Advances in the Theory of Nonlinear Analysis and its Applications 6 (2022) No. 4, 547–564.
https://doi.org/10.31197/atnaa.998275
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications
ISSN: 2587-2648
Peer-Reviewed Scientific Journal

Linearization-Discretization process to solve systems of nonlinear Fredholm integral equations in an infinite-dimensional context

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Abstract

In this paper, we propose a different way for solving systems of nonlinear Fredholm integral equations of the second kind. We construct our new strategy in two steps, through beginning with the linearization phase of the system of Fredholm integral equations by applying Newton method, then we pass to the discretization phase for some involved integral operator using Nyström method. The convergence analysis of our new method is proved under some necessary conditions. At last, a numerical application to approach a nonlinear Fredholm integro-differential equation by using this new process is taken to confirm its advantage.

Mathematics Subject Classification (2010): 45G15 45B05 49M15 65J15

Key words and phrases: System of nonlinear equations Fredholm integral equations Newton method Nyström method Integro-differential equation

1. Introduction

Fredholm integral equations emerge from the modeling of the Spatial Spread of an Epidemic, what's more, different physical and organic models [11]. For solving this kind of nonlinear problems, we use the

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Received September 21, 2021; Accepted: September 20, 2022; Online: October 9, 2022.

classical process which starts by discretizing the problem for finding a nonlinear algebraic system, at that point, we linearize the discrete nonlinear system using Newton's method. For instance, many works concerning the Newton method ([6, 12, 3, 4, 9]) and fixed point method ([8, 7]) have been performed in order to solve nonlinear integral equations.

In a recent paper [5], author constructs a very useful numerical process based on the inverse way of classical process, in order to approach a nonlinear Fredholm integral equation of the second kind. This new process starts with the linearization phase and goes on with the discretization phase. However, the numerical results of [5] confirm the high efficacy of this new process compared with the classical process.

In the present paper, we adapt the numerical method proposed in [5] for solving a nonlinear integral equation to solve a system of nonlinear Fredholm equations of the second kind defined in an infinite dimensional context given by the following form:

$$\begin{cases} u_1(t) = \displaystyle\int_0^1 \kappa_1\left(t, s, u_1(s), u_2(s), ..., u_N(s)\right) ds + g_1(t), \\ u_2(t) = \displaystyle\int_0^1 \kappa_2\left(t, s, u_1(s), u_2(s), ..., u_N(s)\right) ds + g_2(t), \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ u_N(t) = \displaystyle\int_0^1 \kappa_N\left(t, s, u_1(s), u_2(s), ..., u_N(s)\right) ds + g_N(t), \end{cases} \tag{1}$$

for all $t \in [0, 1]$, and a given functions g_i.

As well as, to show the effectiveness of this new process, we compare our results with an other existing results using the classical process. For this reason, we denote **option A** to refer our new strategy, and the classical process will be called **option B**. However, we describe the strategy of the both options A and B as follows:

Option A: We propose to begin with the linearization phase to the system (1) in its infinite-dimensional space context by applying the Newton method, then we go to the discretization phase by using Nyström method to approach some involved integral operators.

Option B: We start with the discretization phase on the system (1) by using Nyström method, which leads to a nonlinear finite-dimensional algebraic system, then we apply the classical Newton method on the obtained algebraic system.

The paper is organized as follows: In section 2, we present notations and preliminary results. In section 3, we describe the strategy of option A, and present all necessary propositions and conditions that will be used in the convergence analysis. Section 4, is devoted to the convergence analysis of option A, and section 5, to the convergence analysis of option B. In section 6, we show how to approach a nonlinear Fredholm integro-differential equation by applying option A. In section 7, we present numerical examples which confirm the effectiveness of our new process.

2. Notions and preliminary results

We consider for all $1 \leq i \leq N$ a real Banach spaces $\mathcal{X}_i = C^1([0, 1], \mathbb{R})$ and $\widetilde{\mathcal{X}} = \prod_{i=1}^{N} \mathcal{X}_i$, with Ω_i and $\widetilde{\Omega}$ be a nonempty open subsets of $\mathcal{X}_i$ and $\widetilde{\mathcal{X}}$ respectively. Let $||.||_{\mathcal{X}_i}$ be the norm of the Banach space $\mathcal{X}_i$, and $||.||_{\widetilde{\mathcal{X}}}$

be the norm of $\widetilde{\chi}$ such as

$$\forall Z = (z_1, z_2, ..., z_N) \in \widetilde{\chi}, ||Z||_{\widetilde{\chi}} = \sum_{i=1}^{N} ||z_i||_{\chi_i} = \sum_{i=1}^{N} \left(\|z_i\|_\infty + \|z_i'\|_\infty\right),$$

where $||.||_\infty$ is the norm of the uniform convergence represented as

$$||z_i||_\infty = \sup_{t\in[0,1]} |z_i(t)|, \quad z_i \in \chi_i.$$

We define a nonlinear Fréchet-differentiable operator $K_i : \widetilde{\Omega} \subset \widetilde{\chi} \to \chi_i$:

$$K_i\left(u_1, u_2, ..., u_N\right)(t) = \int_0^1 \kappa_i\left(t, s, u_1(s), u_2(s), ..., u_N(s)\right) ds, \quad u_i \in \Omega_i, \quad t \in [0,1].$$

For all $1 \leq i, j \leq N$, let $T_{ij} := \dfrac{\partial K_i}{\partial u_j}$ denote the Fréchet derivative of K_i associated to u_j, i.e, for all $V = (v_1, v_2, ..., v_N) \in \widetilde{\chi}$,

$$[T_{ij}(V)y_i](t) = \int_0^1 \frac{\partial \kappa_i}{\partial u_j}\left(t, s, V(s)\right) y_i(s) ds, \quad y_i \in \chi_i, \quad t \in [0,1].$$

The Nyström approximation $K_{n,i}$ of order n of the nonlinear operator K_i is given by

$$K_{n,i}(V)(t) = \sum_{p=1}^{n} \omega_{n,p} \kappa_i\left(t, t_p, V(t_p)\right), \quad V \in \widetilde{\chi}, \quad t \in [0,1].$$

The Nyström approximation $T_{n,ij}$ of order n of the linear operator T_{ij} is given by

$$\left[T_{n,ij}(V)y_i\right](t) = \sum_{p=1}^{n} \omega_{n,p} \frac{\partial \kappa_i}{\partial u_j}\left(t, t_p, V(t_p)\right) y_i(t_p), \quad x \in \widetilde{\chi}, \quad y_i \in \chi_i, \quad t \in [0,1].$$

In practice, the trapezoidal numerical integration rule (see[1], pp.109), for all $\kappa \in C^1([0,1]^2 \times \mathbb{R}^N, \mathbb{R})$, gives us the following convergence order: for all $t \in [0,1]$,

$$\left| \int_0^1 \kappa(t, s, U(s)) ds - \sum_{p=1}^{n} \omega_{n,p} \kappa(t, t_p, U(t_p)) \right| = \frac{1}{12n^2} \left| \left[\frac{\partial \kappa(t, s, U(s))}{\partial s} \right]_{s=0}^{s=1} \right| + O(h^4). \tag{2}$$

Now, by using previous notations, the system of nonlinear equations (1) can be rewritten as:

$$\begin{cases} u_1(t) = K_1\left(u_1(t), u_2(t), ...u_N(t)\right) + g_1(t), \\ u_2(t) = K_2\left(u_1(t), u_2(t), ...u_N(t)\right) + g_2(t), \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ u_N(t) = K_N\left(u_1(t), u_2(t), ...u_N(t)\right) + g_N(t), \end{cases} \tag{3}$$

for all $t \in [0,1]$ and a given functions $g_i \in \Omega_i$. As well as, the system (3) can take a clear and simple form as:

$$\text{Find} \quad U \in \widetilde{\Omega} \subset \widetilde{\chi} : \quad U = K(U) + G. \tag{4}$$

where $U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}, G = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{pmatrix}$ and $K = \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_N \end{pmatrix}$.

Let I_{NN} be the identity operator of the space $L(\widetilde{\mathcal{X}})$, where $L(\widetilde{\mathcal{X}})$ denotes the space of all linear boundes operators defined from $\widetilde{\mathcal{X}}$ into $\widetilde{\mathcal{X}}$. For all $\varphi \in \widetilde{\mathcal{X}}$, let $M_T(\varphi) \in L(\widetilde{\mathcal{X}})$ be the Fréchet derivative of the operator K that we give it as the following form:

$$\forall h \in \widetilde{\mathcal{X}}, \quad M_T(\varphi)h = \begin{pmatrix} T_{11}(\varphi) & T_{12}(\varphi) & \dots & T_{1N}(\varphi) \\ T_{21}(\varphi) & T_{22}(\varphi) & \dots & T_{2N}(\varphi) \\ \vdots & \vdots & \ddots & \vdots \\ T_{N1}(\varphi) & T_{N2}(\varphi) & \dots & T_{NN}(\varphi) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{N} T_{1j}(\varphi)h_j \\ \vdots \\ \sum_{j=1}^{N} T_{Nj}(\varphi)h_j \end{pmatrix}$$

where

$$||M_T(\varphi)h||_{\widetilde{\mathcal{X}}} = \sum_{i=1}^{N} \left\| \sum_{j=1}^{N} T_{ij}(\varphi)h_j \right\|_{\mathcal{X}_i}.$$

We suppose some conditions that will play an important role in the proof of the convergence analysis. For all $1 \leq i, j \leq N$, we assume that

$$\begin{cases} (i) \text{ Equation (4) has a unique solution } U = (u_1, u_2, ..., u_N) \in \widetilde{\mathcal{X}}, \\ (ii) \quad (I_{NN} - M_T(U)) \text{ is invertible, and } ||(I_{NN} - M_T(U))^{-1}|| \leq \eta < +\infty, \\ (iii) \quad \dfrac{\partial \kappa_i}{\partial u_j} \in C^2([0,1]^2 \times \mathbb{R}^N, \mathbb{R}), \\ (i\nu) \quad R = \sum_{i=1}^{N} R_i > 0 \text{ is such that } B_R(U) = \prod_{i=1}^{N} B_{R_i}(u_i) \subset \widetilde{\Omega}, \end{cases} \tag{5}$$

with $B_R(U)$ is the ball of center U and radius R for the norm $||.||_{\widetilde{\mathcal{X}}}$, and $B_{R_i}(u_i)$ is the ball of center u_i and radius $R_i > 0$ for the norm $||.||_{\mathcal{X}_i}$.

3. Description of new process (Option A)

We propose the Newton method to linearize equation (4) as a premier phase, by the following scheme:

$$\left(I_{NN} - M_T(U^{(k)})\right)\left(U^{(k+1)} - U^{(k)}\right) = -U^{(k)} + K\left(U^{(k)}\right) + G, \qquad U^{(0)} \in \widetilde{\mathcal{X}}, \quad k = 0, 1, \dots \quad . \tag{6}$$

In practise, we need to calculate $\left(I_{NN} - M_T(U^{(k)})\right)^{-1}$ in each iteration, but this operator cannot be found exactly. For this reason, we go to apply the discretization phase by using Nyström method in order to approximate the involved integral operators in the scheme (6).

Let $U_n^{(k)} = \left(U_{n,1}^{(k)}, U_{n,2}^{(k)}, ..., U_{n,N}^{(k)}\right) \in \widetilde{\mathcal{X}}$ be the approximation of $U^{(k)} = \left(U_1^{(k)}, U_2^{(k)}, ..., U_N^{(k)}\right) \in \widetilde{\mathcal{X}}$ obtained by Nyström method. So, the discretization of the scheme (6) is presented as follows

$$\left(I_{NN} - M_{T_n}(U_n^{(k)})\right)\left(U_n^{(k+1)} - U_n^{(k)}\right) = -U_n^{(k)} + K\left(U_n^{(k)}\right) + G, \qquad U_n^{(0)} \in \widetilde{\Omega}, \tag{7}$$

or in the matrix form

$$\begin{pmatrix} I - T_{n,11}(U_n^{(k)}) & -T_{n,12}(U_n^{(k)}) & \dots & -T_{n,1N}(U_n^{(k)}) \\ -T_{n,21}(U_n^{(k)}) & I - T_{n,22}(U_n^{(k)}) & \dots & -T_{n,2N}(U_n^{(k)}) \\ \vdots & \vdots & \ddots & \vdots \\ -T_{n,N1}(U_n^{(k)}) & -T_{n,N2}(U_n^{(k)}) & \dots & I - T_{n,NN}(U_n^{(k)}) \end{pmatrix} \begin{pmatrix} U_{n,1}^{(k+1)} - U_{n,1}^{(k)} \\ U_{n,2}^{(k+1)} - U_{n,2}^{(k)} \\ \vdots \\ U_{n,N}^{(k+1)} - U_{n,N}^{(k)} \end{pmatrix} =$$

$$
-\begin{pmatrix} U_{n,1}^{(k)} \\ U_{n,2}^{(k)} \\ \vdots \\ U_{n,N}^{(k)} \end{pmatrix} + \begin{pmatrix} K_1(U_n^{(k)}) \\ K_2(U_n^{(k)}) \\ \vdots \\ K_N(U_n^{(k)}) \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{pmatrix}
$$

However, we can rewritten (7) as, for all $1 \le i \le N$:

$$
U_{n,i}^{(k+1)}(t) - \sum_{j=1}^{N}\sum_{p=1}^{n} \omega_{n,p} \frac{\partial \kappa_i}{\partial u_j}\left(t, t_p, U_n^{(k)}(t_p)\right) U_{n,j}^{(k+1)}(t_p) = f_{n,i}^{(k)}(t), \tag{8}
$$

where,

$$
f_{n,i}^{(k)}(t) = -\sum_{j=1}^{N}\sum_{p=1}^{n} \omega_{n,p} \frac{\partial \kappa_i}{\partial u_j}\left(t, t_p, U_n^{(k)}(t_p)\right) U_{n,j}^{(k)}(t_p) + K_i\left(U_n^{(k)}\right)(t) + g_i(t). \tag{9}
$$

We defined the vector $X_N^{(k+1)} = (x_1^{(k+1)}, x_2^{(k+1)}, ..., x_N^{(k+1)}) \in \mathbb{R}^{n\times N}$ for saving the collocation of our discretized approximation $U_n^{(k+1)}(t) \in \widetilde{\mathcal{X}}$ in the nodes $(t_p)_{1\le p\le n}$, and for all $1 \le i \le N$ we denote by

$$
x_i^{(k+1)}(p) = U_{n,i}^{(k+1)}(t_p).
$$

The solution of system of equations defined in $(8) - (9)$ is gotten by two steps :

Step1. Solve the linear algebraic system

$$
(I_{NN} - A_n^{(k)})X_N^{(k+1)} = b_n^{(k)},
$$

where for all $1 \le i, j \le N$ and $1 \le p, l \le n$

$$
[A_n^{(k)}]_{ij}(l,p) = \omega_{n,p} \frac{\partial \kappa_i}{\partial u_j}\left(t_l, t_p, x_1^{(k)}(p), x_2^{(k)}(p), ..., x_N^{(k)}(p)\right), \tag{10}
$$

$$
b_{n,i}^{(k)}(l) = -\sum_{j=1}^{N}\sum_{p=1}^{n} \omega_{n,p} \frac{\partial \kappa_i}{\partial u_j}\left(t_l, t_p, x_1^{(k)}(p), x_2^{(k)}(p), ..., x_N^{(k)}(p)\right) x_j^{(k)}(p) \tag{11}
$$

$$
+ \int_0^1 \kappa_i\left(t_l, s, U_{n,1}^{(k)}(s), U_{n,2}^{(k)}(s), ..., U_{n,N}^{(k)}(s)\right) ds + g_i(t_l).
$$

Step2. For all $1 \le i \le N$, we recover $U_{n,i}^{(k+1)}$ by the natural interpolation formula

$$
\begin{aligned}
U_{n,i}^{(k+1)}(t) &= \sum_{j=1}^{N}\sum_{p=1}^{n} \omega_{n,p} \frac{\partial \kappa_i}{\partial u_j}\left(t, t_p, x_1^{(k)}(p), x_2^{(k)}(p), ..., x_N^{(k)}(p)\right)\left(x_j^{(k+1)}(p) - x_j^{(k)}(p)\right) \\
&+ \int_0^1 \kappa_i\left(t, s, U_{n,1}^{(k)}(s), U_{n,2}^{(k)}(s), ..., U_{n,N}^{(k)}(s)\right) ds + g_i(t).
\end{aligned} \tag{12}
$$

Before studying the convergence of Option A, we give some properties of operators M_T and M_{T_n}.

3.1. Properties of operators M_T and M_{T_n}

Proposition 3.1. *Under the assumption (5)(iii), we have M_T is Lipschitzian over $B_R(U)$ where*

$$
\lambda_R = 2\sup\left\{ \sup_{1\le j\le N} \sup_{(t,s,\widetilde{\psi}_j)\in[0,1]^2\times D_R} \sum_{i=1}^{N} \left| \frac{\partial^2 \kappa_i}{\partial u_j^2}(t,s,\widetilde{\psi}_j) \right|, \sup_{1\le j\le N} \sup_{(t,s,\widetilde{v}_j)\in[0,1]^2\times D_R} \sum_{i=1}^{N} \left| \frac{\partial^3 \kappa_i}{\partial u_j^2 \partial t}(t,s,\widetilde{v}_j) \right| \right\},
$$

is the Lipschitz constant, and

$$D_R = [-||U||_{\widetilde{\chi}} - R, ||U||_{\widetilde{\chi}} + R].$$

Proof. Let $\varphi = (\varphi_1, \varphi_2, ..., \varphi_N), \phi = (\phi_1, \phi_2, ..., \phi_N) \in B_R(U)$ and $h \in \widetilde{\chi}$, we have

$$\begin{aligned}
\|(M_T(\varphi) - M_T(\phi))h\|_{\widetilde{\chi}} &= \sum_{i=1}^{N} \left\| \sum_{j=1}^{N} \left[T_{ij}(\varphi) - T_{ij}(\phi)\right] h_j \right\|_{\chi_i} \leq \sum_{i=1}^{N}\sum_{j=1}^{N} \|\left[T_{ij}(\varphi) - T_{ij}(\phi)\right] h_j\|_{\chi_i} \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \left(|| \left[T_{ij}(\varphi) - T_{ij}(\phi) h_j\right] ||_\infty + || \frac{d}{dt}(\left[T_{ij}(\varphi) - T_{ij}(\phi)\right] h_j) ||_\infty \right) \\
&\leq \sup_{t\in[0,1]} \sum_{i=1}^{N}\sum_{j=1}^{N} \int_0^1 \left| \left(\frac{\partial \kappa_i}{\partial u_j}(t,s,\varphi(s)) - \frac{\partial \kappa_i}{\partial u_j}(t,s,\phi(s)) \right) \right| |h_j(s)| ds + \\
&+ \sup_{t\in[0,1]} \sum_{i=1}^{N}\sum_{j=1}^{N} \int_0^1 \left| \left(\frac{\partial^2 \kappa_i}{\partial u_j \partial t}(t,s,\varphi(s)) - \frac{\partial^2 \kappa_i}{\partial u_j \partial t}(t,s,\phi(s)) \right) \right| |h_j(s)| ds,
\end{aligned}$$

and by the assumption $(5)(iii)$, we can apply the Mean Value Theorem, then for all $1 \leq i, j \leq N, \exists \psi_j, \upsilon_j \in [\varphi_j, \phi_j]$ $\big($The line segment joining two points $\varphi_j, \phi_j \in B_{R_j}(u_j)\big)$ such that

$$\left| \frac{\partial \kappa_i}{\partial u_j}(t,s,\varphi(s)) - \frac{\partial \kappa_i}{\partial u_j}(t,s,\phi(s)) \right| \leq \sup_{\psi_j \in [\varphi_j,\phi_j]} \left| \frac{\partial^2 \kappa_i}{\partial u_j^2}(t,s,u_1(s),...,\psi_j(s),...,u_N(s)) \right| ||\varphi_j - \phi_j||_\infty,$$

$$\left| \frac{\partial^2 \kappa_i}{\partial u_j \partial t}(t,s,\varphi(s)) - \frac{\partial^2 \kappa_i}{\partial u_j \partial t}(t,s,\phi(s)) \right| \leq \sup_{\upsilon_j \in [\varphi_j,\phi_j]} \left| \frac{\partial^3 \kappa_i}{\partial u_j^2 \partial t}(t,s,u_1(s),...,\upsilon_j(s),...,u_N(s)) \right| ||\varphi_j - \phi_j||_\infty,$$

it's not difficult to demonstrate that, for all $1 \leq j \leq N$, we have:

$$||\widetilde{\psi}||_{\widetilde{\chi}} = \widetilde{\psi}_j \in D_R = [-R - ||U||_{\widetilde{\chi}}, R + ||U||_{\widetilde{\chi}}], \text{where} \quad \widetilde{\psi} = (u_1, ..., \psi_j, ..., u_N) \in B_R(U),$$

$$||\widetilde{\upsilon}||_{\widetilde{\chi}} = \widetilde{\upsilon}_j \in D_R = [-R - ||U||_{\widetilde{\chi}}, R + ||U||_{\widetilde{\chi}}], \text{where} \quad \widetilde{\upsilon} = (u_1, ..., \upsilon_j, ..., u_N) \in B_R(U),$$

and by these notations, we can compose

$$\sup_{t\in[0,1]} \sup_{\psi_j \in [\varphi_j,\phi_j]} \left| \frac{\partial^2 \kappa_i}{\partial u_j^2}(t,s,u_1(s),...,\psi_j(s),...,u_N(s)) \right| \leq \sup_{(t,s,\widetilde{\psi}_j)\in[0,1]^2\times D_R} \left| \frac{\partial^2 \kappa_i}{\partial u_j^2}(t,s,\widetilde{\psi}_j) \right| ||\varphi_j - \phi_j||_\infty,$$

$$\sup_{t\in[0,1]} \sup_{\upsilon_j \in [\varphi_j,\phi_j]} \left| \frac{\partial^3 \kappa_i}{\partial u_j^2 \partial t}(t,s,u_1(s),...,\upsilon_j(s),...,u_N(s)) \right| \leq \sup_{(t,s,\widetilde{\upsilon}_j)\in[0,1]^2\times D_R} \left| \frac{\partial^3 \kappa_i}{\partial u_j^2 \partial t}(t,s,\widetilde{\upsilon}_j) \right| ||\varphi_j - \phi_j||_\infty.$$

So, as for all $1 \leq j \leq N, ||h_j||_\infty \leq ||h||_{\widetilde{\chi}}$, we have

$$\begin{aligned}
\|(M_T(\varphi) - M_T(\phi))\| \leq &\sum_{i=1}^{N}\sum_{j=1}^{N} \sup_{(t,s,\widetilde{\psi}_j)\in[0,1]^2\times D_R} \left| \frac{\partial^2 \kappa_i}{\partial u_j^2}(t,s,\widetilde{\psi}_j) \right| ||\varphi_j - \phi_j||_\infty \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N} \sup_{(t,s,\widetilde{\upsilon}_j)\in[0,1]^2\times D_R} \left| \frac{\partial^3 \kappa_i}{\partial u_j^2 \partial t}(t,s,\widetilde{\upsilon}_j) \right| ||\varphi_j - \phi_j||_\infty.
\end{aligned}$$

Obviously, $\sum_{j=1}^{N} ||\varphi_j - \phi_j||_\infty \leq ||\varphi - \phi||_{\widetilde{\chi}}$, what's more, we take

$$\lambda_R = 2\sup\left\{ \sup_{1\leq j\leq N} \sup_{(t,s,\widetilde{\psi}_j)\in[0,1]^2\times D_R} \sum_{i=1}^{N} \left| \frac{\partial^2 \kappa_i}{\partial u_j^2}(t,s,\widetilde{\psi}_j) \right|, \sup_{1\leq j\leq N} \sup_{(t,s,\widetilde{\upsilon}_j)\in[0,1]^2\times D_R} \sum_{i=1}^{N} \left| \frac{\partial^3 \kappa_i}{\partial u_j^2 \partial t}(t,s,\widetilde{\upsilon}_j) \right| \right\},$$

to discover at last that

$$\|(M_T(\varphi) - M_T(\phi))\| \le \lambda_R ||\varphi - \phi||_{\widetilde{\mathcal{X}}}.$$

□

Proposition 3.2. *Assume that* (5)(iii)*, there exists a constant* C_N *such that, for all* $V \in B_R(U)$

$$\|M_T(V) - M_{T_n}(V)\| \le \frac{C_N}{n^2}, \tag{13}$$

where

$$C_N = \frac{1}{12n^2} \sup_{t\in[0,1]} \sum_{i=1}^{N}\sum_{j=1}^{N} \left\{ \left| \left[\frac{\partial^2 \kappa_i(t,s,V(s))}{\partial v_j \partial s} \right]_{s=0}^{s=1} \right| + \left| \left[\frac{\partial^3 \kappa_i(t,s,V(s))}{\partial v_j \partial s \partial t} \right]_{s=0}^{s=1} \right| \right\}$$

Proof. Let $h \in \mathcal{X}$ be the direction of the operator M_T. For all $V = (v_1, v_2, ..., v_N) \in B_R(U)$, we have

$$\begin{aligned}
\|(M_T(V) - M_{T_n}(V))\, h\|_{\widetilde{\mathcal{X}}} &= \sum_{i=1}^{N} \left\| \sum_{j=1}^{N} [T_{ij}(V) - T_{n,ij}(V)]\, h_j \right\|_{\mathcal{X}_i} \\
&\le \sum_{i=1}^{N}\sum_{j=1}^{N} \|[T_{ij}(V) - T_{n,ij}(V)]\, h_j\|_{\mathcal{X}_i} \\
&= \sup_{t\in[0,1]} \sum_{i=1}^{N}\sum_{j=1}^{N} |([T_{ij}(V) - T_{n,ij}(V)]\, h_j)\,(t)| \\
&+ \sup_{t\in[0,1]} \sum_{i=1}^{N}\sum_{j=1}^{N} \left| \frac{d}{dt} ([T_{ij}(V) - T_{n,ij}(V)]\, h_j)\,(t) \right| \\
&\le \sup_{t\in[0,1]} \left(\sum_{i=1}^{N}\sum_{j=1}^{N} |[T_{ij}(V) - T_{n,ij}(V)]\,(t)| \right) ||h_j||_\infty \\
&+ \sup_{t\in[0,1]} \left(\sum_{i=1}^{N}\sum_{j=1}^{N} \left| \frac{d}{dt} [T_{ij}(V) - T_{n,ij}(V)]\,(t) \right| \right) ||h_j||_\infty,
\end{aligned}$$

and by the trapezoidal rule (2), we have for all $1 \le i, j \le N$

$$|(T_{ij}(V) - T_{n,ij}(V))(t)| \le \frac{1}{12n^2} \left| \left[\frac{\partial^2 \kappa_i(t,s,V(s))}{\partial v_j \partial s} \right]_{s=0}^{s=1} \right|,$$

$$|\frac{d}{dt}(T_{ij}(V) - T_{n,ij}(V))(t)| \le \frac{1}{12n^2} \left| \left[\frac{\partial^3 \kappa_i(t,s,V(s))}{\partial v_j \partial s \partial t} \right]_{s=0}^{s=1} \right|,$$

we finish up at last that

$$\|M_T(V) - M_{T_n}(V)\| \le \frac{1}{12n^2} \sup_{t\in[0,1]} \sum_{i=1}^{N}\sum_{j=1}^{N} \left\{ \left| \left[\frac{\partial^2 \kappa_i(t,s,V(s))}{\partial v_j \partial s} \right]_{s=0}^{s=1} \right| + \left| \left[\frac{\partial^3 \kappa_i(t,s,V(s))}{\partial v_j \partial s \partial t} \right]_{s=0}^{s=1} \right| \right\}.$$

□

Proposition 3.3. *Assume that* (5) *holds. Let* $r := \min\left(R, \frac{1}{2\lambda_R \eta}\right)$*, where* λ_R *is defined in Proposition* 3.1*. Then for all* $V = (v_1, v_2, ..., v_N) \in B_r(U), I_{NN} - M_T(V)$ *is invertible and*

$$||(I_{NN} - M_T(V))^{-1}|| \leq 2\eta.$$

Proof. For all $V = (v_1, v_2, ..., v_N) \in \widetilde{\chi}$, we have

$$\begin{aligned} I_{NN} - M_T(V) =& I_{NN} - M_T(U) - M_T(V) + M_T(U) \\ =& (I_{NN} - M_T(U)) \left[I_{NN} - (I_{NN} - M_T(U))^{-1} (M_T(V) - M_T(U)) \right], \end{aligned}$$

we have for all $V = (v_1, v_2, ..., v_N) \in B_r(U)$, (Proposition 3.1)

$$\|(M_T(V) - M_T(U))\| \leq \lambda_R r.$$

Then

$$\left\| (I_{NN} - M_T(U))^{-1} (M_T(V) - M_T(U)) \right\| \leq \eta \lambda_R r \leq \tfrac{1}{2},$$

use the Geometric Series Theorem (see [1], pp.516), we conclude that $I_{NN} - M_T(V)$ is invertible such that, for all $V = (v_1, v_2, ..., v_N) \in B_r(U)$

$$(I_{NN} - M_T(V))^{-1} = \left(I_{NN} - (I_{NN} - M_T(U))^{-1} [M_T(V) - M_T(U))] \right)^{-1} (I_{NN} - M_T(U))^{-1},$$

and its inverse is uniformly bounded on $B_r(U)$, where

$$\left\| (I_{NN} - M_T(V))^{-1} \right\| \leq \eta \sum_{m=0}^{\infty} \left\| (I_{NN} - M_T(U))^{-1} (M_T(V) - M_T(U)) \right\|^m \leq 2\eta.$$

□

Proposition 3.4. *Assume that* (5) *holds. Then for n big enough, and for all $V = (v_1, v_2, ..., v_N) \in B_r(U)$, $I_{NN} - M_{T_n}(V)$ is invertible, and there exists $\delta_n \in]0, 1[$, such that*

$$\sup_{V \in B_r(U)} \left\| I_{NN} - (I_{NN} - M_{T_n}(V))^{-1} (I_{NN} - M_T(V)) \right\| \leq \delta_n,$$

$$\sup_{V \in B_r(U)} \left\| (I_{NN} - M_{T_n}(V))^{-1} \right\| \leq 2\eta(1 + \delta_n).$$

Proof. For all $V = (v_1, v_2, ..., v_N) \in B_r(U)$, we have

$$\begin{aligned} I_{NN} - M_{T_n}(V) =& I_{NN} - M_T(V) + M_T(V) - M_{T_n}(V) \\ =& (I_{NN} - M_T(V)) \left(I_{NN} - (I_{NN} - M_T(V))^{-1} [M_{T_n}(V) - M_T(V))] \right), \end{aligned}$$

and as we have in Proposition 3.2

$$\|M_T(V) - M_{T_n}(V)\| \leq \frac{C_N}{n^2} \to 0, n \to +\infty.$$

So, with n adequately large, plainly $\dfrac{C_N}{n^2} < \dfrac{1}{2\eta}$. Then

$$\left\| (I_{NN} - M_{T_n}(V))^{-1} (M_{T_n}(V) - M_T(V)) \right\| \leq \frac{2\eta C_N}{n^2} < 1,$$

and by the Geometric Series Theorem (see [1], pp.516), we have for all $V \in B_r(U)$, $(I_{NN} - M_{T_n}(V))$ is invertible and $\left\| (I_{NN} - M_{T_n}(V))^{-1} \right\| \leq \dfrac{2\eta}{1 - 2\eta\xi_n}$, where $\xi_n = \dfrac{C_N}{n^2}$.

As

$$I_{NN} - (I_{NN} - M_{T_n}(V))^{-1} (I_{NN} - M_T(V)) = (I_{NN} - M_{T_n}(V))^{-1} (M_T(V) - M_{T_n}(V)),$$

we define $\delta_n = \dfrac{2\eta\xi_n}{1 - 2\eta\xi_n}$ and for n large enough, $\delta_n < 1$ we find

$$\sup_{V \in B_r(U)} \left\| I_{NN} - (I_{NN} - M_{T_n}(V))^{-1} (I_{NN} - M_T(V)) \right\| \leq \delta_n,$$

we have

$$\begin{aligned}(I_{NN} - M_{T_n}(V))^{-1} =& (I_{NN} - M_T(V))^{-1} \\ & - \left[I_{NN} - (I_{NN} - M_{T_n}(V))^{-1} (I_{NN} - M_{T_n}(V)) \right] (I_{NN} - M_T(V))^{-1}.\end{aligned}$$

In this way, we close at last that

$$\sup_{V \in B_r(U)} \left\| (I_{NN} - M_{T_n}(V))^{-1} \right\| \leq 2\eta(1 + \delta_n).$$

□

4. Analysis of option A

In this section, we study the convergence of Option A, where we will prove that our approximate solution $U_n^k = (U_{n,1}^k, U_{n,2}^k, ..., U_{n,N}^k) \in \widetilde{\mathcal{X}}$ defined in (12), converges to the exact solution $U = (u_1, u_2, ..., u_N) \in \widetilde{\mathcal{X}}$ defined in (1).

Theorem 4.1. *Assume that the assumptions* (5) *are satisfied, set* $r = \min\left(R, \frac{1}{2\lambda_R \eta}\right)$. *Then there exist* $\delta_n \in]0, 1[$, *and* $\varrho_n > 0$ *such that, if the starting approximation* $U_n^{(0)}$ *is chosen in the closed ball* $B_{\varrho_n}(U)$, *then for all* $k \in \mathbb{N}^*, U_n^{(k)} \in B_{\varrho_n}(U)$, *and*

$$||U_n^{(k)} - U||_{\widetilde{\mathcal{X}}} \leq \varrho_n \left(\tfrac{1+\delta_n}{2}\right)^k \to 0 \quad as \quad k \to \infty.$$

Proof. We have found in Proposition 3.4 that, if $U_n^{(k)} \in B_r(U)$, $I_{NN} - M_{T_n}(U_n^{(k)})$ is invertible. Then $U_n^{(k+1)}$ defined in (7) can given by

$$U_n^{(k+1)} - U = U_n^{(k)} - U - \left(I_{NN} - M_{T_n}(U_n^{(k)}) \right)^{-1} \left(U_n^{(k)} - U - K(U_n^{(k)}) + K(U) \right).$$

Since

$$K(U) - K(U_n^{(k)}) = -\int_0^1 M_T \left((1-x)U_n^{(k)} + xU \right) . \left(U_n^{(k)} - U \right) dx,$$

then, we can write

$$U_n^{(k+1)} - U = \int_0^1 \left[I_{NN} - \left(I_{NN} - M_{T_n}(U_n^{(k)}) \right)^{-1} \left[I_{NN} - M_T \left((1-x)U_n^{(k)} + xU \right) \right] \right] . \left(U_n^{(k)} - U \right) dx,$$

by added $I_{NN} - M_T(U_n^{(k)})$ to and subtracted from $I_{NN} - M_T\left((1-x)U_n^{(k)} + xU\right)$, we get

$$U_n^{(k+1)} - U = \int_0^1 \left[I_{NN} - \left(I_{NN} - M_{T_n}(U_n^{(k)})\right)^{-1}\left(I_{NN} - M_T(U_n^{(k)})\right)\right].\left(U_n^{(k)} - U\right)dx$$
$$+ \int_0^1 \left(I_{NN} - M_{T_n}(U_n^{(k)})\right)^{-1}\left[M_T\left((1-x)U_n^{(k)} + xU\right) - M_T(U_n^{(k)})\right].\left(U_n^{(k)} - U\right)dx,$$

and

$$||U_n^{(k+1)} - U||_{\widetilde{\chi}} \le ||I_{NN} - \left(I_{NN} - M_{T_n}(U_n^{(k)})\right)^{-1}\left(I_{NN} - M_T(U_n^{(k)})\right)||\,||U_n^{(k)} - U||_{\widetilde{\chi}} +$$
$$+ ||\left(I_{NN} - M_{T_n}(U_n^{(k)})\right)^{-1}||\,||U_n^{(k)} - U||_{\widetilde{\chi}} \int_0^1 ||M_T\left((1-x)U_n^{(k)} + xU\right) - M_T(U_n^{(k)})||\,dx.$$

Let $U_n^{(k)} \in B_r(U)$ and according to Proposition 3.4

$$||I_{NN} - \left(I_{NN} - M_{T_n}(U_n^{(k)})\right)^{-1}\left(I_{NN} - M_T(U_n^{(k)})\right)|| \le \delta_n,$$

and since $B_r(U)$ is convex, for all $x \in [0,1], (1-x)U_n^{(k)} + xU \in B_r(U)$, and according to Proposition 3.1

$$||M_T\left((1-x)U_n^{(k)} + xU\right) - M_T(U_n^{(k)})|| \le \lambda_R x ||U_n^{(k)} - U||_{\widetilde{\chi}}.$$

Hence

$$\int_0^1 ||M_T\left((1-x)U_n^{(k)} + xU\right) - M_T(U_n^{(k)})||\,dx \le \frac{1}{2}\lambda_R ||U_n^{(k)} - U||_{\widetilde{\chi}}.$$

We use the second inequality of Proposition 3.4, we have

$$||U_n^{(k+1)} - U||_{\widetilde{\chi}} \le \delta_n ||U_n^{(k)} - U||_{\widetilde{\chi}} + \left(2\eta(1+\delta_n)||U_n^{(k)} - U||_{\widetilde{\chi}}\right)\frac{1}{2}\lambda_R ||U_n^{(k)} - U||_{\widetilde{\chi}}.$$

We define

$$\varrho_n := \min\left\{r, \left(\frac{1-\delta_n}{2\lambda_R\eta(1+\delta_n)}\right)\right\}.$$

Then if $U_n^{(k)} \in B_{\varrho_n}(U)$, $\frac{1}{2}\lambda_R||U_n^{(k)} - U||_{\widetilde{\chi}} \le \frac{1-\delta_n}{4\eta(1+\delta_n)}$. Hence

$$||U_n^{(k+1)} - U||_{\widetilde{\chi}} \le \left(\frac{1+\delta_n}{2}\right)||U_n^{(k)} - U||_{\widetilde{\chi}},$$

since $1 + \delta_n < 2$ the previous inequality implies that $U_n^{(k+1)} \in B_{\varrho_n}(U)$ and that

$$||U_n^{(k)} - U||_{\widetilde{\chi}} \le \varrho_n \left(\frac{1+\delta_n}{2}\right)^k \to 0, \quad as \quad k \to \infty.$$

□

5. Analysis of option B

This option begins by the discretization of the system (1), where the Nyström method leads to finding $V_n(t) = (V_{n,1}(t), V_{n,2}(t), ..., V_{n,N}(t)) \in \widetilde{\mathcal{X}}$ such that

$$\begin{cases} V_{n,1}(t) = \sum_{p=1}^{n} \omega_{n,p}\kappa_1 \left(t, t_p, V_{n,1}(t_p), V_{n,2}(t_p), ..., V_{n,N}(t_p)\right) + g_1(t), \\ V_{n,2}(t) = \sum_{p=1}^{n} \omega_{n,p}\kappa_2 \left(t, t_p, V_{n,1}(t_p), V_{n,2}(t_p), ..., V_{n,N}(t_p)\right) + g_2(t), \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ V_{n,N}(t) = \sum_{p=1}^{n} \omega_{n,p}\kappa_N \left(t, t_p, V_{n,1}(t), V_{n,2}(t_p), ..., V_{n,N}(t_p)\right) + g_N(t), \end{cases} \tag{14}$$

for all $t \in [0, 1]$ and a given functions $g_i \in \Omega_i$.

Proposition 5.1. *Assume that* (5) *holds. Then*

$$||V_n - U||_{\widetilde{\mathcal{X}}} \leq \frac{C}{n^2},$$

where

$$C = \frac{c}{12} \sup_{t\in[0,1]} \sum_{i=1}^{N} \left\{ \left| \left[\frac{\partial \kappa_i(t, s, U(s))}{\partial s} \right]_{s=0}^{s=1} \right| + \left| \left[\frac{\partial^2 \kappa_i(t, s, U(s))}{\partial s \partial t} \right]_{s=0}^{s=1} \right| \right\}.$$

Proof. For proving this proposition, we use the numerical integration rule (2), and the estimation ((3.6) pp.29) of [1].

□

Now by applying the collocation technique to each equation of system (14), we obtain a nonlinear algebraic system of equations in $\mathbb{R}^{n\times N}$. Let

$$x_j^{(\infty)}(l) = V_{n,j}(t_l), \quad j = 1, ..., N, \quad l = 1, ..., n.$$

For $1 \leq i \leq N$. Let $O_{n,i}$ be open subset of $\mathbb{R}^n$ and let $F_n = (F_{n,1}, F_{n,2}, ..., F_{n,N})$ be the nonlinear operator from some open subset $\widetilde{O}_N = \prod_{i=1}^{N} O_{n,i}$ of $\mathbb{R}^{n\times N}$ into $\mathbb{R}^{n\times N}$ defined by

$$F_{n,i}(X)(l) = x_i(l) - \sum_{p=1}^{n} \omega_{n,p}\kappa_i \left(t_l, t_p, X(p)\right) - g_i(t_l), \tag{15}$$

where $X = (x_1, x_2, ..., x_N) \in \widetilde{O}_N$ and $l = 1, ..., n$. The problem is set up as:

$$\begin{cases} \text{Find} \quad X_{n,N}^{\infty} = (x_1^{\infty}, x_2^{\infty}, ..., x_N^{\infty}) \in \mathbb{R}^{n\times N}, \\ F_n(X_{n,N}^{\infty}) = \Big(F_{n,1}(X_{n,N}^{\infty}), F_{n,2}(X_{n,N}^{\infty}), ..., F_{n,N}(X_{n,N}^{\infty})\Big) = 0_{\mathbb{R}^{n\times N}}. \end{cases} \tag{16}$$

We have F_n is differentiable and its differentiable is presented by a block of matrices as

$$D_{F_n} = \begin{pmatrix} [D_{F_n}]_{11} & \dots & [D_{F_n}]_{1N} \\ [D_{F_n}]_{21} & \dots & [D_{F_n}]_{2N} \\ \vdots & \ddots & \vdots \\ [D_{F_n}]_{N1} & \dots & [D_{F_n}]_{NN} \end{pmatrix}$$

where for all $1 \leq i, j \leq N, [D_{F_n}]_{ij}$ is a matrix of size $n \times n$, and for all $1 \leq l, p \leq n$

$$[D_{F_n}]_{ij}(l,p) = \begin{cases} 1 - \omega_{n,p} \frac{\partial \kappa_i}{\partial u_j} \left(t_l, t_p, x_1^{(\infty)}(p), x_2^{(\infty)}(p), ..., x_N^{(\infty)}(p) \right), & \text{if } l = p, \\ -\omega_{n,p} \frac{\partial \kappa_i}{\partial u_j} \left(t_l, t_p, x_1^{(\infty)}(p), x_2^{(\infty)}(p), ..., x_N^{(\infty)}(p) \right), & \text{if } l \neq p. \end{cases}$$

We solve the problem (16) by using the Newton-Raphson method. The iterate $X_{n,N}^{(k+1)}$ solves

$$D_{F_n} X_{n,N}^{(k+1)} = d_n^{(k)}, \quad k = 1, 2, ... \quad ,$$

where for all $1 \leq l \leq n, 1 \leq i \leq N$, we have

$$d_{n,i}^{(k)}(l) = -\sum_{j=1}^{N} \sum_{p=1}^{n} \omega_{n,p} \frac{\partial \kappa_i}{\partial u_j} \left(t_l, t_p, X_{n,N}^{(k)}(p) \right) x_j^{(k)}(p) + \sum_{p=1}^{n} \omega_{n,p} \kappa_i \left(t_l, t_p, X_{n,N}^{(k)}(p) \right) + g_i(t_l). \tag{17}$$

We recover the approximation $V_n^{(k+1)} = \left(V_{n,1}^{(k+1)}, V_{n,2}^{(k+1)}, ..., V_{n,N}^{(k+1)} \right) \in \widetilde{\mathcal{X}}$ with the natural interpolation formula:

$$V_{n,i}^{(k+1)}(t) = \sum_{p=1}^{n} \omega_{n,p} \kappa_i \left(t, t_p, X_{n,N}^{(k+1)}(p) \right) + g_i(t), \quad i = 1, ..., N, \quad t \in [0,1]. \tag{18}$$

After these steps, we need to prove the convergence of the iterates $V_n^{(k+1)}$ toward the solution V_n of the system (14). Let $||.||_n$ be the vector norm in $\mathbb{R}^n$, and $||.||_{n,N}$ be the vector norm in $\mathbb{R}^{n \times N}$ such as:

$$\forall V = (v_1, v_2, ..., v_N) \in \mathbb{R}^{n \times N}, ||V||_{n,N} = \sum_{j=1}^{N} ||v_j||_n = \sum_{j=1}^{N} \sum_{p=1}^{n} |v_j(p)|.$$

Let $|||.|||$ be the matrix norm in $\mathcal{M}_{n \times N}(\mathbb{R})$ such as

$$\forall M \in \mathcal{M}_{n \times N}(\mathbb{R}), |||M||| = \max_{1 \leq i \leq N} \sum_{j=1}^{N} \max_{1 \leq l \leq n} \sum_{p=1}^{n} |M_{ij}(l.p)|,$$

and $S_r(W)$ the ball of center W and radius r in $\mathbb{R}^{n \times N}$ for the norm $||.||_{n,N}$.
We define the vector $W = (w_1, w_2, ..., w_N) \in \mathbb{R}^{n \times N}$ from the exact solution U by

$$w_i(l) = u_i(t_l), \quad i = 1, ..., N, \quad l = 1, ..., n.$$

For $V_{n,N} = (v_1, v_2, ..., v_N) \in \mathbb{R}^{n \times N}$, we define $\widetilde{V}_{n,N} = (\widetilde{v}_{n,1}, \widetilde{v}_{n,2}, ..., \widetilde{v}_{n,N}) \in \widetilde{\mathcal{X}}$ by

$$\widetilde{v}_{n,i}(t) = \sum_{p=1}^{n} \omega_{n,p} \kappa_i(t, t_p, V_{n,N}(p)) + g_i(t), \quad i = 1, ..., N, \quad t \in [0,1]. \tag{19}$$

Lemma 5.2. *For all* $V_{n,N} \in S_\varrho(W) = \prod_{i=1}^{N} S_{\varrho_i}(w_i) \subset \mathbb{R}^{n \times N}$,

$$||\widetilde{V}_{n,N} - U||_{\widetilde{\mathcal{X}}} \leq C_\varrho ||V_{n,N} - W||_{n,N} + O\left(\frac{N}{n^2} \right). \tag{20}$$

Proof. We have

$$||\widetilde{V}_{n,N} - U||_{\widetilde{\mathcal{X}}} = \sum_{i=1}^{N} ||\widetilde{v}_{n,i} - u_i||_{\mathcal{X}_i} \leq \sum_{i=1}^{N} \left(||K_{n,i}(\widetilde{V}_{n,N}) - K_{n,i}(U)||_{\mathcal{X}_i} + ||K_{n,i}(U) - K_i(U)||_{\mathcal{X}_i} \right).$$

For all $1 \leq i \leq N$, we use (2) and the regularity of κ_i, for writing:

$$||K_{n,i}(U) - K_i(U)||_{\chi_i} = O\left(\frac{1}{n^2}\right), \quad \text{and} \quad ||K_{n,i}(\widetilde{V}_{n,N}) - K_{n,i}(U)||_{\chi_i} \leq (C_{\varrho_i} + C'_{\varrho_i})||v_i - w_i||_n,$$

and by taking $C_\varrho = \max_{1\leq i\leq N}(C_{\varrho_i} + C'_{\varrho_i})$ and $\varrho = \sum_{i=1}^{N} \varrho_i$, we have finished the proof. □

In the following step, we fixed $n >> N$ such that the Propositions 3.1 − 5.1 are satisfied, and we choose the positive number ϱ such that

$$\varrho C_\varrho + O\left(\frac{N}{n^2}\right) \leq r, \tag{21}$$

with r is the parameter defined in Proposition 3.3 .Then

$$\forall V_{n,N} \in S_\varrho(W) \subset \mathbb{R}^{n\times N} \Longrightarrow \widetilde{V}_{n,N} \in B_r(U) \subset \widetilde{\chi}.$$

As for all $1 \leq l \leq n$, we have

$$\left(D_{F_n}(V_{n,N}).h\right)(l,.) = (I_{NN} - M_{T_n}(\widetilde{V}_{n,N}))(t_l,.)\widetilde{h}(t_l), \quad \forall h \in \mathbb{R}^{n\times N}, \tag{22}$$

and by using the Proposition 3.4, we can find that $D_{F_n}(V_N)$ is invertible and

$$\exists \eta_n > 0, \quad |||(D_{F_n}(V_{n,N}))^{-1}||| \leq \eta_n, \quad \forall V_{n,N} \in S_\varrho(W). \tag{23}$$

Similarly to Proposition 3.1, we can demonstrate that

$$|||D_{F_n}(X) - D_{F_n}(Y)||| \leq \lambda_\varrho ||X - Y||_{n,N}, \quad \forall X, Y \in S_\varrho(W), \tag{24}$$

$$\lambda_\varrho = 2 \max_{1\leq i\leq N} \sup \left\{ \sup_{1\leq j\leq N} \sup_{(t,s,Z_j)\in[0,1]^2\times D_R} \left| \frac{\partial^2 \kappa_i}{\partial u_j^2}(t,s,Z_j) \right|, \sup_{1\leq j\leq N} \sup_{(t,s,Z'_j)\in[0,1]^2\times D_R} \left| \frac{\partial^3 \kappa_i}{\partial u_j^2 \partial t}(t,s,Z'_j) \right| \right\},$$

and

$$I_\varrho = [-\varrho - ||W||_{n,N}, \varrho + ||W||_{n,N}].$$

Theorem 5.3. *Let $V_n^{(k+1)}$ be the iterate solution defined in* (18). *Assume that the assumption* (5) *are satisfied. Let r be the parameter defined in Proposition* 3.3 *and ϱ satisfy* (21) *. For $V_n^{(0)} \in S_\varrho(W)$, let the positives constants $r_n, \beta_n, \eta_n, \lambda_\varrho$ and τ_n be given with the accompanying properties:*

$$S_{r_n}(V_n^{(0)}) \subset S_\varrho(W), \quad \tau_n = \frac{\beta_n \eta_n \lambda_\varrho}{2} < 1, \quad r_n = \frac{\beta_n}{1 - \tau_n},$$

the inequalities (23) *and* (24) *are satisfied, and*

$$|||(D_{F_n}(V_n^{(0)}))^{-1} D_{F_n}(V_n^{(0)})||| \leq \beta_n.$$

Then $V_n^{(k)} \in S_{r_n}(V_n^{(0)})$ and

$$||V_n^{(k)} - U||_{\widetilde{\chi}} \leq c\beta_n \frac{\tau_n^{2^k - 1}}{1 - \tau_n^{2^k}} + \frac{c}{n^2}. \tag{25}$$

Proof. It is the Newton Theorem for Several Variables, and the proof is well detailed in [10] (see Theorem 5.3.2, pp.270). □

Remark 5.4. ***(Comparison between option A and B)***
The difference between the option A and B is due to the fact that integrals on the right-hand side of the system of equations (8) − (9) *in option A are approximated by the Nyström method, ie, for all $1 \leq i \leq N$*

$$\int_0^1 \kappa_i\left(t, s, U_n^{(k)}(s)\right) ds \approx \sum_{q=1}^{m} \omega_{m,q} \kappa_i \left(t, t_q, U_n^{(k)}(t_q)\right), \quad U_n^{(k)} \in \widetilde{\chi}, \quad k = 1, 2, \ldots \quad ,$$

where we choose a finer grid according the number of nodes m in the subdivision too big to n ($m >> n$). This choice that gave option A the preference over the other option B.

6. Application on nonlinear integro-differential equation

This part explains how to apply option A in order to approach a nonlinear Fredholm integro-differential equation. Consider the following integro-differential equation.

$$u(t) = \int_0^1 \kappa\left(t, s, u(s), u'(s), ..., u^{(N-1)}(s)\right) ds + g(t), \quad t \in [0,1], \tag{26}$$

according the technique described in the paper [2], we derive this equation $(N-1)$ times, then we obtain the system

$$\begin{cases} u_1(t) = u(t) = \displaystyle\int_0^1 \kappa\left(t, s, u_1(s), u_2(s), ..., u_N(s)\right) ds + g(t), \\ u_2(t) = u'(t) = \displaystyle\int_0^1 \frac{\partial \kappa}{\partial t}\left(t, s, u_1(s), u_2(s), ..., u_N(s)\right) ds + g'(t), \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ u_N(t) = u^{(N-1)}(t) = \displaystyle\int_0^1 \frac{\partial^{(N-1)}\kappa}{\partial t^{(N-1)}}\left(t, s, u_1(s), u_2(s), ..., u_N(s)\right) ds + g^{(N-1)}(t), \end{cases} \tag{27}$$

for all $t \in [0,1]$. If we set, for all $1 \le i \le N$, $g_i = g^{(i-1)}$ and $\kappa_i = \dfrac{\partial^{(i-1)}\kappa}{\partial t^{(i-1)}}$, so, the system (27) will be equivalent to the system (1). However, we can apply now the new process (Option A) in order to approach the solution of our integro-differential equation, that we will see in the next numerical examples.

7. Numerical Examples

In this section, to examine the effectiveness of our new process (option A), compared to the classical process (option B), we will treat two examples. In the first example, we solve by using option A, the same nonlinear Fredholm integro-differential equation presented in [2], and we compare our results with its results. However, we mention that the results of the paper [2], have been obtained according the classical process (option B). In the second example, we solve a system of nonlinear integral equations by using the both options A and B, also we compare between the obtained results.

Let $(U_{n,1}^{(k)}, U_{n,2}^{(k)}, ..., U_{n,N}^{(k)}) \in \widetilde{\mathcal{X}}$ and $(V_{n,1}^{(k)}, V_{n,2}^{(k)}, ..., V_{n,N}^{(k)}) \in \widetilde{\mathcal{X}}, k \in \mathbb{N}^*$, the k order approximative solution of our system of equations (1) according to the scheme (12) of option A, and to the scheme (18) of option B, respectively.

First, let $n \in \mathbb{N}^*$, and considering the equidistant subdivision Δ_n of $[0,1]$ defined by:

$$\Delta_n = \left\{ t_p = p\,h,\, h = \frac{1}{n}, p = 1, ..., n \right\}.$$

We define the stopping condition on the parameter k as:

$$\text{For Option A :} \qquad S_A^k = \sum_{i=1}^{N} \max_{1\le p\le n} |U_{n,i}^{(k+1)}(t_p) - U_{n,i}^{(k)}(t_p)| \le 10^{-09}.$$

$$\text{For Option B :} \qquad S_B^k = \sum_{i=1}^{N} \max_{1\le p\le n} |V_{n,i}^{(k+1)}(t_p) - V_{n,i}^{(k)}(t_p)| \le 10^{-09}.$$

We denote the obtained error using the both options A and B by:

$$\text{For Option A :} \qquad e_A = \sum_{i=1}^{N} \max_{1\le p\le n} |u_{i,ext}(t_p) - U_{n,i}^{(k)}(t_p)|.$$

For Option B : $$e_B = \sum_{i=1}^{N} \max_{1\le p\le n} |u_{i,ext}(t_p) - V_{n,i}^{(k)}(t_p)|.$$
where, $U = (u_{1,ext}, u_{2,ext}, ..., u_{N,ext}) \in \widetilde{\mathcal{X}}$ is the exact solution of the initial system of equations (1). We pass now to the numerical examples.

Example 1: Consider the nonlinear Fredholm integro-differential equation presented in [2]

$$u(t) = \frac{1}{5}\int_0^1 sin[2(s+t+u(s)) + (1-s)e^s - u'(s)]ds + g(t), \quad \forall t \in [0,1], \tag{28}$$

with $u \in C^1([0,1],\mathbb{R})$ and $g(t) = te^t - \frac{1}{5}[sin^2(1+t) - sin^2(t)]$.

As we have described in section 6, we notice by $u(t) = u_1(t)$, $u'(t) = u_2(t)$, $g_1(t) = g(t)$ and $g_2(t) = g'(t) = (1+t)e^t - \frac{2}{5}[\cos(1+t)\sin(1+t) - \cos(t)\sin(t)]$, then we obtain the following system

$$\begin{cases} u_1(t) = \frac{1}{5}\int_0^1 sin[2(s+t+u_1(s)) + (1-s)e^s - u_2(s)]ds + g_1(t), \\ u_2(t) = \frac{2}{5}\int_0^1 cos[2(s+t+u_1(s)) + (1-s)e^s - u_2(s)]ds + g_2(t), \end{cases} \tag{29}$$

where $U = (te^t, (1+t)e^t)$ is its exact solution. However, we solve this system (29) by using option A and compare our results with the results obtained in [2].

Example 2: Consider the following system of equations, for all $t \in [0,1]$,

$$\begin{cases} u_1(t) = \int_0^1 \frac{u_1(s)^2}{2+t+u_2(s)u_3(s)}ds + t + \frac{1}{3}\log\left(\frac{t+1}{t+2}\right), \\ u_2(t) = \int_0^1 \frac{tu_3(s)}{2+t+u_1(s)+u_2(s)}ds - t\left(1 + \frac{1}{3(t+2)}\right), \\ u_3(t) = \int_0^1 \frac{2tu_2(s) - t}{5+u_1(s)+u_3(s)}ds + t^2 + t\log\left(\frac{7}{5}\right), \end{cases} \tag{30}$$

where $U = (t, -t, t^2)$ is its exact solution. In the same way of example 1, we solve this system (30) by using the both options A and B, then we compare between the obtained results.

	The Error	
n	Option A	Option B ([2])
5	9.6417E-05	8.5244E-02
10	2.3565E-05	4.2957E-02
50	9.2595E-07	8.6250E-03
100	2.3097E-07	4.3136E-03
500	9.2225E-09	8.6289E-04

Table 1: Numerical results of example 1.

The Error		
n	Option A	Option B
5	3.8751E-04	4,8409E-02
10	9.4735E-05	9,2095E-03
50	3.7221E-06	3.1248E-04
100	9.2821E-07	7.6564E-05
500	3.6736E-08	3.0139E-06

Table 2: Numerical results of example 2.

The errors of both options A and B applied on example 1 and 2 are shown in Tables (1) and (2), respectively, which confirm that option A is more accurate than option B. However, in Figures (1) and (3) we can see that the approximate solutions using option A converge to the exact solutions. Furthermore, Figures (2) and (4) show us the distance between two successive iterates using option A and B for example 1 and 2, respectively, which proves that option A has a linear convergence, worse than option B. So, we conclude that our numerical results are similar to the results of [5], which assure that our vision is reasonable.

Conclusion

In this work, we have constructed a Linearization-Discretization process for solving a system of nonlinear Fredholm integral equations defined in an infinite dimensional context. As well as, we have proposed the necessary conditions which guarantee the convergence analysis of this new process.
However, the numerical tests show that our new process should be preferred to the classical method. The reason for this behavior is obviously that the sequence U_n^k constructed by using option A converges to the exact solution U. On the contrary, the sequence V_n^k constructed by using option B converges to V_n, which is just the solution of the discretized problem (14) obtained by the Nyström method.

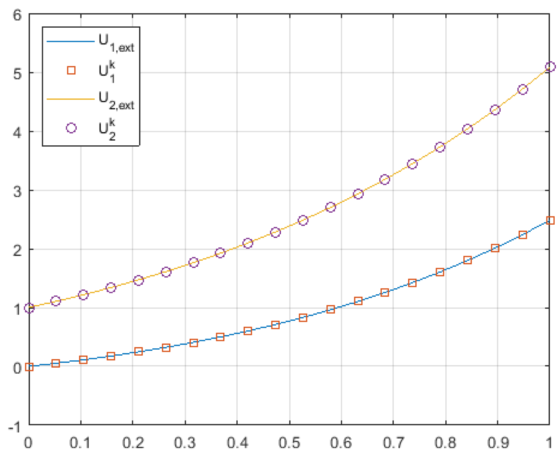

Figure 1: Approximate solutions of example 1, using option A with $n = 20$ and $m = 180$.

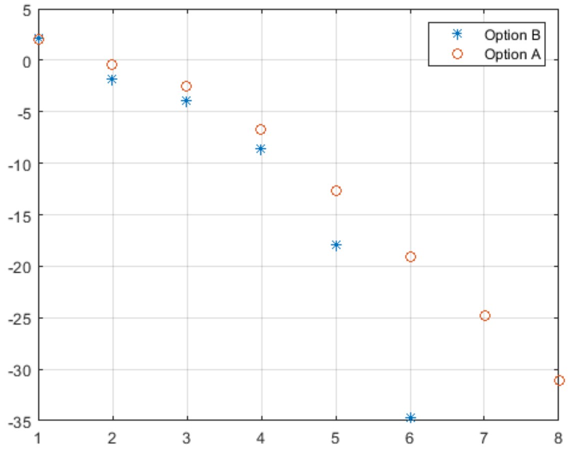

Figure 2: Graph of log_{10} of the distance between two successive iterates (example 1).

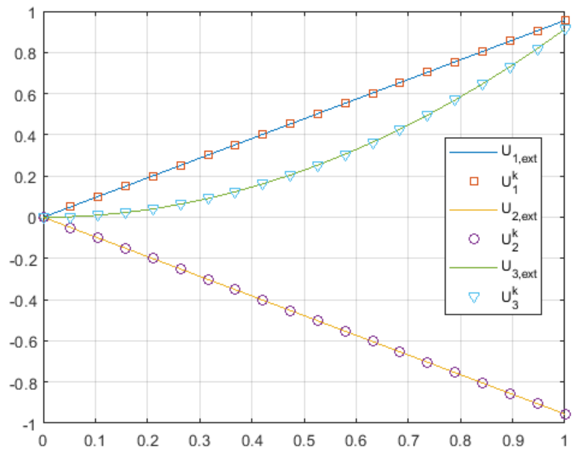

Figure 3: Approximate solutions of example 2, using option A with $n = 20$ and $m = 180$.

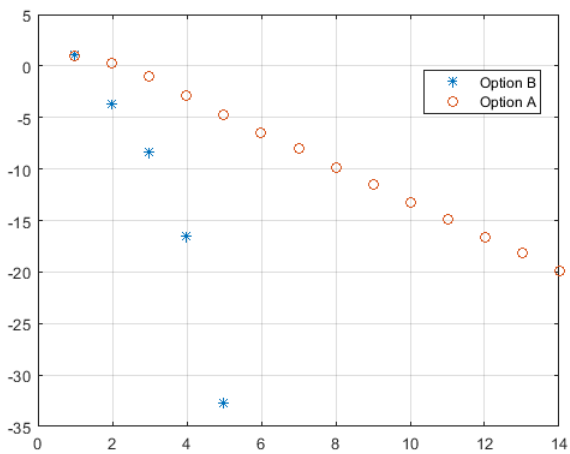

Figure 4: Graph of log_{10} of the distance between two successive iterates (example 2).

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