Journal for the Education of Gifted
Young Scientists, 9(4), 283-297, Dec 2021
e-ISSN: 2149- 360X
jegys.org

Young Wise Publishing
youngwisepub.com
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Research Article

Grade 6 teachers' s mathematical knowledge for teaching the concept of fractions

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Article Info

Received: 27 July 2021
Revised: 27 Sept 2021
Accepted: 09 October 2021
Available online: 15 Dec 2021

Keywords:
Constructivism
Fraction concept
Mathematical Knowledge for Teaching (MKT)
Pedagogical Content Knowledge (PCK)

Abstract

This article reports on two case studies in which we explored two Grade 6 teachers' mathematical knowledge for teaching the concept of the fraction. We were interested in the mathematical knowledge teachers need to have and are able to use to teach fractions. Of the two teachers who were observed and interviewed, one (Rose) explained the concept of the fraction by emphasising an understanding of mathematical concepts. She did this by using various modes of representation to teach the concept of the fraction and fractional manipulatives. On the other hand, the second teacher (Eddy) focused on procedural knowledge. Eddy used a traditional method to teach fractions, encouraging learners to memorise rules without necessarily understanding them. The learners followed the rules blindly because Eddy did not tell them how the rules originated.

To cite this article:
Moloto, M., & Machaba, F. (2021). Grade 6 teachers's mathematical knowledge for teaching the concept of fractions. *Journal for the Education of Gifted Young Scientists*, *9*(4), 283-297. DOI: http://dx.doi.org/10.17478/jegys.1000495

Introduction

The content area of fractions has long proved to be complicated and troublesome for learners to master. Van de Walle, Karp and Bay-Williams (2010, p.313) identified several possible factors contributing to the poor understanding of fractions. They provide reasons for learners' difficulties with fractions as follows:

- fractions have many descriptions, such as part-whole, measurement, division operator;
- the written format of fractions is strange to learners;
- the conceptual understanding of fractions is ignored in instructions; and
- whole-number knowledge is overgeneralized by learners.

Pienaar (2014) argues that one of the reasons teachers experience difficulties when teaching fractions is the way in which mathematics as a subject is viewed in the South African curriculum. Acknowledging the reasons provided above, we believe that because the concept of fractions is one of the topics in the mathematics curriculum that learners find difficult to master, it is important that learners are taught the concept meaningfully and effectively. Teachers' mathematical knowledge for teaching fractions plays a significant role in this case, especially in primary schools or at the elementary level. The Curriculum Assessment and Policy Statement (CAPS) for Foundation Phase Mathematics, Grade R-3 (Department of Basic Education [DBE], 2011) outlines Grade 2 fraction sub-topics such as the use and

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naming of unitary fractions including halves, thirds, and fifths, recognising fractions in diagrammatic form and writing fractions as one half ($\frac{1}{2}$). This implies that the teaching of fractions in primary schools starts in Grade 2 and progresses into the higher grades. As these fractional concepts advance, at Grade 5 level learners are expected to have mastered fractional concepts such as comparing and ordering fractions to at least twelfths, adding and subtracting fractions with the same denominators, mixed numbers, and recognising and using the equivalency of fractions as outlined in the CAPS: Intermediate Phase Mathematics, Grade 4-6 (DBE, 2011).

Problem of Statement

It appears that many South African teachers struggle to master the content of the mathematics they teach (Bansilal, Brijlall & Mkhwanazi, 2014). Taylor and Vinjevold (1999), Carnoy, Chisholm and Chilisa (2012) observe that over the past years, ongoing low learner performance in mathematics has led to increasing interest in understanding how teachers' pedagogical practices and content knowledge contribute to patterns of poor academic performance. Research and evaluation of mathematics interventions point to a lack of foundational mathematical knowledge as one of the key factors in poor performance.

In addition, Fleisch (2008) maintains that poor performance starts early in the foundation phase where learners acquire basic skills that they need to further their studies. This is where primary school teachers should equip learners with the necessary mathematical knowledge, skills, and attitudes. The most pressing question is why these learners have only superficial and inadequate knowledge of fractions when the curriculum advocates the teaching of an understanding of fractions.

In South Africa, mathematics performance in Grade 5 is not satisfactory. In the years 2011–2013, the DBE (DBE 2011, 2013) administered the Annual National Assessment (ANA) in an effort to improve the quality of education and to identify the weaknesses or knowledge gap that South African learners were facing in Mathematics. The ANA reports for 2011 and 2012 showed that the performance in mathematics of Grade 6 learners was below 50% (DBE, 2011, 2012). An analysis of this report suggested that teachers' poor content knowledge when teaching fractions and their incorrect methods of teaching fractions were two of the reasons for South African learners' poor performance in national assessments in mathematics.

Theoretical Framework and Literature Review

There will not be any effective teaching and learning if teachers do not know the subject they are teaching. Ball et al. (2008) argue that teachers must know their subject well, or they will be unlikely to have the information they need to help their learners learn. Ball et al. (2008) add that simply knowing the subject well is not good enough for teaching as teachers should know mathematics in ways that are useful in making sense of learners' mathematics work and in choosing powerful ways to represent the subject in a way that is understandable for learners

In this article we have used Mathematical Knowledge for Teaching (MKT) as a theorical framework. Mathematical knowledge for teaching refers to the knowledge that is specific to the teaching profession as opposed to the kind of knowledge used by other professions such as engineering and accounting. Teachers need to have adequate, in-depth mathematical knowledge to teach their subject. Ball et al. (2005) ask what teachers need to know, and to be able to do, to successfully teach mathematics.

Ball, Hill and Bass (2005) focus explicitly on *how* teachers need to know the content they are teaching. They argue that teachers need the *how* and *wher*e to *use* mathematical knowledge in the practice of their teaching. In their study, they observed the demands of teaching mathematics and concluded that these require mathematical knowledge and skill.

Ball, Thames and Phelps (2008) refer to mathematical knowledge for teaching as the knowledge required in everyday tasks, such as explaining, defining, and representing concepts to learners, listening to learners' talk, working with learners' thinking or ideas, commenting on learners' work and controlling their work. This suggests that everyday tasks should be carried out effectively. The teaching of fractions demands that teachers have the mathematical knowledge and skills to teach the concept , in this case to Grade 6 learners. Mathematical knowledge for teaching requires fractional mathematical reasoning, which most adults do not regularly require.

Many of the subtopics that form part of learning and teaching fractions, such as comparing and ordering common fractions, including tenths and hundredths, adding and subtracting fractions in which one denominator is a multiple of another, identifying whether fractions are proper, improper or mixed, converting fractions to percentages and decimals, and equivalent fractions are introduced in Grade 6. These require a teacher with a deep understanding of fractions to explain concepts so that learners understand them.

Teachers should know how to introduce, explain and represent fraction concepts using models or concrete objects to encourage abstract thinking in their learners. These concepts should be taught or conveyed to learners in a way that allows them to grasp or understand them. These researchers believe that before teachers can teach algorithms or procedural methods to solve fractions, they should consider the conceptual understanding of fractions.

Ball et al. (2008) outline the domains of mathematical knowledge for teaching (MKT) that teachers need to carry out their work as teachers. They indicate that teachers require a great deal of knowledge and expertise in teaching the subject matter, as shown in Figure 2.1 below.

According to Ball et al. (2008), the teacher's knowledge, as indicated in Figure 1, is divided into two domains namely *subject matter knowledge* and *pedagogical content knowledge*. Subject matter knowledge has three domains. The first is common content knowledge (CCK), referred to as the mathematical knowledge that anyone might have. Examples of common content knowledge include knowledge of algorithms and procedures such as adding fractions, comparing fractions, changing improper to proper fractions and recognising wrong answers.

Specialised content knowledge (SCK) is defined as the mathematical knowledge and skill needed specifically by teachers in their work of teaching; it is also used when assessing learners' errors. The last domain is horizon content knowledge (HCK). Pedagogical content knowledge, according to Shulman (1986), also has sub-domains, namely knowledge of content and teaching (KCT), knowledge of content and students (KCS) and knowledge of content and curricula (KCC). All these categories – common content knowledge (CCK), horizon content knowledge, specialised content knowledge (SCK), knowledge of content and students (SCK), knowledge of content and teaching (KCT) and knowledge of content and curriculum form the practice-based theoretical framework of MKT.

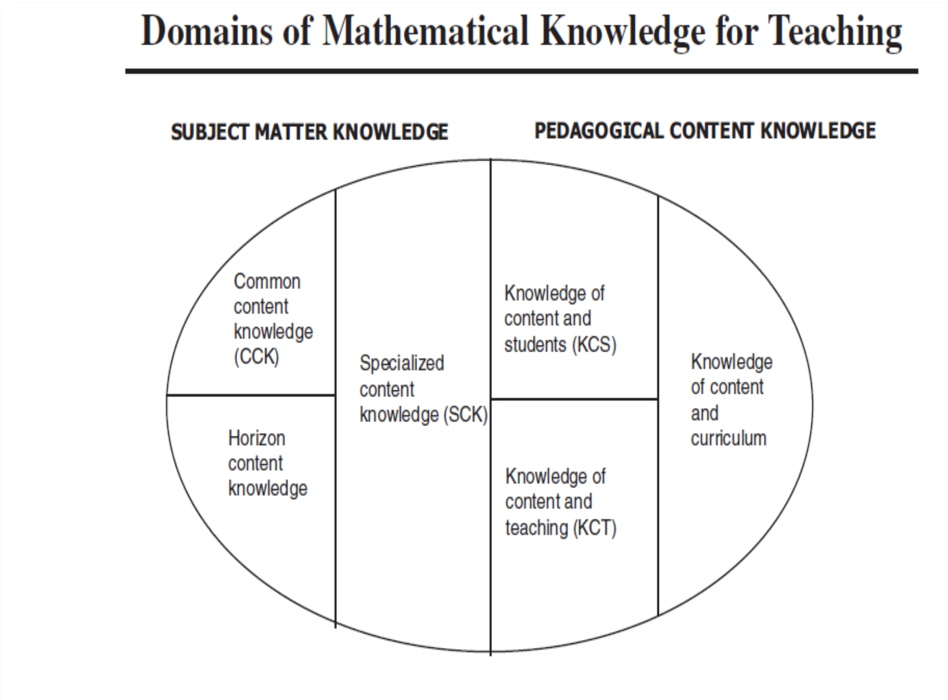


Figure 1.
Domains of Mathematical Knowledge for Teaching (MKT)

Among the sub-domains identified by Ball et al. (2008), *specialised content knowledge* requires the teacher to have a deeper understanding of fractions that will allow him/her to explain new ideas, work out mathematical fraction problems in various ways and analyse learners' explanations. This *specialised content knowledge* is unique and exceptional since it allows teachers to apply various ways of solving mathematical problems. It is vital because it does not channel learners' thinking but allows them instead to explore.

It would be difficult for a teacher to teach learners about fractional mathematics concepts without knowing the content. It is generally accepted that what a teacher teaches, and how this is taught is a task requiring the teacher's own knowledge of the subject. Mathematics teachers should be knowledgeable about the mathematics they are teaching. Ball et al. (2005) argue that specialised content knowledge includes the teacher's ability to use content knowledge to access different representations and the knowledge of different methods for solving mathematics problems that may arise in their teaching.

Balls' notion and Shulman's pedagogical content knowledge (PCK) are intertwined or interwoven. In his presidential address, Shulman (1986) pointed out that teaching entails more than knowing the subject matter. He indicated that besides the content knowledge and curricular knowledge, teachers need a third type of knowledge. He recognised a special domain of teacher knowledge which is referred to as pedagogical content knowledge (PCK). He regarded the knowledge of teaching and the knowledge of the subject matter as equally important. Shulman (1986) advised teachers not to separate content knowledge from pedagogy because both are needed if they are to carry out their work effectively. He argues that teachers need to know and understand more of their subject than other users because teaching requires a transformation of knowledge into a form that learners can understand.

Shulman's (1986) notion of PCK is viewed as the ability of teachers to use their knowledge of mathematics to break down, represent, formulate, explain, illustrate and make the concepts understandable to learners. This emphasises the idea that mathematics teachers should use their mathematical knowledge to explain fractional concepts and deliver them to learners in a way that learners fully comprehend.

Shulman (1986) also points out that teaching involves more than knowing the subject matter; teaching also entails transforming this knowledge for the learners in an understandable manner. He further argues that besides knowing the content well, the teacher needs to know how to deliver or convey his or her knowledge in such a way that the learners understand it. This means the teacher should know the mathematical concepts of fractions well to be able to deliver them to learners in a way that they can be comprehended. Teachers are, therefore, urged to apply Shulman's (1986) ideas when teaching fractions.

Teacher content knowledge should represent a deep understanding of the concepts to be mastered by learners. Adler and Davis (2006) argue that teachers' mathematical knowledge is an important factor in learners' success. Teachers' mathematical knowledge has an impact on their classroom teaching. Adler and Davis (2006) argue that a teacher requires a deep and broad understanding of mathematics.

Ball et al. (2004) propose eight categories of mathematical teaching that teachers frequently engage with. These eight categories are the tasks of teaching that occur most often in teachers' work. Kazima, Pillay and Adler (2008) reduced the initial eight categories/aspects to six because they concluded that some of the them overlapped.
The six categories identified by Kazima, Pillay and Adler (2008) are as follows:

- *Defining,* which implies that the teacher provides a definition of a concept to learners.
- *Explaining,* which means that teachers explain problems to learners.
- *Representation,* which means that teachers represent an idea in a variety of ways.
- *Working with learners' ideas,* which means teachers engage with learners' expected and unexpected mathematical ideas.
- *Restructuring learners' tasks,* which refers to simplifying a problem or making it more complex.
- *Questioning*, which refers to posing and responding to questions as the lesson proceeds.

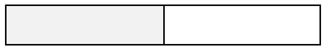
Three of these six categories, namely Defining, Explaining, and Representation were used in this study as indicators of teachers' mathematical knowledge or lack thereof.

Best Practices in Teaching Fractions

Ball et al. (2005) point out that for teachers to teach mathematics well, they need to break down or simplify their mathematical ideas to make them accessible to learners. This means that teachers need to know how to do mathematics and how to use mathematics in practice (Adler, 2004). Van de Walle (2013) observes that fractions are complex but important concepts in mathematics; they are used frequently in various measurements and calculations. The teaching of fractions requires teachers to shift their emphasis from the learning of rules to the development of a strong conceptual basis for fractions.

The Development and Definition of the Fraction Concept

Bassarear and Moss (2016) note that the word "fraction" is derived from the Latin word *fractus*, which comes from the word *frangere*, meaning to break. A fraction is a breaking of something that is a whole into smaller, equal parts. When we work with learners, we talk about the concept of a half. What does it mean and what is a half? In most cases, we take for granted that learners know the meaning of the word and conclude that they understand it. At some point in their lives they have shared things such as a pizza or a pie. Learners know how much half a pizza is. They know that when sharing a pizza with someone, the two pieces should be the same size (equal). Once it has been established that they understand the concept, we can build on it. If they know that a half is one of two parts that are the same size, then they should be able to understand that thirds are three parts of the same size, fourths are four equal parts and so on. This forms the basic construct for fractions. Van de Walle (2016) indicates, however, that the concept of fraction tells us only about the relationship between the part and the whole. Figure 2 provides an example of a fraction as part of a whole, as indicated by Van de Walle (2016):



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Area Model

This whole is a **rectangle**; the whole is divided into two **equal parts**. **Each** part is half of the whole; Two **parts**, therefore, make one whole.

Teaching Fractions by Unpacking of the Concept of Fractions

The unpacking or explanation of fractions requires a teacher to have a deep understanding of the concept. In this case, the teacher should start with learners' prior knowledge. Tall (1989) refers to learners' prior knowledge as cognitive roots that are essential for developing a concept by connecting and laying the foundation for learners' conceptual thinking. On the other hand, Essien (2009) talks of the first encounter, emphasising the importance of starting with learners' prior knowledge and connecting it to new knowledge. These researchers argue that in any pedagogical practice, the first counter needs to be addressed using mathematical concepts. The unpacking of a concept also requires the teacher to design the first counters as well as cognitive roots. The teacher should explain what a fraction is to learners. The teacher could pose questions to learners to arouse their interest about the topic or simply to establish their prior knowledge of fractions.

The teacher will then involve learners by defining, explaining and representing what a fraction is. At this stage, the teacher should use models to help learners understand fractions; a variety of models may be used to foster deep conceptual understanding, such as area models, length models and set models. Cramer and Wyberg (2009) indicate that the effective use of models in fraction tasks plays a significant role; learners appear to explore when a variety of models is used, and this builds their understanding of fractions (Cramer & Wyberg, 2009).

Different and appropriate representations of models of fractions broaden and deepen learners' understanding and help them to learn more easily. Van de Walle (2004) identified uses for models in the classroom, for instance to help learners develop new concepts and to make connections between concepts and symbols, and to assess learners' understanding.

When introducing fractions, the fraction symbol should be delayed until the fraction concept is stable. Van de Walle (2009) points out that the fraction symbol can prove to be a confusing notation for children, so learners should instead be encouraged to write the fraction names in words, for example 3 quarters or three quarters instead of $\frac{3}{4}$.

When unpacking the fraction concept, teachers are encouraged to refrain from using traditional methods of teaching. For instance, it seems that learners are encouraged to memorise rules without knowing where they come from. This may result in learners simply becoming blind followers of the rules without understanding them. Teachers are urged to use models instead of enforcing rules to overcome this. Using models makes fractions more concrete for the learner, and not just numbers on top of each other with no meaning. The learner will be able to estimate the answer before calculating, and evaluating the reasonableness of the final answer. Learners should be motivated to discover the fraction concept on their own by drawing or folding a piece of paper into equal parts and explaining these parts.

In this regard, Stohlmann, Cramer, Moore and Maiorca (2013) argue that if learners are taught the procedural way of working out fractions first, they are **less likely** to master the fraction concept. They believe that understanding the fraction concept first is **more powerful** and more generative than remembering mathematical procedures.

Teaching of Equivalent Fractions

Lamon (2002) explains that equivalence between fractions refers to the fact that many different fractions can be used to name the same quantity, depending on how the quantity is subdivided. Van de Walle (2016) adds that equivalent fractions are a way of describing the same amount using different sized parts; equivalence is about naming the same fractions in more than one way. Van de Walle (2016) argues that models may be used to develop conceptual understanding of equivalence, as illustrated in Figure 3 below. He explains that two fractions are equivalent if they are representations of the same amount.

Van de Walle and Lovin (2006) expand on the concept of equivalence when they state that to help learners create an understanding of equivalent fractions they should use models to find different names for a fraction. They (Van de Walle and Lovin, 2006, p. 66) provide the following important point about equivalent fractions: "Two equivalent fractions are two ways of describing the same amount by using different-sized fractional parts. For example, in the fraction $\frac{6}{8}$, if the eighths are taken in twos, then each pair of eighths is a fourth. The six-eighths then can be shown as $\frac{2}{8} = \frac{1}{4}$

and $\frac{6}{8} = \frac{3}{4}$ can be shown as three fourths". Figure 3 illustrates this:

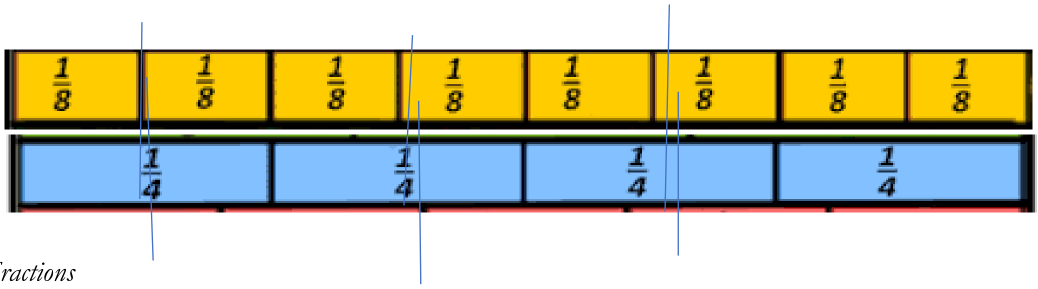

Figure 3.
Equivalence Fractions

According to Gould (2005), set models can be used to develop the concept of equivalence. Petit et al. (2010) believe that length models are very important in developing learners' understanding of fractions; length or measurement models can be paper folding strips, where one piece is measured in terms of the smallest strip. Each length is a different colour for ease of identification. Petit et al. (2010) note that strips of paper are also length models; these can be folded to produce fraction strips by learners. The teacher is required to do the activity with the learners to develop the concept of equivalent fractions and demonstrate or show equivalence in fractions. Working with learners will allow them to develop the concept on their own. Van de Walle et al. (2010), Cramer and Wyberg (2009), and Lamon (2008) emphasise that concrete representation is key if learners are to comprehend fractions. Learners can be asked to use crayons to colour the strips. Learners can also be asked to take a piece of paper from an exercise book; the teacher will then instruct them to cut the piece of paper into nine strips that are exactly equal in size and shape.

Teaching Comparing and Ordering of Fractions

Petit et al. (2010) explain that when looking to see whether two or more fractions are equal, we are comparing them by identifying which is smaller or bigger than the others. Van de Walle (2016) adds that comparing fractions means checking which part of the same whole is bigger or smaller than another part. When comparing fractions, the whole must always be the same. A fraction wall chart can also be used when comparing fractions (Van de Walle, 2016). Using a fraction wall allows learners to see that all fractions have the same whole, and that $\frac{1}{2}$ is greater than $\frac{1}{3}$ and also greater than $\frac{1}{4}$ and that $\frac{1}{3}$ is greater than $\frac{1}{4}$ and so on, as illustrated in Figure 4

Petit et al. (2010) note that comparing fractions using rules can be effective in arriving at the correct answer, but if learners are taught these rules before they master the fraction concept of relative sizes their chance of making mistakes is high. They stress that using rules requires no thought about the size of a fraction. If learners are taught these rules before they are encouraged to think about the relative size of different fractions, they may be less likely to develop a number sense when it comes to fraction size.

Petit et al. (2010) regard the number line model as a good one to help learners to develop a better understanding of the relative sizes of fractions. They argue that the number line should extend beyond 1 when comparing fractions like $4\frac{1}{2}$. The main aim of using models is to give learners a grounded understanding of the concept, and to avoid simple memorisation of an algorithm method. Learners should have a sound understanding of comparing fractions and their ordering. Cramer and Whitney (2010) recommend that teachers help learners to understand the meaning of the fractions, to make sense of them and to avoid rote procedures. Learners should also be able to understand that fractions are numbers, as well as learning how to use models. Fazio and Siegler (2011) argue that misconceptions with fractions stem from a lack of conceptual understanding.

When using the fraction wall illustrated in Figure 4, learners should notice that the fraction with the bigger denominator is the smallest when comparing fractions with the same numerators.

Exercises like this, using greater than signs **(>)** and less than signs **(<)** can be given to learners to work out.

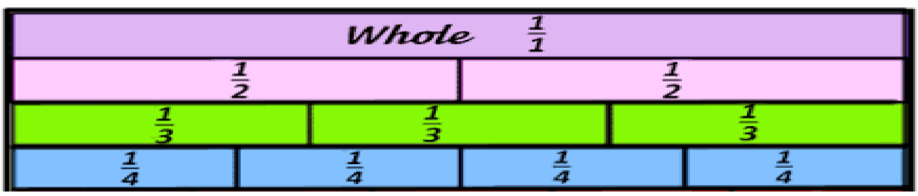

Figure 4.
Fraction Chart

Once learners have mastered visual representations, they should be able to create their own visuals to work out fractions

Method

Research Design

The study took a mainly qualitative approach. A qualitative research approach was since the study investigated the practice of intermediate phase teachers from specific schools when teaching fractions. Using a qualitative research approach, the study reports on two case studies in which the researchers explored intermediate phase teachers' mathematical knowledge to introduce, unpack, develop, and define fractions for Grade 6 learners.

Participants

The sample in this study comprised two Grade 6 mathematics teachers from neighbourhood schools. These two teachers were purposefully and conveniently selected as the nearest individuals to serve as participants and based on their considerable experience of teaching mathematics, as indicated in Table 1:

Table 1.
Structures of Participants

Participants	Years of experience in teaching	No. of years teaching mathematics to Grade 6	Qualifications	Gender
Eddy	15 years	10	Primary teacher's Diploma/Degree	Male
Rose	25 years	16	Primary teacher's Diploma/Degree	Female

Data Collection

The data were collected using two methods: observing and interviewing teachers who offered mathematics at a primary school. The main data collection method was observation. A researcher observed the teachers in practice, taking the role of a non-participant observer (complete observer) in the classroom. Two teachers were observed as they went about their work of teaching fractions to Grade 6 learners. Three double lessons per teacher per school were observed. One lesson is 30 minutes in length, and six lessons by each teacher were observed, with a total of 12 lessons. However, this article reports on one observed lesson only by each teacher, which concerned comparing fractions. The lesson observation notes were transcribed and chunked into evaluative events/episodes.

Data Analysis

Once the data from the observations and interviews had been collected, the researcher used as observation schedule to check whether the categories identified during the observation corresponded to the categories condensed by Kazima (2008) from the eight aspects developed by Ball et al. (2004).

Kazima et al. (2008) reduced the categories/aspects to three because they concluded that some of them overlapped. The three categories selected were as follows:

- ***Defining,*** which means that the teacher provides learners with a definition of a concept.
- ***Explaining,*** which means teachers explains problems to learners.
- ***Representation,*** which means teachers represents an idea in various ways.

Results

Eddy's Lesson Observations

Category 1: How do teachers introduce, define, explain and represent the concept of fractions to Grade 6 learners?

Comparing and Ordering Fractions

Below is a extract from a lesson in which the teacher (Eddy) defined, explained and represented the concept of fractions in his teaching. LS represents "learners" and L represents "a learner in class", for example, L1 represents Learner 1, L2 represents Learner 2 etc.

00:08–00:09

1.	**Eddy:**	Good morning, class.
2.	**LS**:	Good morning, sir.
3.	**Eddy**:	Sit down.
4.	**LS**:	(Sit down and listen to the teacher.)
5.	**Eddy**:	Today we are going to learn about comparing and ordering of fractions [turns to the board and writes] ($\frac{5}{7}$; $\frac{2}{7}$; $\frac{6}{7}$; $\frac{4}{7}$; $\frac{1}{7}$). Look at these fractions. What do you realise?

00:10

6.	**L1**:	They have the same denominators.
7.	**Eddy**:	Yes, they have the same denominators. Comparing fractions like thisis easy because if they have the same denominators, the fraction with the bigger numerator is the biggest. So, who can come and arrange them for us?
8.	**L2**:	[Stands and goes to the chalkboard, writes] $\frac{6}{7}$; $\frac{5}{7}$; $\frac{4}{7}$; $\frac{2}{7}$; $\frac{1}{7}$
9.	**Eddy**:	Good, this is how we order and compare fractions. The same applies if they have the same numerators; the fraction with the bigger denominator is the smallest. [writes $\frac{1}{3}$; $\frac{1}{5}$; $\frac{1}{7}$; $\frac{1}{4}$; $\frac{1}{2}$ on the board. Someone, come and arrange these fractions from the smallest to the biggest.
10.	**L3**:	[Stands and goes to the board and writes] $\frac{1}{7}$; $\frac{1}{5}$; $\frac{1}{4}$; $\frac{1}{3}$; $\frac{1}{2}$

00:20

11.	**Eddy**:	You are correct. Clap hands for him. Now because you understand, let us continue comparing fractions with different numerators and denominators. When comparing fractions with different numerators and denominators, we should make them to have the same denominators by looking for the LCM which means lowest common multiple for example [writes on the board] fractions that are multiples of the other $\frac{2}{3}$ and $\frac{1}{6}$ we should multiply $\frac{1}{6}$ by $\frac{2}{2}$ like $\frac{2}{3} \times \frac{2}{2} = \frac{4}{6}$ now because the denominators are the same, we can compare them and the answer is $\frac{2}{3} > \frac{1}{6}$. If they are not multiples of the others find the LCM. (Comparing and Ordering Fractions, Lesson 1)

It is evident from this lesson (line 7) that the teacher, Eddy, went on to explain a rule by saying, *Yes, they have the same denominators. Comparing fractions like this is easy because if they have the same denominators, the fraction with the bigger numerator is the biggest. So, who can come and arrange them for us?* After the teacher had provided an explanation of comparing and ordering of fractions with the same denominator, he then asked if there was any learner who could provide an answer to his explanation of a "rule". It appears that while Eddy was explaining the rules for comparing fractions, he did not explain how this rule originated. He was encouraging learners to master "rules and procedures" at the expense of developing the concept of fractions for them. This kind of teaching encourages memorisation rather than conceptual understanding.

We believe that if the teacher wanted learners to master the concept of comparing and ordering fractions, he should not have started by foregrounding the "rule" *if they have the same denominators, the fraction with the bigger numerator is the biggest.* The teacher could have used teaching strategies that would have allowed learners to discover the rule by themselves. For example, the teacher could have used a number line representation or a diagram representation to develop the concept of comparing and ordering fractions. The teacher's explanation of the rule should have come as a reinforcement of what learners had already discovered through their own investigation. It was no surprise that L2 (in line 8) provided the correct answer $\frac{6}{7}$; $\frac{5}{7}$; $\frac{4}{7}$; $\frac{2}{7}$; $\frac{1}{7}$.

Similarly, as reflected in lines 9 to 11, the teacher used a similar strategy of foregrounding procedures, routines and rules rather than developing the concept so that learners could discover these rules by themselves. Again, as in line 11, Eddy used procedures in his teaching to compare fractions with different denominators.

When Eddy said "*Now because you understand*" it was his assumption that they had understood because he had provided them with the rules and procedures and they repeated these rules, and some of them (L1 and L2) arrived at the correct answer. Eddy assumed that all learners were at the same level of understanding, without checking whether the other learners understood. If fact, it was not even certain that L1 and 2 had understood the concept of comparing and ordering fractions based on algorithms and routine procedures. Based on his assumption, Eddy moved on to compare fractions with different numerators and denominators. Even at this stage, he taught procedures, saying:

> When comparing fractions with different numerators and denominators, we should make them to have the same denominators by looking for the LCM which means lowest common multiple for example [writes on the board] fractions that are multiple of the other $\frac{2}{3}$ and $\frac{1}{6}$ we should multiply $\frac{1}{6}$ by $\frac{2}{2}$, at the expense of developing the concept. (Eddie)

The teacher used only one form of representation in his lesson, which was number representation. He dominated the lesson by providing rules and procedures without letting learners discover them by themselves. The teacher could have approached this lesson differently if he wanted learners to develop the concept, using a number line, drawing or fraction chart to compare and order fractions. In this lesson, there was no evidence of different representations to teach the comparing and ordering of fractions. When asked to reflect on his lesson during the interview, Eddy implied that teaching by emphasising procedures and rules without considering learners' understanding was the norm in his teaching. This is supported by an extract from the interview:

Researcher:	You mentioned that you have ten years' experience in teaching mathematics in Grade 6, and you have taught fractions many times. Do you always teach fractions the way you did this year in the lessons that I observed? Does your teaching develop learners to master fractional concepts?
Eddy:	This is the way I normally teach these learners because I just assume that the lower grade teachers have already introduced the fractional concept, therefore I am teaching them rules on how to convert fractions and so on.

From the extract above, it appears that Eddy's teaching had always been dominated by the teaching of rules, without much focus on the understanding of the concepts. He said, *This is the way I normally teach these learners.* Eddy's justification for teaching rules without ensuring understanding of the concept was based on the assumption that teachers of the lower grades had already introduced these concepts and ensured an understanding among learners. Eddy had been teaching in this way for the past ten years.

Rose's Lesson

The second teacher observed was Rose from school B. Her lesson went as follows:

Category 1: How do teachers define, explain and represent the concept of fractions to Grade 6 learners?

Unpacking the concept of fractions

The extract below illustrates how the teacher (Rose) defined, explained, and represented the concept of fractions in her teaching.

00:23

5.	**Rose**:	[Gives learners an A4 paper sheet each.] This is the A4 paper, I am going to tell you what to do and you should listen.
6.	**LS**:	[Listening and paying attention to the teacher.)**]**
7.	**Rose**:	Fold your A4 paper once and make sure the two parts are on top of each other. Are you following?
8.	**LS:**	Yes, ma'am

00:26

9.	**Rose**:	Unfold your papers. What do you notice?
10.	**LS:**	(Raising their hands.)
11.	**Rose**:	L1 tell us what can you about the A4 paper.
12.	**L1**:	The paper has 2 parts now.
13.	**Rose**:	Do you agree with her, learners?
14.	**LS:**	Yes, ma'am
15.	**Rose**:	What else can you say?
16.	**L2**:	Two parts are equal. We had a whole of the A4 paper but now it is divided into 2.
17.	**Rose**:	What else can you say about the paper?
18.	**L3**:	We now have halves no longer a whole.

00:39

20.	**Rose**:	You are correct. Each part is half of the paper. Now let us fold it twice and unfold and see what happens. How many parts do you see now?
21.	**L3:**	I see four parts.
22.	**Rose:**	[Poses a question to L3 who has just answered.] What can you say about these parts?
23.	**L3:**	They are equal.
24.	**Rose:**	[Orders learners.] Fold the paper three times now.
25.	**LS:**	[Doing what was ordered.]
00:45		
26.	**Rose:**	Unfold your paper. (Learners do what the teacher has told them.) How many parts do you see?
27.	**LS4:**	There are eight equal parts.
28.	**Rose:**	Do you agree with him?
29.	**LS**	[Answers by shouting.] Yes, ma'am.
00:49		
30.	**Rose:**	Learners, continue to fold the paper four times and unfold, then five times and unfold and see how many parts you see.
00:54		
31.	**Rose:**	So, learners, we were dealing with fractions all this while and a fraction is part of a whole, half is part of a whole, a quarter is a part of a whole etc., do you understand?
32.	**LS:**	Yes ma'am.

(Fractional Concept, lesson 1,t ime interval 00:20–00:54)

The extract above illustrates how Rose executed her teaching of mathematical work. The aspects of defining, explaining, and representing were observed in Rose's lessons. In Lesson 1, it was clear that the teacher wanted learners to develop a conceptual understanding of fractions and to recognise the fractional concept on their own. In line 7:

"*Fold your A4 paper once and make sure the two parts are on top of each other*". Line 9, *"Unfold your papers. What do you notice?"* Line 20*, "Now let us fold it twice and unfold and see what happens. How many parts do you see now*?" Line 30, "*Learners, continue to fold the paper four times and unfold, then five times and unfold and see how many parts you see?"*

There was evidence of verbal representation such as half of an A4 paper. In terms of different representations, the teacher could have used other representations such as circular, rectangular or square diagrams, i.e. a diagrammatical representation, on the board, shading some parts of a whole for learners to form a clear picture of other representations. Using A4 paper may lead learners to think that this is the only object to use in the development of fractional concepts.

At this stage of developing the concept of fractions, the Rose had done well because at this stage the teacher should only use verbal expressions, which is what Rose did. The teacher and the learners showed each other halves, fourth, eighths and sixteenths. The only thing she could have added was to allow learners to develop other fraction names such as thirds, fifths, sixths themselves, and the rest would have followed.

It was also evident that the teacher explained and defined fractions for her learners: Line 31 "So*, learners, we were dealing with fractions all this while and a fraction is part of a whole, half is part of a whole, a quarter/fourth is a part of a whole etc., do you understand ?"*

It appears that this was how Rose generally taught mathematical concepts; when asked if she always taught fractions as she had in this lesson, she replied:

Rose: This is how I am teaching. When I introduce a lesson, I make sure that learners understand the concept before teaching them the rules. I want learners to master the concept first. I have realised that mastering the concept is important because after mastering the concepts they learn with ease when teaching them the *how* part of working out fractional rules.

It is evident from the above extract that Rose, when introducing a lesson, would make sure that learners understood the concept before teaching them the rules. Rose seemed to have found the secret of teaching learners a concept with understanding and the benefit thereof. She said, "*They learn with ease."* When asked what she would do to develop conceptual understanding in her teaching of fractions, she said that in the absence of teaching resources she had to improvise, as indicated in the interview extract below:

Researcher: From your knowledge and point of view, what do you think are the main things a teacher needs to know in order to develop conceptual understanding of fractions for learners to master?

Rose: From my perspective, for learners to understand better, teaching resources should be available and if they are not available, as a teacher I must improvise.

Discussion

The theoretical lens that informed this study on the notion of teaching fractions to Grade 6 learners was drawn from Ball et al.'s (2008) framework, Shulman's (1986) pedagogical content knowledge (PCK) and the constructivist theory.

This section responds to the first research question: How do teachers unpack/introduce, define, explain and represent the concept of fractions to Grade 6 learners? This study revealed that Eddy merely engaged in **explaining** procedures to learners when teaching fractions. He was spoon-feeding learners with mathematical rules; he wanted learners to memorise the rules without understanding where the rules came from. The memorised rules may be forgotten in the long run. Stohlmann et al. (2013) advise teachers not to teach the procedural way of working out fractions, firstly because if they do so, the learners are **less likely** to master the fraction concept. Stohlmann et al. (2013) emphasise that teachers should refrain at all costs from encouraging the memorisation of rules. Ball and Bass (2005) add that for teachers to teach mathematics well, they need to unpack or decompress their mathematical ideas to make them accessible to learners. This implies that if teachers are to teach the concept of fractions, they should know what is expected of them. Skemp (1976) points out that instrumental understanding refers merely to being able to apply a sequence of steps without necessarily knowing why they are being applied in that way, or what they mean; that is, rules without reasons. In contrast, relational understanding is knowing what to do and why, which means that learners should be told where and how the rules originated. Hiebert (1996) also notes that mathematical tasks that encourage learners to use procedures that are not actively linked to meaning or that consist of memorisation are viewed as of lower-level cognitive demand in the learning of mathematical fraction concepts.

Eddy's learners were passive participants in the class. Constructivism (Piaget, 1964) perceives learners as creators of their own learning and as active participants in the learning process; Eddy's learners should have been discovering rules on their own and making sense of mathematics.

The study revealed that Eddy failed to provide learners with the opportunity to discover the mathematical rules on their own (Stohlmann, Cramer, Moore and Maiorca, 2013). They sat and listened to what their teacher was saying. His lessons were teacher-centred and learners were passive participants. Few learners repsonded to the questions he posed, and the majority were passive. Eddy asked: *"Learners, how do we know that a fraction is [a] common fraction?"* L1 answered: "*We know if there is a top number and a bottom number.*" Blaise (2011) observes that teachers use the teacher-centred approach with direct instruction in behaviourism.

In Eddy's first lesson, comparing and ordering of fractions, it was evident that he used procedures, routines and rules that applied to fractions with the same denominator. For example, he said, "*When comparing fractions with different numerators and denominators, we should make them to have the same denominators by looking for the LCM".* To develop the concept of comparing and ordering fractions, he should have used different models such as linear or circular models to represent the concept meaningfully. His old-fashioned approach to teaching may have led to learners making mistakes and developing misconceptions. Sarwadi and Shahrill (2014) embrace the Piagetian view that when learners fail to assimilate or accommodate, a gap is formed in the learning of the concept and this leads to misconceptions.

Furthermore, Eddy's teaching did not resonate with the theory of constructivism (Piaget, 1964), which states that the teacher's role is that of facilitator and motivator. In his teaching, Eddy was not facilitating learning by encouraging learners to take control of their own learning. His teaching was teacher-centred rather than learner-centred (Machaba, 2017). He failed to allow learners to construct their own knowledge and understanding using their existing experiences. The fraction concepts were not fully developed in any of Eddy's lessons; only the mathematical rules were emphasised, which is regarded as poor delivery of content. Van de Walle (2016) notes that rushing to procedures may cause learners to make errors and form misconceptions, and this could hamper their conceptual understanding. Teacher Eddy should have developed the fraction concepts using several models for learners to develop a solid and deeper understanding of fractional concepts. The researcher is of the view that Eddy's way of teaching suggested that his mathematical knowledge for teaching was inadequate because in all the lessons observed, he failed to develop the concept of fractions meaningfully for learners.

The findings reveal that Eddy could not unpack the concept in such a way that learners developed conceptual understanding, although he could explain and define fractional concepts in such a way that learners would understand.

In Ball et al.'s (2008) terms, we could say that Eddy possessed the *content knowledge*, but not the specialised content knowledge that requires the teacher to have a deeper understanding of fractions that would have allowed him to explain new ideas and work out fractional mathematics problems in a variety of ways, and analyse learners' explanations. Olivier (1989) argues that errors are indicators of the existence of misconceptions and happen as a result of many factors, for example the way in which teachers teach fractions. From a constructivist point of view, errors are intelligent constructs of knowledge by learners.

Ball et al. (2008) believe that a teachers should have knowledge of the subject they are teaching. Shulman's (1986) notion of PCK is viewed as the knowledge to teach the subject matter, the knowledge to formulate and present the subject matter so that it is comprehensible to learners. The frameworks of both Ball and Shulman indicate that teachers should know their learners and understand their common difficulties, errors and misconceptions, which means that they should have specialised content knowledge (SCK) and knowledge of the curriculum and their students (KCS). Ball et al. (2008) emphasise that mathematics teachers should use their mathematical knowledge to unpack fractional concepts and deliver them to learners in a way that they will fully comprehend.

On the other hand, analysis revealed that Rose wanted her learners to develop a conceptual understanding and recognise the fraction concept independently. From a constructivist perspective, learners construct knowledge and understanding on their own, connecting their web of ideas. In Rose's first lesson on fraction concepts, she developed the concept successfully by involving learners in an activity in which each learner was folding and unfolding a piece of paper. In this lesson, learners were active participants in their learning, and this resonates with constructivism as the theory specifies that learners are active agents of their own learning process. Rose knew that learners should master the fraction concept before being introduced to algorithms. This is supported by Van de Walle (2009), who argues that teachers should not rush to algorithms as they can delay learners' understanding of the concept.

Rose used verbal instructions and learners were able to understand that they were folding the paper in half, or into a fourth, sixth etc. Learners discovered the concept of fractions on their own. Analysis revealed that there was a clear indication that learners were developing the concept of fractions with their teacher. Learners were actively engaged in this activity, and they were able to discover the fraction concepts on their own. Analysis revealed that Rose used linear modelling when her learners were folding and unfolding the paper to develop and name the fractions, and this is supported by Petit, Laird, Marsden and Ebby (2010) who found that the length model for fractional concepts was important in developing learners' understanding of fractions and naming them. This is supported by Lamon (2008), who points out that the naming of fractions helps learners to use the correct language and to understand the concept of fractions.

In her second lesson on fractional notations, Rose used a circular area model, demonstrating part, whole and equal sized parts with an apple (Cramer and Wyberg, 2009). This resonates with Van de Walle (2007) who believes that teachers should emphasise fractional parts as equal shares or equal sized portions of a whole or unit.

Once she felt that her learners understood the concept, Rose moved to fractional notation where she used an apple as a model. This resonates with Cramer et al. (2008) who support the idea of using a circular area model because they found that these were effective in developing the fractional concept. Rose cut up an apple in front of her learners and asked them to watch what she was doing. She said: "*Look at me, all of you, I cut it like this" (*showing learners how she cut it). The apple was cut or divided into four parts and one part of the apple was given to a learner, with Rose stating: "*I give Mpho this part"* (referring to one part). She then asked the class questions about the parts of the apple such as "*How many parts was this apple divided into?"* Through her teaching, her learners discovered that when we talk about fractions, we are actually referring to **equal-sized parts**. One of her learners (L5) responded *that "[it]t was divided into four* ***equal parts."*** Rose then gave the learners the notation of the concept of fractions symbolically, for example $\frac{1}{4}$. This corresponds to Van de Walle (2016), who argues that representation at this stage is **symbolic.** This is where she should have told learners that $\frac{1}{4}$ is called one-fourth of an apple, not one over four, however. As Siebert and Gaskin (2006) and Cramer and Whitney (2010) emphasise, teachers should avoid expressions such as one out of two, two out of six and so on when teaching learners learners.

These learners understood that the top number represents parts that were used, considered, or taken out, whereas the bottom number indicates the number of equal parts into which the apple was cut. The bottom number also gives the fraction a name, for example a fourth. Rose explained the fraction concept successfully, as is supported by Ball and Bass (2005) who argue that if teachers are to teach mathematics well they need to break down their mathematical ideas so that they are accessible to learners. Rose unpacked the fraction concept in a way learners could understand. Shulman's (1986) notion of pedagogical content knowledge is understood as the knowledge of teaching the subject matter, formulating and presenting the subject matter so that it is understandable for learners.

Ball et al. (2008) refer to mathematical knowledge for teaching as the knowledge required in everyday tasks such as explaining, defining and representing concepts to learners. They argue that teachers require a great deal of knowledge and expertise to carry out the work of teaching the subject matter. Shulman (1986) warns teachers not to separate content knowledge from pedagogy because both are needed if teachers are to carry out their work effectively. He argues that teachers need to know and understand more of their subject than other users because teaching requires a transformation of knowledge into a form that learners can understand.

Conclusion

This study's findings revealed that of the two teachers observed, Eddy did not meet all six criteria when unpacking the fraction concept. His mathematical knowledge was inadequate to explain the concepts successfully. He used a traditional approach to teach fractions, encouraging learners to memorise rules without necessarily understanding them. He used a teacher-centred approach with direct instruction. The learners followed the rules blindly because he did not tell them how the rules originated. On the other hand, Rose followed a learner-centred approach, characterised by: a variety of productive questions; increasedlearner involvement; social, verbal, concrete physical and experiential engagement with fraction concepts; and active construction of ideas by learners.

From the findings, the researchers made the following recommendations.

Regarding the use of models, it is recommended that teachers use different representations such as area models, circular, rectangular models, set models and length models to develop the concept of fractions successfully. Cramer and Wyberg (2009) found that the effective use of models in fraction tasks plays a significant role. Learners seem to explore when a variety of models is used, which builds learners' understanding of fractions (Cramer & Wyberg, 2009).

It is therefore recommended that school procurement committees purchase fractional charts to make the teaching of fractions more effective. It is also recommended that schools have internet facilities to download information related to fractions.

Acknowledgements

This research was supported by the University of South Africa.

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