned} I = &\left[v^2x_2^2\kappa^4 + \left(1 + vx_1 + vx_2\frac{\kappa'}{\kappa} - vx_3\tau\right)^2\kappa^2 + \left(vx_2\tau + vx_3\frac{\kappa'}{\kappa}\right)^2\kappa^2\right]ds^2 \\ &+ 2\kappa^2\left(x_2 + vx_1x_2 + vx_2^2\frac{\kappa'}{\kappa} + vx_3^2\frac{\kappa'}{\kappa}\right)dsdv + dv^2 \end{aligned}.$$

Again, the coefficients for the second fundamental form of the surface can be computed by the equations

$$L = \frac{\det\left[\gamma_{ss}, \gamma_s, \gamma_v\right]}{\left\|\gamma_s \times \gamma_v\right\|},\ M = \frac{\det\left[\gamma_{sv}, \gamma_s, \gamma_v\right]}{\left\|\gamma_s \times \gamma_v\right\|},\ N = \frac{\det\left[\gamma_{vv}, \gamma_s, \gamma_v\right]}{\left\|\gamma_s \times \gamma_v\right\|}. \tag{12}$$

The partial derivatives in the equation (12) are obtained as

$$\begin{aligned} \gamma_{sv} &= -\left(x_2k_1 + x_3k_2\right)N + x_1k_1N_1 + x_1k_2N_2 \\ \gamma_{vv} &= 0. \end{aligned}$$

$$\begin{aligned}\gamma_{ss} = &\left(-v\left(x_2 k_1' + x_3 k_2' + x_1\left(k_1^2 + k_2^2\right)\right)\right)N \\ &+\left(k_1\left(1 - v x_2 k_1 - v x_3 k_2\right) + v x_1 k_1\right)N_1 \\ &+\left(k_2\left(1 - v x_2 k_1 - v x_3 k_2\right) + v x_1 k_2'\right)N_2\end{aligned} \tag{13}$$

In addition, the normal vector field of the surface is $n = \dfrac{\gamma_s \times \gamma_v}{\left\|\gamma_s \times \gamma_v\right\|}$ and when the derivatives in equations (9) and (10) are substituted, the norm

$$\left\|\gamma_s \times \gamma_v\right\| = \sqrt{v^2 x_1^2\left(x_3 k_1 - x_2 k_2\right)^2 + \left(\left(1 - v\left(x_2 k_1 + x_3 k_2\right)\right)x_3 - v x_1^2 k_2\right)^2 + \left(\left(1 - v\left(x_2 k_1 + x_3 k_2\right)\right)x_2 - v x_1^2 k_1\right)^2}$$

is found. Then, the normal vector of the surface is obtained as

$$n = \frac{\begin{bmatrix} x_3 + v x_1 x_3 - v x_3^2 \tau - v x_2^2 \tau, -v x_2 x_3 \kappa^2 - v x_2 x_1 \tau - v x_1 x_3 \dfrac{\kappa'}{\kappa} \\ , -v x_2^2 \kappa^2 - x_1 - v x_1^2 - v x_1 x_2 \dfrac{\kappa'}{\kappa} + v x_1 x_3 \tau \end{bmatrix}}{\sqrt{\begin{aligned}&\left(x_3 + v x_1 x_3 - v x_3^2 \tau - v x_2^2 \tau\right)^2 + \left(-v x_2 x_3 \kappa^2 - v x_2 x_1 \tau - v x_1 x_3 \dfrac{\kappa'}{\kappa}\right)^2 \\ &+ \left(-v x_2^2 \kappa^2 - x_1 - v x_1^2 - v x_1 x_2 \dfrac{\kappa'}{\kappa} + v x_1 x_3 \tau\right)^2\end{aligned}}}. \tag{14}$$

Thus, the coefficients of the second fundamental form are computed as

$$\begin{aligned}
L &= \frac{\begin{bmatrix} \kappa^2\left(v x_3 - v x_1 x_3 - x_3 + v^2 x_1 x_3^2 - v^2 x_1^2 x_3 - v x_1 x_3\right) + \kappa^2 \tau\left(-v^2 x_3^3 + v^2 x_1 x_3^2 + v x_3^2 - v^2 x_2^2 x_3 - v^2 x_1 x_2^2\right) \\ -v^2 x_2 x_3^2 \kappa \kappa' \tau + \dfrac{\kappa'}{\kappa}\tau\left(-3v^2 x_1^2 x_2 - v x_1 x_2 - v^2 x_1 x_2 x_3\right) + \left(\dfrac{\kappa'}{\kappa}\right)^2 \tau\left(-2v^2 x_1 x_3 + v^2 x_1 x_3^2\right) - v^2 \kappa^4 \tau^2 x_2^3 \\ +\tau\left(-x_1 - 2v x_1^2 - v^2 x_1^3\right) - v^2 x_1 x_2^2 \dfrac{\kappa'}{\kappa}\tau^2 - v^2 x_2^2 x_3 \kappa^4 \\ +\tau^2\left(-v x_1 x_2 - v^2 x_1^2 x_2 + 2v^2 x_1^2 x_3\right) + \tau^3\left(-v x_1 x_2^2 + v x_1 x_2 x_3 - v^2 x_1 x_3^2\right) \end{bmatrix}}{\sqrt{\left(x_3 + v x_1 x_3 - v x_3^2 \tau - v x_2^2 \tau\right)^2 + \left(-v x_2 x_3 \kappa^2 - v x_2 x_1 \tau - v x_1 x_3 \dfrac{\kappa'}{\kappa}\right)^2 + \left(-v x_2^2 \kappa^2 - x_1 - v x_1^2 - v x_1 x_2 \dfrac{\kappa'}{\kappa} + v x_1 x_3 \tau\right)^2}} \\
M &= \frac{-x_2 x_3^2 \kappa^2 - v x_2^2 \kappa \kappa' - x_1 x_3 \dfrac{\kappa'}{\kappa} + v x_1 x_2 \dfrac{\kappa'}{\kappa}\tau - v x_1 x_2^2 \dfrac{\kappa'}{\kappa}\tau - x_1 x_2 \tau}{\sqrt{\left(x_3 + v x_1 x_3 - v x_3^2 \tau - v x_2^2 \tau\right)^2 + \left(-v x_2 x_3 \kappa^2 - v x_2 x_1 \tau - v x_1 x_3 \dfrac{\kappa'}{\kappa}\right)^2 + \left(-v x_2^2 \kappa^2 - x_1 - v x_1^2 - v x_1 x_2 \dfrac{\kappa'}{\kappa} + v x_1 x_3 \tau\right)^2}} \\
N &= 0.
\end{aligned} \tag{15}$$

Now, a couple of theorems will be considered in the direction of the computed elements.

Theorem 3.1.1. Let $\gamma: I \times R \to R^3$ be a ruled surface given by $\gamma(s,v) = \lambda(s) + vX(s)$ in the Euclidean 3-space. The necessary and sufficient condition for this surface to be developable is to satisfy the equation $\tau = -\left(\frac{x_3}{x_1}\kappa^2 + \frac{x_3}{x_2}\frac{\kappa'}{\kappa}\right)$.

Proof. A ruled surface is developable if and only if the distribution parameter is zero. When $P_X = 0$ and $\kappa, x_1, x_2 \neq 0$ in (5), the equation $\tau = -\left(\frac{x_3}{x_1}\kappa^2 + \frac{x_3}{x_2}\frac{\kappa'}{\kappa}\right)$ is obtained.

Theorem 3.1.2. Let $\gamma: I \times R \to R^3$ be a ruled surface parametrized by $\gamma(s,v) = \lambda(s) + vX(s)$ in the Euclidean 3-space. If the base curve λ of the surface γ is the striction line at the same time, the condition $x_2 = 0$ or $\kappa = 0$ is satisfied.

Proof. For the base curve λ is to be the striction curve, the inner product $\left\langle \frac{d\lambda}{ds}, X' \right\rangle = 0$ in Definition 2.7 should be satisfied. Therefore, when $x_2 = 0$ or $\kappa = 0$ is substituted in equation (6), the proof will be completed.

3.2. The Characterization of the Ruled Surfaces with Modified orthogonal frame in the Special Situations

3.2.1. The ruled surfaces with the direction vector $X(s) = Sp\{e_1, e_2\}$

When the special situation $x_1^2 + x_2^2 = 1, x_3 = 0$ is taken in the general equation of the ruled surfaces with the modified orthogonal frame defined in (1), the ruled surface $\gamma(s,v) = \lambda(s) + v\left(x_1 e_1 + x_2 e_2\right)$ is strained by the vectors e_1 and e_2 is formed.

Theorem 3.2.1.1. In equation (5)**,** $x_3 = 0$, and if this is substituted in the distribution parameter formula, the corollary is

$$P_x = \frac{x_1 x_2 \tau}{x_2^2 \kappa^4 + \left(x_1 + x_2 \frac{\kappa'}{\kappa}\right)^2 + x_2^2 \tau^2}. \tag{16}$$

Theorem 3.2.1.2. Let $\gamma: I \times R \to R^3$ be a ruled surface given by $\gamma(s,v) = \lambda(s) + vX(s)$ in the Euclidean 3-space. The necessary and sufficient condition for this surface to be developable is to satisfy the equalities $x_1 = 0$, $x_2 = 0$ or $\tau = 0$.

Proof. The necessary and sufficient condition for a ruled surface to be developable is that its parameter is zero. When $P_X = 0$ and $\kappa, x_1, x_2 \neq 0$ in equation (16), the proof will be completed.

Theorem 3.2.1.3. Let $\gamma : I \times R \rightarrow R^3$ be a ruled surface given by $\gamma(s,v) = \lambda(s) + vX(s)$ in the Euclidean 3-space. If the base curve λ of the surface γ is the striction line at the same time; the condition $x_2 = 0$ is satisfied where $\kappa \neq 0$.

Proof. When the value $x_2 = 0$ is substituted in equation (6), the proof will be completed.

Theorem 3.2.1.4. Let $\gamma_{\{e_1,e_2\}}$ be a ruled surface generated by the vector $Sp\{e_1,e_2\}$ of the base curve λ. The unit normal vector of the surface $\gamma_{\{e_1,e_2\}}$ is

$$U_{\{e_1,e_2\}} = \frac{\left(-vx_2^2\tau, -vx_1x_2\tau, -vx_2^2\kappa^2 - x_1 - vx_1^2 - vx_1x_2\dfrac{\kappa'}{\kappa}\right)}{\sqrt{v^2x_2^4\tau^2 + v^2x_1^2x_2^2\tau^2 + \left(vx_2^2\kappa^2 + x_1 + vx_1^2 + vx_1x_2\dfrac{\kappa'}{\kappa}\right)^2}}.$$

Proof. When the value $x_3 = 0$ is substituted in the equation (14), the unit normal vector for the special situation has been computed.

3.2.2 Ruled surfaces with the direction vector $X(s) = Sp\{e_1, e_3\}$

When the special situation $x_1^2 + x_3^2 = 1, x_2 = 0$ is taken in the general equation of the ruled surfaces with the modified orthogonal frame defined in equation (1), the equation of the ruled surface is strained by the vectors $\gamma(s,v) = \lambda(s) + v\left(x_1e_1 + x_3e_3\right)$, e_1 and e_3 is formed.

Theorem 3.2.2.1. The value will be $x_2 = 0$ in equation (5). When it is substituted in the formula of the distribution parameter, the result is

$$P_x = -\frac{x_1x_3\dfrac{\kappa'}{\kappa}}{\left(x_1 - x_3\tau\right)^2 + x_3^2\left(\dfrac{\kappa'}{\kappa}\right)^2}. \tag{17}$$

Theorem 3.2.2.2. Let $\gamma : I \times R \rightarrow R^3$ be a ruled surface given by $\gamma(s,v) = \lambda(s) + vX(s)$ in the Euclidean 3-space. Thus, the necessary and sufficient condition for this surface to be developable, the equations $x_1 = 0$, $x_3 = 0$ or κ=const. have to be satisfied.

Proof. The necessary and sufficient condition for a surface to be developable is that the distribution parameter is zero. When $P_X = 0$ where $\kappa \neq 0$ in equation (17), the proof will be completed.

Theorem 3.2.2.3. Let $\gamma_{\{e_1,e_3\}}$ be the ruled surface generated by the vector $Sp\{e_1,e_3\}$ of the base curve λ. The unit normal vector of the surface $\gamma_{\{e_1,e_3\}}$ is

$$U_{\{e_1,e_3\}} = \frac{\left(x_3 + vx_1x_3 - vx_3^2\tau, -vx_1x_3\frac{\kappa'}{\kappa}, -x_1 - vx_1^2 + vx_1x_3\tau\right)}{\sqrt{\left(x_3 + vx_1x_3 - vx_3^2\tau\right)^2 + v^2x_1^2x_3^2\left(\frac{\kappa'}{\kappa}\right)^2 + \left(-x_1 - vx_1^2 + vx_1x_3\tau\right)^2}}.$$

Proof. When the value $x_2 = 0$ is substituted in the equation (14), the unit normal vector for the special situation has been computed.

3.2.3. The ruled surfaces with the direction vector $X(s) = Sp\{e_2, e_3\}$

When the special situation $x_2^2 + x_3^2 = 1,\ x_1 = 0$ is taken in the general equations of the ruled surfaces with the modified orthogonal frame defined in (1), the equation of the ruled surface $\gamma(s,v) = \lambda(s) + v\left(x_2e_2 + x_3e_3\right)$ is spanned by the vectors e_2 and e_3 is formed.

Theorem 3.2.3.1. The value will be $x_1 = 0$ in equation (5). When this denotation is substituted in the distribution parameter, the result is

$$P_x = -\frac{x_2x_3\kappa^2}{x_2^2\kappa^4 + \left(x_2\frac{\kappa'}{\kappa} - x_3\tau\right)^2 + \left(x_2\tau + x_3\frac{\kappa'}{\kappa}\right)^2}. \tag{18}$$

Theorem 3.2.3.2. Let $\gamma : I \times R \to R^3$ be a ruled surface given by $\gamma(s,v) = \lambda(s) + vX(s)$ in the Euclidean 3-space. The necessary and sufficient condition for this surface to be developable is to satisfy the equations $x_2 = 0$, $x_3 = 0$ or κ=const..

Proof. The necessary and sufficient condition for a surface to be developable is that the distribution parameter is zero. When $P_X = 0$ where $\kappa \neq 0$ in equation (18), the proof becomes completed.

Theorem 3.2.3.3. Let $\gamma : I \times R \to R^3$ be a ruled surface parametrized by $\gamma(s,v) = \lambda(s) + vX(s)$ in the Euclidean 3-space. If the base curve λ of the surface γ is a striction line at the same time, the condition $x_2 = 0$ is satisfied.

Proof. When the values $x_2 = 0$ or $\kappa = 0$ are substituted in equation (6), the proof will be satisfied.

Theorem 3.2.3.4. Let $\gamma_{\{e_2,e_3\}}$ be the ruled surface generated by the vector $Sp\{e_2,e_3\}$ of the base curve λ. The unit normal vector of the surface $\gamma_{\{e_2,e_3\}}$ is

$$U_{\{e_2,e_3\}} = \frac{\left(x_3 - vx_3^2\tau - vx_2^2\tau, -vx_2x_3\kappa^2, -vx_2^2\kappa^2\right)}{\sqrt{\left(x_3 - vx_3^2\tau - vx_2^2\tau\right)^2 + v^2x_2^2x_3^2\kappa^4 + v^2x_2^4\kappa^4}}.$$

Proof. If the value $x_1 = 0$ is substituted in the equation (14), the unit normal vector for the special situation has been computed.

Incase $X(s) = e_1$, it was not analyzed because there were no significiant results.

3.2.4. The ruled surfaces with the direction vector $X(s) = e_2$

If the special situation $x_1 = x_3 = 0$ of the ruled surfaces with the modified orthogonal frame defined in equation (1) is taken, the equation of the ruled surface $\gamma(s,v) = \lambda(s) + vx_2e_2$ is formed.

Theorem 3.2.4.1. Let $\gamma(s,v)$ be a ruled surface with the direction $X = e_2$. The condition $x_1 = x_3 = 0$ is obtained by the distribution parameter equation of the ruled surface $\gamma(s,v)$

$$P_x = \frac{0}{x_2^2\kappa^4 + x_2^2\left(\frac{\kappa'}{\kappa}\right)^2 + x_2^2\tau^2} = 0 \tag{19}$$

in equation (5). That the distribution parameter is zero will show that this ruled surface is developable.

Corollary 3.2.4.2. The coefficients of the first fundamental form of the ruled surface $\gamma(s,v)$ with the direction $X = e_2$ are $E = v^2x_2^2\kappa^4 + \left(1 + vx_2\frac{\kappa'}{\kappa}\right)^2\kappa^2 + vx_2\kappa^2\tau^2,\ F = \kappa^2\left(x_2 + vx_2^2\frac{\kappa'}{\kappa}\right),\ G = x_2^2\kappa^2,$ and the coefficients of the second fundamental form are computed as $L = \frac{-v^2x_2^3\kappa^4\tau^2}{vx_2^2\sqrt{\kappa^4+\tau^2}},\quad M = \frac{-vx_2^2\kappa\kappa'}{vx_2^2\sqrt{\kappa^4+\tau^2}},$ $N = 0.$

Theorem 3.2.4.3. The unit normal vector of the ruled surface $\gamma(s,v)$ with the direction $X = e_2$ is $U_{e_2} = \dfrac{\left(-vx_2^2\tau, 0, -vx_2^2\kappa^2\right)}{\sqrt{v^2x_2^4\tau^2 + v^2x_2^4\kappa^4}}$.

Proof. If the value $x_1 = x_3 = 0$ is substituted in equation (14), the unit normal vector for the special situation has been computed.

3.2.5. The ruled surfaces with the direction $X(s) = e_3$

When the special situation $x_1 = x_2 = 0$ is taken in the general equation of the ruled surfaces with a modified orthogonal frame defined in equation (1), the ruled surface $\gamma(s,v) = \lambda(s) + vx_3e_3$ is formed.

Theorem 3.2.5.1. Let $\gamma(s,v)$ be a ruled surface with the direction $X = e_2$. The condition $x_1 = x_2 = 0$ is obtained by the distribution parameter $P_x = 0$ of the ruled surface $\gamma(s,v)$ from equation (5). That the distribution parameter is zero will show that this surface is developable.

Corollary 3.2.5.2. The coefficients of the first fundamental form of the ruled surface $\gamma(s,v)$ with the direction $X = e_3$ are $E = v^2x_3^2\kappa^2\tau^2 + v^2x_3^2\left(\kappa'\right)^2$, $F = vx_3^2\kappa\kappa'$, $G = x_3^2\kappa^2$, and the coefficients of the fundamental form are $L = \dfrac{\kappa^2\left(vx_3 - x_3\right) + \kappa^2\tau\left(-v^2x_3^3 + vx_3^2\right)}{x_3 - vx_3^2\tau}$, $M = \dfrac{0}{x_3 - vx_3^2\tau} = 0$, $N = 0.$

Theorem 3.2.5.3. The unit normal vector of the ruled surface $\gamma(s,v)$ with the direction $X = e_3$ is

$$U_{e_3} = \frac{\left(x_3 - vx_3^2\tau, 0, 0\right)}{\sqrt{\left(x_3 - vx_3^2\tau\right)^2}} = 1.$$

Proof. If the value $x_1 = x_2 = 0$ is substituted in equation (14), the unit normal vector for the special situation has been computed.

3.3 Ruled Surfaces with the Modified orthogonal frame formed by τ

In this section, a new ruled surface whose direction is formed by the elements of the modified orthogonal frame is defined and the differential properties such as distribution parameter, striction line, Gauss and mean curvature are given.

Let the curve $\alpha(s): I \subset \mathbb{R} \to E^3$ be a regular curve in Euclidean 3-space. The base curve $\lambda(s)$ defined by $\lambda(s) = e_1(s)$ is the tangent vector of a regular curve $\alpha(s)$. Thus, the ruled surface is given by $\gamma : I \times R \to R^3$, $(s,v) \to \gamma(s,v) = \lambda(s) + vX(s)$. When the direction vector

$X(s)=Sp\{e_1,e_2,e_3\}$ is chosen in a way that it will be formed by the elements of the modified orthogonal frame, where $\forall x_1,x_2,x_3\in R$, the direction vector $X(s)$ is;

$$\begin{aligned} &X(s)=x_1e_1+x_2e_2+x_3e_3, \qquad for \qquad (x_1^2+x_2^2+x_3^2=1)\\ &\gamma(s,v)=\lambda(s)+v\left(x_1e_1+x_2e_2+x_3e_3\right). \end{aligned} \tag{20}$$

If the first derivative of the direction vector is taken, the denotation

$$X'=-x_2\kappa\tau e_1+\left(x_1\frac{\kappa}{\tau}+x_2\frac{\tau'}{\tau}-x_3\tau\right)e_2+\left(x_2\tau+x_3\frac{\tau'}{\tau}\right)e_3 \tag{21}$$

is found. From the Definition 2.2, the tangent of the curve is

$$\lambda'(s)=e_1'=e_2 \tag{22}$$

and when equations (20), (21), and (22) are substituted in Definition 2.6, the distribution parameter $\lambda(s)$ is found by the equation

$$P_x=\frac{x_1x_2\tau+x_1x_3\dfrac{\tau'}{\tau}+x_2x_3\kappa\tau}{x_2^2\kappa^2\tau^2+\left(x_1\dfrac{\kappa}{\tau}+x_2\dfrac{\tau'}{\tau}-x_3\tau\right)^2+\left(x_2\tau+x_3\dfrac{\tau'}{\tau}\right)^2}. \tag{23}$$

The position vector of the striction line $\bar{\lambda}$ from Definition 2.7 is obtained by

$$\bar{\alpha}=\alpha+\frac{x_1x_2\kappa\tau e_1+x_2^2\kappa\tau e_2+x_2x_3\kappa\tau e_3}{x_2\kappa^2\tau^2+\left(x_1\dfrac{\kappa}{\tau}+x_2\dfrac{\tau'}{\tau}-x_3\tau\right)^2+\left(x_2\tau+x_3\dfrac{\tau'}{\tau}\right)^2}. \tag{24}$$

The first fundamental form of the ruled surface can be given as follows:

$$\begin{aligned} \gamma_s=&-vx_2\kappa\tau e_1+\left(\frac{\kappa}{\tau}+vx_1\frac{\kappa}{\tau}+vx_2\frac{\tau'}{\tau}-vx_3\tau\right)e_2\\ &+\left(vx_2\tau+vx_3\frac{\tau'}{\tau}\right)e_3+\left(vx_2\tau+vx_3\frac{\kappa'}{\kappa}\right)e_3 \end{aligned} \tag{25}$$

$$\gamma_v=x_1e_1+x_2e_2+x_3e_3. \tag{26}$$

When the corollaries obtained in equations (25) and (26) are substituted in equation (8), the coefficients

$$E = v^2\left(x_2^2+x_3^2\right)\left(\tau^4+\left(\tau'\right)^2\right)+v^2x_2^2\kappa^2\tau^2+\kappa^2\left(1+2vx_1+v^2x_1^2\right)+2v\kappa'\left(x_2+vx_1x_2\right)-2v\kappa\tau^2\left(x_3+vx_1x_2\right)$$
$$F = x_2\kappa\tau+vx_2^2\tau\tau'+vx_3^2\tau\tau',\quad G = x_1^2+\tau^2\left(x_2^2+x_3^2\right) \tag{27}$$

are obtained. Again, the coefficients for the second fundamental form of the surface can be computed by the equations $L=\dfrac{\det\left[\gamma_{ss},\gamma_s,\gamma_v\right]}{\left\|\gamma_s\times\gamma_v\right\|}$, $M=\dfrac{\det\left[\gamma_{sv},\gamma_s,\gamma_v\right]}{\left\|\gamma_s\times\gamma_v\right\|}$, $N=\dfrac{\det\left[\gamma_{vv},\gamma_s,\gamma_v\right]}{\left\|\gamma_s\times\gamma_v\right\|}$.

The partial derivatives in these equations are obtained as

$$\gamma_{ss}=\left(-v\left(x_2k_1'+x_3k_2'+x_1\left(k_1^2+k_2^2\right)\right)\right)+\left(k_1\left(1-vx_2k_1-vx_3k_2\right)+vx_1k_1\right)N_1+\left(k_2\left(1-vx_2k_1-vx_3k_2\right)+vx_1k_2'\right)N_2$$
$$\gamma_{sv}=-\left(x_2k_1+x_3k_2\right)N+x_1k_1N_1+x_1k_2N_2,\qquad \gamma_{vv}=0. \tag{28}$$

In addition, let the normal vector field of the surface be $n=\dfrac{\gamma_s\times\gamma_v}{\left\|\gamma_s\times\gamma_v\right\|}$. Then, when the derivatives in equations (25) and (26) are substituted in the equations, the normal vector of the surface is obtained as

$$n=\frac{\left[\begin{array}{l}\left(x_3\dfrac{\kappa}{\tau}+vx_1x_3\dfrac{\kappa}{\tau}-vx_3^2\tau+vx_2^2\tau\right)e_1+\left(vx_2x_3\kappa\tau+vx_1x_2\tau+vx_1x_3\dfrac{\tau'}{\tau}\right)e_2\\+\left(vx_1x_3\tau-vx_2^2\kappa\tau-x_1\dfrac{\kappa}{\tau}-vx_1^2\dfrac{\kappa}{\tau}-vx_1x_2\dfrac{\tau'}{\tau}\right)^2\end{array}\right]}{\sqrt{\begin{array}{l}\left(x_3\dfrac{\kappa}{\tau}+vx_1x_3\dfrac{\kappa}{\tau}-vx_3^2\tau+vx_2^2\tau\right)^2+\left(vx_2x_3\kappa\tau+vx_1x_2\tau+vx_1x_3\dfrac{\tau'}{\tau}\right)^2\\+\left(vx_1x_3\tau-vx_2^2\kappa\tau-x_1\dfrac{\kappa}{\tau}-vx_1^2\dfrac{\kappa}{\tau}-vx_1x_2\dfrac{\tau'}{\tau}\right)^2\end{array}}}. \tag{29}$$

Thus, the coefficients of the second form are computed as

$$L=\frac{\left[\begin{array}{l}\kappa^2\tau\left(2vx_3^2+v^2x_1x_3^2+vx_1x_3^2+v^2x_1x_2^2\right)-\dfrac{\kappa}{\tau}\left(x_3+vx_1x_3\right)-vx_2x_3\kappa^2\dfrac{\tau'}{\tau}-vx_1x_3\left(1+v\right)\dfrac{\kappa^3}{\tau}\\+\kappa\tau\tau'\left(4v^2x_2x_3^2-v^2x_1x_2^2\right)-2v^2x_2^2x_3\dfrac{\kappa\left(\tau'\right)^2}{\tau}+\kappa\tau^3\left(2v^2x_2^2x_3-v^2x_3^3\right)+vx_1x_2x_3\tau\left(\tau'\right)^2\\+\kappa^2\tau\tau'\left(-v^2x_2x_3^2-vx_2x_3\right)+\dfrac{\kappa\tau'}{\tau}\left(-3vx_1x_2-2v^2x_1^2x_2-v^2x_1^2-vx_1^2-v^2x_1^3\right)\\+\dfrac{\left(\tau'\right)^2}{\tau}\left(3v^2x_1x_3^2-2v^2x_1x_2^2-v^2x_1^2x_2\right)+\tau\tau'\left(5v^2x_1x_2x_3+vx_1x_2x_3+v^2x_1^2x_3\right)\\+v^2x_1\tau^3\left(x_2^2-x_3^2\right)-2vx_1x_3\dfrac{\kappa\left(\tau'\right)^2}{\tau^3}\left(1+vx_1\right)+\left(\dfrac{\tau'}{\tau}\right)^3\left(-v^2x_1x_2x_3-vx_1x_2x_3\right)\\+\dfrac{\kappa^2}{\tau}\left(-x_1-2vx_1^2-v^2x_1^3\right)+2vx_1x_3\kappa\tau\left(1+vx_1\right)\end{array}\right]}{\sqrt{\begin{array}{l}\left(x_3\dfrac{\kappa}{\tau}+vx_1x_3\dfrac{\kappa}{\tau}-vx_3^2\tau+vx_2^2\tau\right)^2+\left(vx_2x_3\kappa\tau+vx_1x_2\tau+vx_1x_3\dfrac{\tau'}{\tau}\right)^2\\+\left(vx_1x_3\tau-vx_2^2\kappa\tau-x_1\dfrac{\kappa}{\tau}-vx_1^2\dfrac{\kappa}{\tau}-vx_1x_2\dfrac{\tau'}{\tau}\right)^2\end{array}}}$$

$$M=\frac{\begin{bmatrix}\kappa^2(vx_1^2x_2-x_2x_3)+\kappa\tau'\left(vx_1^2x_3+vx_1x_2^2\right)-vx_1x_2x_3\kappa\tau^2-vx_1x_2^2\tau'\\+\tau^2x_1x_2x_3+\kappa\left(-x_1x_2-vx_1^2x_2\right)+\frac{\kappa\tau'}{\tau^2}\left(-x_1x_3-vx_1^2x_3\right)\end{bmatrix}}{\sqrt{\begin{matrix}\left(x_3\frac{\kappa}{\tau}+vx_1x_3\frac{\kappa}{\tau}-vx_3^2\tau+vx_2^2\tau\right)^2+\left(vx_2x_3\kappa\tau+vx_1x_2\tau+vx_1x_3\frac{\tau'}{\tau}\right)^2\\+\left(vx_1x_3\tau-vx_2^2\kappa\tau-x_1\frac{\kappa}{\tau}-vx_1^2\frac{\kappa}{\tau}-vx_1x_2\frac{\tau'}{\tau}\right)^2\end{matrix}}},\ N=0. \tag{30}$$

Now, a couple of theorems will be considered in the direction of these elements.

Theorem 3.3.1. Let $\gamma: I\times R\rightarrow R^3$ be a ruled surface given by $\gamma(s,v)=\lambda(s)+vX(s)$ in the Euclidean 3-space. The necessary and sufficient condition for this surface to be developable is that the equality $\kappa=-\left(\frac{x_1}{x_3}+\frac{x_1}{x_2}\frac{\tau'}{\tau^2}\right)$ is satisfied.

Proof. The necessary and sufficient condition for a surface to be developable is that the distribution parameter is zero. When $P_X=0$ where $\tau\neq 0$ in (23), the equation $\kappa=-\left(\frac{x_1}{x_3}+\frac{x_1}{x_2}\frac{\tau'}{\tau^2}\right)$ is obtained.

Theorem 3.3.2. Let $\gamma: I\times R\rightarrow R^3$ be a ruled surface that is parametrized by $\gamma(s,v)=\lambda(s)+vX(s)$ in the Euclidean 3-space. If the base curve λ of the surface γ is a striction line at the same time, the condition $x_2=0$ or $\kappa=0$ is satisfied where $\tau\neq 0$.

Proof. For the base curve λ to be the striction line, the equation has to be $\left\langle\frac{d\lambda}{ds},X'\right\rangle=0$ in Definition 2.7. Therefore, when the values $x_2=0$ or $\kappa=0$ are substituted in equation (24), the proof will be completed.

3.4.The Characterization of the Ruled Surfaces with the Modified orthogonal frame in Spacial Situations

3.4.1. *The ruled surfaces with the direction vector* $X(s)=Sp\{e_1,e_2\}$

When the special situation $x_1^2+x_2^2=1, x_3=0$ is taken in the general equation of the ruled surfaces with the modified orthogonal frame defined in equation (20), the ruled surface $\gamma(s,v)=\lambda(s)+v\left(x_1e_1+x_2e_2\right)$ strained by the vectors e_1 and e_2 is formed.

Theorem 3.4.1.1. The value in equation (23) will be $x_3=0$. If this denotation is substituted in the formula of the distribution parameter, the result is

$$P_x = \frac{x_1 x_2 \tau}{x_2^2 \kappa^2 \tau^2 + \left(x_1 \frac{\kappa}{\tau} + x_2 \frac{\tau'}{\tau} \right)^2 + x_2^2 \tau^2}. \tag{31}$$

Theorem 3.4.1.2. Let $\gamma : I \times R \to R^3$ be a ruled surface given by $\gamma(s,v) = \lambda(s) + vX(s)$ in the Euclidean 3-space. The necessary and sufficient condition for this surface to be developable is to satisfy the equations $x_1 = 0$ or $x_2 = 0$ where $\tau \neq 0$.

Proof. The necessary and sufficient condition for a ruled surface to be developable is that the distribution parameter is zero. When $P_X = 0$ in equation (31) where $\tau \neq 0$, the proof becomes completed.

Theorem 3.4.1.3. Let $\gamma : I \times R \to R^3$ be a ruled surface that is parametrized by $\gamma(s,v) = \lambda(s) + vX(s)$ in the Euclidean 3-space. If the base curve λ of the surface γ is a striction line at the same time, the condition $x_2 = 0$ or $\kappa = 0$ where $\tau \neq 0$ is satisfied.

Proof. When the values $x_2 = 0$ or $\kappa = 0$ are substituted in equation (24), the proof will be satisfied.

Theorem 3.4.1.4. Let $\gamma_{\{e_1,e_2\}}$ be a ruled surface of the base curve λ generated by the vector $Sp\{e_1,e_2\}$. The unit normal vector of the surface $\gamma_{\{e_1,e_2\}}$ is

$$U_{\{e_1,e_2\}} = \frac{v x_2^2 \tau e_1 + v x_1 x_2 \tau e_2 + \left(-v x_2^2 \kappa \tau - x_1 \frac{\kappa}{\tau} - v x_1^2 \frac{\kappa}{\tau} - v x_1 x_2 \frac{\tau'}{\tau} \right) e_3}{\sqrt{v^2 x_2^4 \tau^2 + v^2 x_1^2 x_2^2 \tau^2 + \left(v x_2^2 \kappa \tau + x_1 \frac{\kappa}{\tau} + v x_1^2 \frac{\kappa}{\tau} + v x_1 x_2 \frac{\tau'}{\tau} \right)^2}}.$$

Proof. When the value $x_3 = 0$ is substituted in equation (29), the unit normal vector for the special situation becomes computed.

3.4.2 The ruled surfaces with the direction vector $X(s) = Sp\{e_1, e_3\}$

When the special situation $x_1^2 + x_3^2 = 1,\ x_2 = 0$ is taken in the general equation of the ruled surfaces with the modified orthogonal frame defined in (20), the ruled surfaces $\gamma(s,v) = \lambda(s) + v(x_1 e_1 + x_3 e_3)$ is spanned by the vectors e_1 and e_3 is formed.

Theorem 3.4.2.1. The value in equation (23) will be $x_2 = 0$. When this value is substituted in the formula of the distribution parameter, the result is

$$P_x = -\frac{x_1 x_3 \dfrac{\tau'}{\tau}}{\left(x_1 \dfrac{\kappa}{\tau} - x_3 \tau\right)^2 + x_3^2 \left(\dfrac{\tau'}{\tau}\right)^2}. \tag{32}$$

Theorem 3.4.2.2. Let $\gamma : I \times R \to R^3$ be a ruled surface given by $\gamma(s,v) = \lambda(s) + vX(s)$ in the Euclidean 3-space. The necessary and sufficient condition for this surface to be developable is to satisfy the equations $x_1 = 0$ or $x_3 = 0$ where $\tau \neq 0$.

Proof. The necessary and sufficient condition for this surface to be developable is that the distribution parameter is zero. When $P_X = 0$ in equation (32), the proof will be completed.

Theorem 3.4.2.3. Let $\gamma_{\{e_1,e_3\}}$ be a ruled surface of the base curve λ formed by the vector $Sp\left\{e_1, e_3\right\}$. Thus, the unit normal vector of the surface $\gamma_{\{e_1,e_3\}}$ is

$$U_{\{e_1,e_3\}} = \frac{\left(x_3 \dfrac{\kappa}{\tau} + v x_1 x_3 \dfrac{\kappa}{\tau} - v x_3^2 \tau, v x_1 x_3 \dfrac{\tau'}{\tau}, -x_1 \dfrac{\kappa}{\tau} - v x_1^2 \dfrac{\kappa}{\tau} + v x_1 x_3 \tau\right)}{\sqrt{\left(x_3 \dfrac{\kappa}{\tau} + v x_1 x_3 \dfrac{\kappa}{\tau} - v x_3^2 \tau\right)^2 + v^2 x_1^2 x_3^2 \left(\dfrac{\tau'}{\tau}\right)^2 + \left(-x_1 \dfrac{\kappa}{\tau} - v x_1^2 \dfrac{\kappa}{\tau} + v x_1 x_3 \tau\right)^2}}.$$

Proof. When the value $x_2 = 0$ is substituted in equation (29), the unit normal vector for the special situation becomes computed.

3.4.3. The ruled surfaces with the direction $X(s) = Sp\left\{e_2, e_3\right\}$

When the special situation $x_2^2 + x_3^2 = 1, x_1 = 0$ is taken in the general equation of the ruled surfaces with the modified orthogonal frame defined in (20), the ruled surface $\gamma(s,v) = \lambda(s) + v\left(x_2 e_2 + x_3 e_3\right)$ is spanned by the vectors e_2 and e_3 occurs.

Theorem 3.4.3.1. The value in equation (23) will be $x_1 = 0$**.** When this denotation is substituted in the formula of the distribution parameter, the result is

$$P_x = -\frac{x_2 x_3 \kappa \tau}{x_2^2 \kappa^2 \tau^2 + \left(x_2 \dfrac{\tau'}{\tau} - x_3 \tau\right)^2 + \left(x_2 \tau + x_3 \dfrac{\tau'}{\tau}\right)^2}. \tag{33}$$

Theorem 3.4.3.2. Let $\gamma : I \times R \to R^3$ be a ruled surface given by $\gamma(s,v) = \lambda(s) + vX(s)$ in the Euclidean 3-space. The necessary and sufficient condition for this surface to be developable is to satisfy the equalities $x_2 = 0$, $x_3 = 0$ or $\kappa \neq 0$.

Proof. The necessary and sufficient condition for this surface to be developable is that the distribution parameter is zero. When $P_X = 0$ in equation (33) where $\tau \neq 0$, the proof will be completed.

Theorem 3.4.3.3. Let $\gamma : I \times R \to R^3$ be a ruled surface parametrized by $\gamma(s,v) = \lambda(s) + vX(s)$ in the Euclidean 3-space. If the base curve λ of the surface γ is a striction curve at the same time, the condition $\kappa = 0$ or $x_2 = 0$ is satisfied.

Proof. When the value $\kappa = 0$ or $x_2 = 0$ is substituted in equation (24), the proof will be completed.

Theorem 3.4.3.4. Let $\gamma_{\{e_2,e_3\}}$ be a ruled surface of the base curve λ formed by the vector $Sp\left\{e_2, e_3\right\}$. The unit normal vector of the surface $\gamma_{\{e_2,e_3\}}$ is

$$U_{\{e_2,e_3\}} = \frac{\left[\left(x_3\frac{\kappa}{\tau} + vx_2x_3\frac{\tau'}{\tau} - vx_3^2\tau + vx_2^2\tau - vx_2x_3\frac{\tau'}{\tau}\right)e_1 + vx_2x_3\kappa\tau e_2 - \left(x_1\frac{\kappa}{\tau} + vx_1^2\frac{\kappa}{\tau}\right)e_3\right]}{\sqrt{\left(x_3\frac{\kappa}{\tau} + vx_2x_3\frac{\tau'}{\tau} - vx_3^2\tau + vx_2^2\tau - vx_2x_3\frac{\tau'}{\tau}\right)^2 + v^2x_2^2x_3^2\kappa^2\tau^2 + \left(x_1\frac{\kappa}{\tau} + vx_1^2\frac{\kappa}{\tau}\right)^2}}.$$

Proof. When the value $x_1 = 0$ is substituted in equation (29), the unit normal vector for the special situation becomes computed.

The case $X(s) = e_1$ was not analyzed because there were no significant results.

3.4.4. The ruled surfaces with the direction vector $X(s) = e_2$

If the general situation $x_1 = x_3 = 0$ is taken in the general equation of the ruled surfaces with a modified orthogonal frame defined in equation (20), the ruled surface $\gamma(s,v) = \lambda(s) + vx_2e_2$ is formed.

Theorem 3.4.4.1. Let $\gamma(s,v)$ be a ruled surface with the direction $X = e_2$. The distribution parameter of the ruled surface $\gamma(s,v)$ for the condition $x_1 = x_3 = 0$ in (21) is equal to zero. That the distribution parameter is zero will show that this ruled surface is developable.

Corollary 3.4.4.2. The coefficients of the first fundamental form of the ruled surface $\gamma(s,v)$ with direction $X = e_2$ are $E = x_2^2\left(\tau^4 + (\tau')^2\right)v^2 + v^2x_2^2\kappa^2\tau^2 + 2vx_2\kappa\tau'$, $F = x_2\kappa\tau + vx_2^2\tau\tau'$, $G = x_2^2\kappa^2$, and the coefficients of the second fundamental form are computed as

$$L = \frac{-v^2 x_2^3 \kappa^4 \tau^2}{\sqrt{v^2 x_2^4 \tau^2 + v^2 x_2^4 \kappa^2 \tau^2}}, \; M = \frac{-v x_2^2 \kappa \kappa'}{\sqrt{v^2 x_2^4 \tau^2 + v^2 x_2^4 \kappa^2 \tau^2}}, \; N = 0.$$

Theorem 3.4.4.3. The unit normal vector of the ruled surface $\gamma(s,v)$ with the direction $X = e_2$ is $U_{e_2} = \dfrac{\left(-v x_2^2 \tau, 0, -v x_2^2 \kappa \tau\right)}{\sqrt{v^2 x_2^4 \tau^2 + v^2 x_2^4 \kappa^2 \tau^2}}$.

Proof. If the value $x_1 = x_3 = 0$ is substituted in equation (29), the unit normal vector for the special situation will be computed.

3.4.5. The ruled surfaces with the direction vector $X(s) = e_3$

When the special situation $x_1 = x_2 = 0$ is taken in the general equation of the ruled surfaces with the direction with $X = e_3$ using the modified orthogonal frame defined in (20), the ruled surface equation will be $\gamma(s,v) = \lambda(s) + v x_3 e_3$.

Theorem 3.4.5.1. Let $\gamma(s,v)$ be a ruled surface with the direction $X = e_2$. The condition $x_1 = x_3 = 0$ is obtained by the equation of distribution parameter $P_x = 0$ of the ruled surface $\gamma(s,v)$ in equation (23). The distribution parameter is zero will show that this ruled surface is a developable ruled surface.

Corollary 3.4.5.2. The coefficients of the first fundamental form of the ruled surface $\gamma(s,v)$ with the direction $X = e_3$ are $E = v^2 x_3^2 \left(\tau^4 + (\tau')^2\right) - 2 v x_3 \kappa \tau^2$, $F = v x_3^2 \tau \tau'$, $G = x_3^2 \tau^2$, and the coefficients of the second fundamental form are computed as $L = \dfrac{\kappa^2 (v x_3 - x_3) + \kappa^2 \tau \left(-v^2 x_3^3 + v x_3^2\right)}{x_3 - v x_3^2 \tau}$, $M = 0$, $N = 0$.

Theorem 3.4.5.3. The unit normal vector of the ruled surface $\gamma(s,v)$ with the direction $X = e_3$ is

$$U_{e_3} = \frac{\left(x_3 \dfrac{\kappa}{\tau} - v x_3^2 \tau, 0, 0\right)}{\sqrt{\left(x_3 - v x_3^2 \tau\right)^2}} = 1.$$

Proof. When the value $x_1 = x_2 = 0$ is substituted in equation (29), the unit normal vector for the special situation is computed.

4.Numerical Examples

Example 4.1. Let's find the ruled surface $\alpha(s)=(\sin(s)\cos(s),s,\sin(s))$ with a modified orthogonal frame of the space curve and then compute the distribution parameter, striction line, unit normal vector, and the Gauss and mean curvature of this ruled surface. When $\|\alpha'(s)\|\neq 1$, the curve $\alpha(s)$ is a curve with non unit speed. Thus, the elements of the Serret-Frenet frame of this curve are

$$T=\left(\frac{\cos(2s)}{\sqrt{4\cos(s)^4-3\cos(s)^2+2}},\frac{1}{\sqrt{4\cos(s)^4-3\cos(s)^2+2}},\frac{\cos(s)}{\sqrt{4\cos(s)^4-3\cos(s)^2+2}}\right),$$

$$N=\left(\begin{array}{l}-\dfrac{\left(\cos(s)^2+\frac{5}{2}\right)\sqrt{2}\sin(s)\cos(s)}{\sqrt{4\cos(s)^4-3\cos(s)^2+2}\sqrt{\sin(s)^2\left(2\cos(s)^4+10\cos(s)^2+1\right)}},\dfrac{\sin(s)\cos(s)\sqrt{2}\left(8\cos(s)^2-3\right)}{2\sqrt{\sin(s)^2\left(2\cos(s)^4+10\cos(s)^2+1\right)}\sqrt{4\cos(s)^4-3\cos(s)^2+2}},\\ \dfrac{\sqrt{2}\left(2\cos(s)^4-1\right)\sin(s)}{\sqrt{\sin(s)^2\left(2\cos(s)^4+10\cos(s)^2+1\right)}\sqrt{4\cos(s)^4-3\cos(s)^2+2}}\end{array}\right)$$

$$B=\left(-\frac{\sqrt{2}\sin(s)}{2\sqrt{\sin(s)^2\left(2\cos(s)^4+10\cos(s)^2+1\right)}},-\frac{\sqrt{2}\sin(s)\left(\cos(s)^2+\frac{1}{2}\right)}{\sqrt{\sin(s)^2\left(2\cos(s)^4+10\cos(s)^2+1\right)}},\frac{2\sqrt{2}\sin(s)\cos(s)}{\sqrt{\sin(s)^2\left(2\cos(s)^4+10\cos(s)^2+1\right)}}\right)$$

The curvature and the torsion are $\kappa=\dfrac{\sin(s)^2\sqrt{2}\left(2\cos(s)^4+10\cos(s)^2+1\right)}{\left(4\cos(s)^4-3\cos(s)^2+2\right)\sqrt{\sin(s)^2\left(2\cos(s)^4+10\cos(s)^2+1\right)}}$,

$\tau=-\dfrac{2\sin(s)\sqrt{4\cos(s)^4-3\cos(s)^2+2}}{2\cos(s)^4+10\cos(s)^2+1}$. Therefore, the elements of the modified orthogonal frame by using curvature are

$$e_1=\frac{\left(\cos(2s),1,\cos(s)\right)}{\sqrt{4\cos(s)^4-3\cos(s)^2+2}},$$

$$e_2=\frac{\left(-\left(\cos(s)^2+\frac{5}{2}\right)\sin(2s),\frac{1}{2}\sin(2s)\left(8\cos(s)^2-3\right),2\sin(2s)\cos(s)^3-2\sin(s)\right)}{\left(4\cos(s)^4-3\cos(s)^2+2\right)^{\frac{3}{2}}}$$

$$e_3=\frac{\left(\sin(s),-2\cos(s)^2\sin(s)-\sin(s),\sin(2s)\right)}{-4\cos(s)^4+3\cos(s)^2-2}$$

Now, let's give the ruled surface

$$\gamma(s,v)=e_1(s)+v(e_1(s)+e_3(s))$$

$$=\frac{\begin{pmatrix}\left((2v+2)\cos(s)^2-v-1\right)\sqrt{4\cos(s)^4-3\cos(s)^2+2}-\sin(s)v,\\ (v+1)\sqrt{4\cos(s)^4-3\cos(s)^2+2}-2v\sin(s)\left(\cos(s)^2+\frac{1}{2}\right),\\ \left(\sqrt{4\cos(s)^4-3\cos(s)^2+2}\,v+4\sin(s)v+\sqrt{4\cos(s)^4-3\cos(s)^2+2}\right)\cos(s)\end{pmatrix}}{4\cos(s)^4-3\cos(s)^2+2}.$$

The Gauss and mean curvatures are computed by $K\neq 0$ and $H\neq 0$.

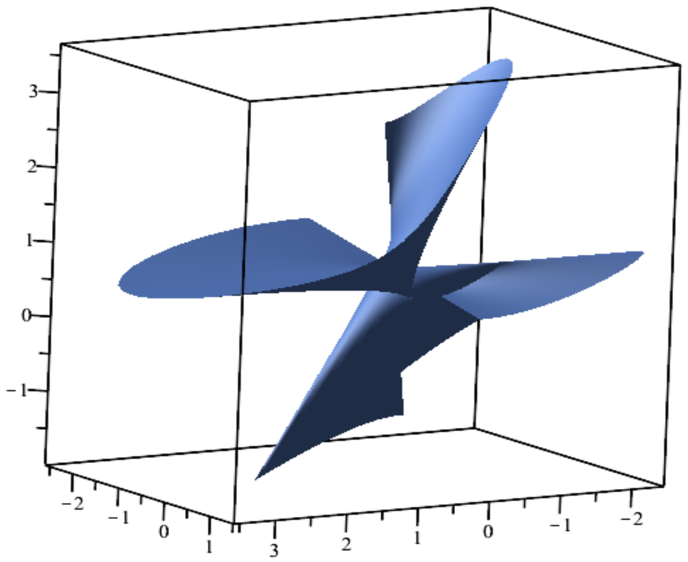

Figure 1 The ruled surface for the curve $\alpha(s)=(\sin(s)\cos(s),s,\sin(s))$

Example 4.2. Let's compute the ruled surface $\alpha(s)=\left(\frac{1}{s},s^3,s^2\right)$ with a modified orthogonal frame of the space curve and then compute the distribution parameter, striction line, unit normal vector, and the Gauss and mean curvature of this ruled surface.

When $\|\alpha'(s)\|\neq 1$, the curve $\alpha(s)$ is not a curve with unit speed. Thus, the elements of the Serret-Frenet frame of this curve are $T=\frac{\left(-1/s^2,3s^2,2s\right)}{\sqrt{9s^4+4s^2+1}}$, $N=\frac{\left(6s+2s^{-1},2s^3+2s^{-3},-3s^4+s^{-4}\right)}{\sqrt{s^4+4s^{-2}+s^{-4}}\sqrt{9s^4+4s^2+s^{-4}}}$

$B=\frac{\left(-s^2,s^{-2},-2s^{-1}\right)}{\sqrt{s^4+4s^{-2}+s^{-2}}}$. The curvature and the torsion are $\kappa=\frac{6(s^8+4s^2+1)}{(9s^8+4s^6+1)\sqrt{s^4+4s^{-2}+s^{-4}}}$,

and $\tau=\frac{18s^8+8s^6+2}{(s^8+4s^2+1)\sqrt{9s^8+4s^6+1}}$. Therefore, the elements of the modified orthogonal frame by using torsion are $e_1 = \frac{\left(-s^{-2}, 3s^2, 2s\right)}{\sqrt{9s^4 + 4s^2 + s^{-4}}}$, $e_2 = \frac{\left(12s^5 + 4s^2, 4s^7 + 4s, -6s^8 + 2\right)}{\left(s^8 + 4s^2 + 1\right)^{3/2}}$,

$$e_3 = \frac{\left(-18s^8 - 8s^6 - 2, 18s^4 + 8s^2 + 2s^{-4}, -36s^5 - 16s^3 - 4s^{-3}\right)}{\left(s^8 + 4s^2 + 1\right)\sqrt{\frac{s^8 + 4s^2 + 1}{s^4}}\sqrt{\frac{9s^8 + 4s^6 + 1}{s^4}}}.$$

Then, given the ruled surface $\gamma(s,v) = e_1(s) + ve_3(s)$ whose direction is $X = e_3$ and the base curve is λ by the equation

$$\gamma(s,v) = \frac{\left(-s^{-4}\left(s^8 + 4s^2 + 1\right)^{3/2} - 18v\left(s^8 + \frac{4}{9}s^6 + \frac{1}{9}\right), 3s^{-4}\left(s^8 + 4s^2 + 1\right)^{3/2} + 2\left(9s^4 + 4s^2 + s^{-4}\right)v, 2\left(s^{-1}\left(s^8 + 4s^2 + 1\right)^{3/2} - 18s^5 v - 8s^3 v - 2s^{-3} v\right)\right)}{\left(s^4 + 4s^{-2} + s^{-4}\right)\sqrt{s^8 + 4s^2 + 1}\sqrt{9s^8 + 4s^6 + 1}}.$$

The Gauss and mean curvatures are calculated by $K = 0$ and $H \neq 0$.

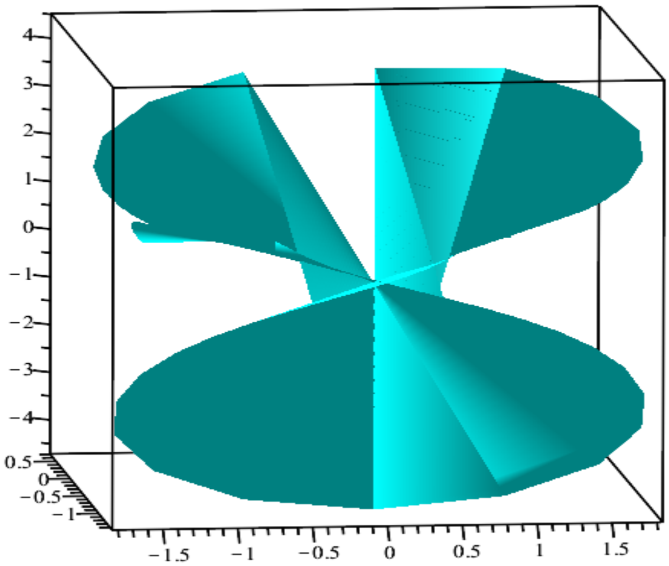

Figure 2 The ruled surface for the curve $\alpha(s) = \left(1/s, s^3, s^2\right)$

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

Author Contributions

The authors contributed equally. The first author performed all theorem’s proof and calculating the examples. The second author put forward the first idea and making computer programme for the examples.

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References

[1] Andaras, C., Recio, T., Tabera, L.F., Sendra, J.R. and Villarino, C., (2011). Proper real reparametrization of rational ruled surfaces. *Computer Aided Geometric Design*, 28(2), 102-113.

[2]Bottema, O., Roth, B., (1979). Theoretical Kinematics. North-Holland Publishing Company, New York, p.558.

[3] Hacısalihoğlu, H.H., (1983). Diferansiyel Geometri II, Gazi Üniversitesi Yayınları.

[4] Sasai T., (1984). The fundamental theorem of analytic space curves and apparent singularities of Fuchsian differential equations. *Tohoku Math. Journal*, 36, 17-24.

[5] Bukcu, B., Karacan, M.K., (2016). On the modified orthogonal frame with curvature and torsion in 3-space. *Mathematical Sciences and Applications E-notes*, 4(1), 184-186.

[6] Bukcu, B., Karacan, M.K., (2016). Spherical curves with modified orthogonal frame. *Journal of New Results in Science*, 60-68.

[7] Lone M.S., Karacan M.K., Tuncer Y., Es H., (2018). Spherical indicatries with the modified orthogonal frame. arXiv:1810.01082v1.

[8] Lone M.S., Karacan M.K., Tuncer Y., Es H., (2018). On some curves with modified orthogonal frame in Euclidean 3-space. *Iranian Journal of Science and Technology, Transactions A: Science*, *43*(4), 1905-1916.

[9] Eren K., Kosal H.H., (2020). Evolution of space curves and the special ruled surfaces with modified orthogonal frame. *AIMS Mathematics*, 5(3), 2027-2039.

[10] Akyigit, M., Eren K., and Kosal H.H., (2021). Tubular surfaces with modified orthogonal frame in Euclidean 3-space. *Honam Mathematical Journal*, 43(3), 453–463.

[11] Arıkan, M., Nurkan, S.M., (2020). Adjoint curve according to modified orthogonal frame with torsion in 3-space. *Usak University Journal of Science and Natural Sciences*, 2, 54-64.

[12] Damar. E., Yuksel, N., Vanlı, T.A., (2017). The ruled surfaces according to type-2 Bishop frame in E^3 . *International Mathematical Forum*, 12(3), 133-143.