J. Algebra Comb. Discrete Appl.
8(3) • 151–160

Received: 15 May 2020
Accepted: 15 May 2021

Journal of Algebra Combinatorics Discrete Structures and Applications

Rotated D_n-lattices in dimensions power of 3^*

Research Article

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Abstract: In this work, we present constructions of families of rotated D_n-lattices which may be good for signal transmission over both Gaussian and Rayleigh fading channels. The lattices are obtained as sublattices of a family of rotated $\mathbb{Z} \oplus \mathcal{A}_2^k$ lattices, where $\mathcal{A}_2^k$ is a direct sum of $k = \frac{3^{r-1}-1}{2}$ copies of the A_2-lattice, using free $\mathbb{Z}$-modules in $\mathbb{Z}[\zeta_{3^r} + \zeta_{3^r}^{-1}]$.

2010 MSC: 11H06, 11R18, 94B12

Keywords: Lattices, Cyclotomic fields, Signal transmission

1. Introduction

A lattice $\Lambda \subseteq \mathbb{R}^n$ is a discrete set generated by integer combinations of n linearly independent vectors in $\mathbb{R}^n$ over $\mathbb{R}$. Its packing density $\Delta(\Lambda)$ is the proportion of the space $\mathbb{R}^n$ covered by congruent disjoint spheres of maximum radius [8]. A lattice Λ has diversity $m \leq n$ if m is the maximum number such that for all $\mathbf{y} = (y_1, \ldots, y_n) \in \Lambda$, with $\mathbf{y} \neq \mathbf{0}$, there are at least m non-zero coordinates. Given a full diversity lattice $\Lambda \subseteq \mathbb{R}^n$, with $m = n$, the minimum product distance is defined as $d_{min}(\Lambda) = \inf\{\prod_{i=1}^{n} |y_i| \text{ for all } \mathbf{y} = (y_1, \ldots, y_n) \in \Lambda, \ with \ \mathbf{y} \neq \mathbf{0}\}$ [5].

Lattices have been considered in different areas, especially in coding theory, and they have been studied in several papers, from different points of view [1–7, 9, 10, 12, 13, 15]. Signal constellations having lattice structure have been studied for signal transmission over both Gaussian and single-antenna Rayleigh fading channel [7]. Usually the problem of finding good signal constellations for a Gaussian

* *This work was supported by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) under Grants No. 432735/2016-0 and 429346/2018-2 and Fapesp (Fundação de Amparo à Pesquisa do Estado de São Paulo) under Grant No. 2013/25977-7.*

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https://doi.org/10.13069/jacodesmath.1000778 ISSN 2148-838X

channel is associated to the search for lattices with high packing density [8]. On the other hand, for a Rayleigh fading channel the efficiency is strongly related to the lattice diversity and minimum product distance [5, 7]. The approach in this work, following [12] and [13] is the use of algebraic number theory to construct rotated D_n-lattices with full diversity via free $\mathbb{Z}$-modules.

In [1, 4, 5] some families of rotated $\mathbb{Z}^n$-lattices for $n = \frac{p-1}{2}$, where $p \geq 5$ is a prime number, and $n = 2^s$, for $s \geq 1$, with full diversity and good minimum product distance are studied for transmission over Rayleigh fading channels. In [12, 13] are studied some families of rotated D_n-lattices with full diversity and good minimum product distance for transmission over both Gaussian and Rayleigh fading channels. In [12] are constructed rotated D_n-lattices for $n = (p-1)/2$, where $p \geq 7$ is a prime and $n = 2^k$, for $k \geq 2$ integer, and in [13] families of rotated D_n-lattices for $n = 2^k(p-1)$, with $k \geq 0$ integer and $p \geq 5$ a prime, and $n = (p-1)(q-1)/4$, where $p, q \geq 5$ are distinct prime numbers.

In this work, we construct families of rotated D_n-lattices with full diversity n for $n = 3^s$, $s \geq 1$, (Propositions 3.4 and 3.5). A D_n-lattice has better packing density $\delta(D_n)$ when compared to $\mathbb{Z}^n$, i.e., D_n has the best lattice packing density for $n = 3, 4, 5$ and $\lim_{n \longrightarrow \infty} \frac{\delta(\mathbb{Z}^n)}{\delta(D_n)} = 0$, and also a very efficient decoding algorithm [8].

2. Algebraic lattices

Let $\{v_1, \ldots, v_m\}$ be a set of linearly independent vectors in $\mathbb{R}^n$ and $\Lambda = \{\sum_{i=1}^m a_i v_i;\ a_i \in \mathbb{Z}\}$ the associated lattice. The set $\{v_1, \ldots, v_m\}$ is called a *basis* for Λ. A matrix M whose rows are these vectors is said to be a *generator matrix* for Λ while the associated *Gram matrix* is $G = MM^t = (\langle v_i, v_j \rangle)_{i,j=1}^m$. The *determinant* of Λ is $\det \Lambda = \det G$ and it is an invariant under change of basis (see [8, p. 4]). Two lattices Λ_1 and Λ_2 are said to be *similar* if there is an orthogonal mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ and a real positive number c such that $c\phi(\Lambda_1) = \Lambda_2$. When $c = 1$ the similar lattices Λ_1 and Λ_2 are said to be *congruent* or *isomorphic*. In this paper, as in [5, 12], we will say that Λ_1 is a *rotated* Λ_2-lattice if Λ_1 and Λ_2 are congruent.

Let $\mathbb{K}$ be a totally real number field of degree n and $\mathcal{O}_{\mathbb{K}}$ its ring of integes. Let σ_i, for $i = 1, \ldots, n$, be the n distinct $\mathbb{Q}$-homomorphisms from $\mathbb{K}$ to $\mathbb{R}$. The *canonical embedding* $\sigma : \mathbb{K} \longrightarrow \mathbb{R}^n$ is defined by $\sigma(x) = (\sigma_1(x), \ldots, \sigma_n(x))$ [14, 16]. It can be shown that if $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{K}}$ is a free $\mathbb{Z}$-module of rank n with $\mathbb{Z}$-basis $\{w_1, \ldots, w_n\}$, then the image $\Lambda = \sigma(\mathcal{I})$ is a lattice in $\mathbb{R}^n$ with basis $\{\sigma(w_1), \ldots, \sigma(w_n)\}$ [16, Chapter 8] and it has full diversity [2, 5]. A Gram matrix for $\sigma(\mathcal{I})$ is $G = \left(Tr_{\mathbb{K}|\mathbb{Q}}(w_i w_j)\right)_{i,j=1}^n$, where $Tr_{\mathbb{K}|\mathbb{Q}}(x) = \sum_{i=1}^n \sigma_i(x)$ for any $x \in \mathbb{K}$ [5]. In what follows let $q(u_i, u_j) = Tr_{\mathbb{K}|\mathbb{Q}}(u_i u_j)$ for any $u_i, u_j \in \mathbb{K}$.

In this paper, we focus on the maximal totally real subfields of the cyclotomic fields $\mathbb{Q}(\zeta_{3^r})$, where ζ_{3^r} is a primitive 3^r-th root of unity, with $r \geq 3$ a positive integer [17].

3. Rotated D_n-lattices via $\mathbb{K} = \mathbb{Q}(\zeta_{3^r} + \zeta_{3^r}^{-1})$, where $r \geq 3$ and $n = 3^{r-1}$

In [13, Proposition 2.7] it was shown that if $\mathbb{K}$ is a totally real Galois extension with $d_{\mathbb{K}}$ an odd integer, then it is impossible to construct rotated D_n-lattices via fractional ideals of $\mathcal{O}_{\mathbb{K}}$. In particular, it is impossible to construct rotated D_n-lattices via fractional ideals of $\mathbb{Z}[\zeta_{3^r} + \zeta_{3^r}^{-1}]$ since $d_{\mathbb{K}} = 3^{\frac{2(r+1)3^{r-1} - 3^r - 1}{2}}$ by [11]. Thus, in this section, we present some families of rotated D_n-lattices using free $\mathbb{Z}$-modules in $\mathbb{Z}[\zeta_{3^r} + \zeta_{3^r}^{-1}]$. Our strategy is to construct these lattices as sublattices of a family of rotated $\mathbb{Z} \oplus \mathcal{A}_2^k$-lattices, where $\mathcal{A}_2^k$ is a direct sum of $k = \frac{3^{r-1}-1}{2}$ copies of the A_2-lattice. In [3] is presented a family of rotated $\mathbb{Z} \oplus \mathcal{A}_2^k$-lattices as the image of a twisted embedding [2] applied to $\mathbb{Z}[\zeta_{3^r} + \zeta_{3^r}^{-1}]$. In Proposition 3.3, we construct a family of rotated $\mathbb{Z} \oplus \mathcal{A}_2^k$-lattices using the canonical embedding, where the Lemma 3.1 and Proposition 3.2 are support for the proof of Proposition 3.3.

Lemma 3.1. *[9] Consider* $e_0 = 1$ *and* $e_i = \zeta_{3^r}^i + \zeta_{3^r}^{-i}$*, for* $i = 1, 2, \dots, 3^{r-1} - 1$.

1. *If* $i = 0, \dots, 3^{r-1} - 1$*, then* $q(e_i, e_i) = \begin{cases} 3^{r-1} & \text{if } i = 0, \\ 2 \cdot 3^{r-1} & \text{otherwise.} \end{cases}$
2. *If* $i = 1, 2, \dots, 3^{r-1} - 1$*, then* $q(e_i, e_0) = 0$.
3. *If* $i, j = 1, \dots, 3^{r-1} - 1$*, with* $i \neq j$*, then*

$$q(e_i, e_j) = \begin{cases} -3^{r-1} & \text{if } i + j = 3^{r-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.2. *Consider* $u_0 = e_0$*,* $u_1 = e_1$ *and for* $i = 2, 3, \dots, 3^{r-1} - 1$

$$u_i = \begin{cases} e_{\frac{i+1}{2}} & \text{if } i \equiv 1 \ (mod\ 2), \\ e_{3^{r-1} - \frac{i}{2}} & \text{otherwise.} \end{cases}$$

1. *If* $i = 0, \dots, 3^{r-1} - 1$*, then* $q(u_i, u_i) = \begin{cases} 3^{r-1} & \text{if } i = 0, \\ 2 \cdot 3^{r-1} & \text{otherwise.} \end{cases}$
2. *If* $i = 1, 2, \dots, 3^{r-1} - 1$*, then* $q(u_i, u_0) = 0$.
3. *If* $i, j = 1, \dots, 3^{r-1} - 1$*, with* $i \neq j$*, then*

$$q(u_i, u_j) = \begin{cases} -3^{r-1} & \text{if } \ i + j \equiv 3 \ (mod\ 4) \text{ and } |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From Lemma 3.1, it follows that $q(u_0, u_0) = q(e_0, e_0) = 3^{r-1}$ and for $i = 1, 2, \dots, 3^{r-1} - 1$, it follows that $q(u_i, u_i) = 2 \cdot 3^{r-1}$ and $q(u_i, u_0) = q(u_i, e_0) = 0$, for $u_i \in \{e_1, e_2, \dots, e_{3^{r-1}-1}\}$. If $i, j = 1, 2, \dots, 3^{r-1} - 1$, with $i \neq j$, then

$$q(u_i, u_j) = \begin{cases} q(e_{\frac{i+1}{2}}, e_{\frac{j+1}{2}}) & \text{if } i, j \equiv 1 \ (mod\ 2), \\ q(e_{\frac{i+1}{2}}, e_{3^{r-1} - \frac{j}{2}}) & \text{if } i \equiv 1 \text{ and } j \equiv 0 \ (mod\ 2), \\ q(e_{3^{r-1} - \frac{i}{2}}, e_{\frac{j+1}{2}}) & \text{if } i \equiv 0 \text{ and } j \equiv 1 \ (mod\ 2), \\ q(e_{3^{r-1} - \frac{i}{2}}, e_{3^{r-1} - \frac{j}{2}}) & \text{if } i, j \equiv 0 \ (mod\ 2). \end{cases}$$

For $i, j \equiv 1 \ (mod\ 2)$, it follows that either $i + j \equiv 0 \ (mod\ 4)$ or $i + j \equiv 2 \ (mod\ 4)$ and $\frac{i+1}{2} + \frac{j+1}{2} \neq 3^{r-1}$. Otherwise, since $i \neq j$, it follows that $i = j = 3^{r-1} - 1$, which is a contradiction. Thus, $q(u_i, u_j) = 0$. For $i \equiv 1 \ (mod\ 2)$ and $j \equiv 0 \ (mod\ 2)$, it follows that $\frac{i+1}{2} + 3^{r-1} - \frac{j}{2} = 3^{r-1}$ if and only if $i = j - 1$. For $i \equiv 0 \ (mod\ 2)$ and $j \equiv 1 \ (mod\ 2)$, it follows that $3^{r-1} - \frac{i}{2} + \frac{j+1}{2} = 3^{r-1}$ if and only if $j = i - 1$. In the last two cases, as $i + j$ is odd, it follows that $i + j \equiv 3 \ (mod\ 4)$, because if $i + j \equiv 1 \ (mod\ 4)$, with $i = j - 1$ (respectively, $j = i - 1$), it follows that j is odd (respectively, i is odd), which is a contradiction. Therefore, $q(u_i, u_j) = -3^{r-1}$ if $i + j \equiv 3 \ (mod\ 4)$ and $|i - j| = 1$. For $i, j \equiv 0 \ (mod\ 2)$, it follows that either $i + j \equiv 0 \ (mod\ 4)$ or $i + j \equiv 2 \ (mod\ 4)$ and $3^{r-1} - \frac{i}{2} + 3^{r-1} - \frac{j}{2} \neq 3^{r-1}$. Otherwise, since $i \neq j$, it follows that $i = j = 3^{r-1} - 1$, which is a contradiction. Thus, $q(u_i, u_j) = 0$. □

Proposition 3.3. *The lattice* $\frac{1}{\sqrt{3^{r-1}}}\sigma(\mathcal{O}_{\mathbb{K}})$ *is a rotated version of* $\mathbb{Z} \oplus \mathcal{A}_2^k$*, where* $\mathcal{A}_2^k$ *is a direct sum of* $k = \frac{3^{r-1}-1}{2}$ *copies of the* A_2*-lattice.*

Proof. From Proposition 3.2, it follows that $\{u_0, u_1, \dots, u_{3^{r-1}-1}\}$ is a $\mathbb{Z}$-basis of $\mathcal{O}_{\mathbb{K}}$ because it is a permutation of the $\mathbb{Z}$-basis $\{e_0, e_1, \dots, e_{3^{r-1}-1}\}$. A generator matrix of the algebraic lattice $\frac{1}{\sqrt{3^{r-1}}}\sigma_\alpha(\mathcal{O}_{\mathbb{K}})$

is given by $M = \frac{1}{\sqrt{3^{r-1}}}N$, where $N = (\sigma_i(u_{j-1}))_{i,j=1}^{3^{r-1}}$, and the associated Gram matrix is given by $G = MM^t = \dfrac{1}{3^{r-1}}\left(q(u_i,u_j)\right)_{i,j=0}^{3^{r-1}-1}$. So,

$$G = \begin{pmatrix} 1 & & & & & & & \\ & 2 & -1 & & & & & \\ & -1 & 2 & & & & & \\ & & & 2 & -1 & & & \\ & & & -1 & 2 & & & \\ & & & & & \ddots & & \\ & & & & & & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}.$$

It follows that the matrix G is a Gram matrix of $\mathbb{Z} \oplus \mathcal{A}_2^k$-lattice. □

In what follows, we split in two cases, i.e., we construct rotated D_n-lattices for $n = 3^{r-1}$, for r even and for r odd.

3.1. Rotated D_n-lattices for $n = 3^{r-1}$, where $r \geq 4$ is even

In this section, we present a construction of rotated D_n-lattices using $\mathbb{Z}$-modules in the totally real number field $\mathbb{K} = \mathbb{Q}(\zeta_{3^r} + \zeta_{3^r}^{-1})$, where r is even. The D_n-lattice is obtained as sublattice of $\mathbb{Z} \oplus \mathcal{A}_2^k$ using $\mathcal{B} = \{u_0, u_1, \ldots, u_{3^{r-1}-1}\}$ a $\mathbb{Z}$-basis of $\mathcal{O}_{\mathbb{K}}$.

Proposition 3.4. *Let $\mathcal{I} = \mathbb{Z}\omega_0 \oplus \mathbb{Z}\omega_1 \oplus \ldots \oplus \mathbb{Z}\omega_{3^{r-1}-1}$ be a free $\mathbb{Z}$-module of $\mathcal{O}_{\mathbb{K}}$, where*

1. $\omega_0 = -4u_0 - 2u_1 - 2u_2$; $\omega_1 = -2u_1 + 2u_2$; $\omega_2 = 4u_0 - 2u_2$;
   $\omega_3 = -2u_0 + 2u_1 + 2u_2 - u_5 + u_6 - u_9 + u_{10}$;

2. *For* $j = 1, 2\ldots, \frac{3^{r-1}-11}{8}$,
   $\omega_{4j} = u_{8j-3} - u_{8j-2} - u_{8j-1} + u_{8j} + u_{8j+1} - u_{8j+2} - u_{8j+3} + u_{8j+4}$;
   $\omega_{4j+1} = -u_{8j-3} + u_{8j-2} + u_{8j-1} - u_{8j} + u_{8j+1} - u_{8j+2} + u_{8j+3} - u_{8j+4}$;
   $\omega_{4j+2} = u_{8j-3} - u_{8j-2} - u_{8j-1} + u_{8j} - u_{8j+1} + u_{8j+2} + u_{8j+3} - u_{8j+4}$;
   $\omega_{4j+3} = u_{8j-1} - u_{8j} - u_{8j+3} + u_{8j+4} - u_{8j+5} + u_{8j+6} - u_{8j+9} + u_{8j+10}$;
   $\omega_{3^{r-1}+1-4j} = -u_{8j-3} - u_{8j-2} - u_{8j-1} - u_{8j} + 3u_{8j+1} + 3u_{8j+2} - u_{8j+3} - u_{8j+4}$;
   $\omega_{3^{r-1}+2-4j} = -u_{8j-3} - u_{8j-2} + 3u_{8j-1} + 3u_{8j} - u_{8j+1} - u_{8j+2} + u_{8j+3} + u_{8j+4}$;
   $\omega_{3^{r-1}+3-4j} = 3u_{8j-3} + 3u_{8j-2} - u_{8j-1} - u_{8j} + u_{8j+1} + u_{8j+2} + u_{8j+3} + u_{8j+4}$;
   If $j \neq 1$,
   $\omega_{3^{r-1}+4-4j} = -u_{8j-9} - u_{8j-8} - 2u_{8j-7} - 2u_{8j-6} + u_{8j-5} + u_{8j-4} - u_{8j-3} - u_{8j-2} - u_{8j+1} - u_{8j+2} - 2u_{8j+3} - 2u_{8j+4}$;

3. *For* $j = \frac{3^{r-1}-3}{8}$,
   $\omega_{4j} = u_3 - u_4 + u_{8j-3} - u_{8j-2} - u_{8j-1} + u_{8j} + u_{8j+1} - u_{8j+2}$;
   $\omega_{4j+1} = -u_3 + u_4 - u_{8j-3} + u_{8j-2} + u_{8j-1} - u_{8j} + u_{8j+1} - u_{8j+2}$;
   $\omega_{4j+2} = -u_3 + u_4 + u_{8j-3} - u_{8j-2} - u_{8j-1} + u_{8j} - u_{8j+1} + u_{8j+2}$;
   $\omega_{4j+3} = 2u_3 - 2u_{8j} - 2u_{8j+1} - 2u_{8j+2}$;
   $\omega_{3^{r-1}+1-4j} = -u_3 - u_4 - u_{8j-3} - u_{8j-2} - u_{8j-1} - u_{8j} + 3u_{8j+1} + 3u_{8j+2}$;
   $\omega_{3^{r-1}+2-4j} = u_3 + u_4 - u_{8j-3} - u_{8j-2} + 3u_{8j-1} + 3u_{8j} - u_{8j+1}$

$$\begin{aligned}
&\quad - u_{8j+2};\\
\omega_{3^{r-1}+3-4j} &= u_3 + u_4 + 3u_{8j-3} + 3u_{8j-2} - u_{8j-1} - u_{8j} + u_{8j+1}\\
&\quad + u_{8j+2};\\
\omega_{3^{r-1}+4-4j} &= -2u_3 - 2u_4 + u_{8j-9} + u_{8j-8} - 2u_{8j-7} - 2u_{8j-6}\\
&\quad + u_{8j-5} + u_{8j-4} - u_{8j-3} - u_{8j-2} - u_{8j+1} - u_{8j+2}.
\end{aligned}$$

Therefore, $\Lambda = \frac{1}{2\sqrt{3^r}}\sigma(\mathcal{I}) \subseteq \mathbb{R}^{3^{r-1}}$ *is a rotated version of the* $D_{3^{r-1}}$*-lattice.*

Proof. From Proposition 3.2, it follows that

$$\begin{aligned}
q(\omega_0,\omega_0) &= Tr_{\mathbb{K}/\mathbb{Q}}(\omega_0\omega_0) = Tr_{\mathbb{K}/\mathbb{Q}}((-4u_0 - 2u_1 - 2u_2)(-4u_0\\
&\quad - 2u_1 - 2u_2) = Tr_{\mathbb{K}/\mathbb{Q}}(16u_0u_0 + 16u_0u_1 + 16u_0u_2\\
&\quad + 4u_1u_1 + 8u_1u_2 + 4u_2u_2) = 16q(u_0,u_0) + 16q(u_0,u_1)\\
&\quad + 16q(u_0,u_2) + 4q(u_1,u_1) + 8q(u_1,u_2) + 4q(u_2,u_2)\\
&= 24\cdot 3^{r-1}.\\
q(\omega_1,\omega_1) &= 4q(u_1,u_1) + 4q(u_2,u_2) - 8q(u_1,u_2) = 24\cdot 3^{r-1}.\\
q(\omega_2,\omega_2) &= 16q(u_0,u_0) + 4q(u_2,u_2) = 24\cdot 3^{r-1}.\\
q(\omega_3,\omega_3) &= 4q(u_0,u_0) + 4q(u_1,u_1) + 8q(u_1,u_2) + 4q(u_2,u_2)\\
&\quad + q(u_5,u_5) - 2q(u_5,u_6) + q(u_6,u_6) + q(u_9,u_9)\\
&\quad - 2q(u_9,u_{10}) + q(u_10,u_{10}) = 24\cdot 3^{r-1}.\\
q(\omega_0,\omega_1) &= q(\omega_0,\omega_3) = q(\omega_1,\omega_3) = 0.\\
q(\omega_0,\omega_2) &= q(\omega_1,\omega_2) = q(\omega_2,\omega_3) = q(\omega_3,\omega_4) = -12\cdot 3^{r-1}.
\end{aligned}$$

Let $j = 1,2,\ldots,\frac{3^{r-1}-3}{8}$. Since $q(u_i,u_j) \neq 0$ if and only if $i+j \equiv 3\ (mod\ 4)$ and $|i-j| = 1$, it follows that

$$\begin{aligned}
q(\omega_{4j},\omega_{4j}) &= q(u_{8j-3},u_{8j-3}) - 2q(u_{8j-3},u_{8j-2}) + q(u_{8j-2},u_{8j-2})\\
&\quad + q(u_{8j-1},u_{8j-1}) - 2q(u_{8j-1},u_{8j}) + q(u_{8j},u_{8j})\\
&\quad + q(u_{8j+1},u_{8j+1}) - 2q(u_{8j+1},u_{8j+2}) + q(u_{8j+2},u_{8j+2})\\
&\quad + q(u_{8j+3},u_{8j+3}) - 2q(u_{8j+3},u_{8j+4}) + q(u_{8j+4},u_{8j+4})\\
&= 24\cdot 3^{r-1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
q(\omega_{3^{r-1}+1-4j},\omega_{3^{r-1}+1-4j} &= q(u_{8j-3},u_{8j-3}) + 2q(u_{8j-3},u_{8j-2})\\
&\quad + q(u_{8j-2},u_{8j-2}) + q(u_{8j-1},u_{8j-1}) + 2q(u_{8j-1},u_{8j})\\
&\quad + q(u_{8j},u_{8j}) + 9q(u_{8j+1},u_{8j+1}) + 18q(u_{8j+1},u_{8j+2})\\
&\quad + 9q(u_{8j+2},u_{8j+2}) + q(u_{8j+3},u_{8j+3}) + 2q(u_{8j+3},u_{8j+4})\\
&\quad + q(u_{8j+4},u_{8j+4}) = 24\cdot 3^{r-1},\\
q(\omega_{4j+1},\omega_{4j+1}) &= q(\omega_{4j+2},\omega_{4j+2}) = q(\omega_{4j+3},\omega_{4j+3}) =\\
&= q(\omega_{3^{r-1}+2-4j},\omega_{3^{r-1}+2-4j}) = q(\omega_{3^{r-1}+3-4j},\omega_{3^{r-1}+3-4j})\\
&= q(\omega_{3^{r-1}+4-4j},\omega_{3^{r-1}+4-4j}) = 24\cdot 3^{r-1},\\
q(\omega_{4j},\omega_{4j+1}) &= -q(\omega_{8j-3},\omega_{8j-3}) + 2q(\omega_{8j-3},\omega_{8j-2})\\
&\quad - q(\omega_{8j-2},\omega_{8j-2}) - q(\omega_{8j-1},\omega_{8j-1}) + 2q(\omega_{8j-1},\omega_{8j})\\
&\quad - q(\omega_{8j},\omega_{8j}) + q(\omega_{8j+1},\omega_{8j+1}) - 2q(\omega_{8j+1},\omega_{8j+2})\\
&\quad + q(\omega_{8j+2},\omega_{8j+2}) - q(\omega_{8j+3},\omega_{8j+3}) + 2q(\omega_{8j+3},\omega_{8j+4})\\
&\quad - q(\omega_{8j+4},\omega_{8j+4}) = -12\cdot 3^{r-1},\\
q(\omega_{4j+1},\omega_{4j+2}) &= q(\omega_{4j+2},\omega_{4j+3}) = q(\omega_{4j+3},\omega_{4(j+1)}) =\\
&= q(\omega_{3^{r-1}+1-4j},\omega_{3^{r-1}+2-4j}) = q(\omega_{3^{r-1}+2-4j},\omega_{3^{r-1}+3-4j})\\
&= q(\omega_{3^{r-1}+3-4j},\omega_{3^{r-1}+4-4j}) = -12\cdot 3^{r-1}.
\end{aligned}$$

Finally, for $k,l = 1,2,\ldots,3^{r-1}-2$, with $l > k+1$, it follows that $q(\omega_k,\omega_l) = 0$. Now, $\mathcal{C} = \{\omega_0,\omega_1,\ldots,\omega_{3^{r-1}-1}\}$ is a basis of a free $\mathbb{Z}$-module $\mathcal{I}$. A generator matrix of the algebraic lattice $\frac{1}{2\sqrt{3^r}}\sigma(\mathcal{I})$ is given by $M = \frac{1}{2\sqrt{3^r}}N$, where $N = (\sigma_i(\omega_{j-1}))_{i,j=1}^{3^{r-1}}$, and the associated Gram matrix is

$$G = MM^t = \frac{1}{12 \cdot 3^{r-1}} \left(q(\omega_i, \omega_j)\right)_{i,j=0}^{3^{r-1}-1} =$$

$$= \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Therefore, G is the Gram matrix of a $D_{3^{r-1}}$-lattice. □

3.2. Rotated D_n-lattices for $n = 3^{r-1}$, where $r \geq 3$ is odd

In this section, we present a construction of rotated D_n-lattices using $\mathbb{Z}$-modules via the totally real number field $\mathbb{K} = \mathbb{Q}(\zeta_{3^r} + \zeta_{3^r}^{-1})$, where r is odd. The D_n-lattice is obtained as sublattice of $\mathbb{Z} \oplus \mathcal{A}_2^k$ using $\mathcal{B} = \{u_0, u_1, \dots, u_{3^{r-1}-1}\}$ a $\mathbb{Z}$-basis of $\mathcal{O}_\mathbb{K}$.

Proposition 3.5. *Let $\mathcal{I} = \mathbb{Z}\omega_0 \oplus \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_{3^{r-1}-1}$ be a free $\mathbb{Z}$-module of $\mathcal{O}_\mathbb{K}$, where*

1. $\omega_0 = -6u_0 - 3u_1 - 3u_3$; $\omega_1 = 6u_0 - 3u_1 - 3u_3$; $\omega_2 = 6u_1$;
2. *For* $3 \leq j \leq \frac{3^{r-1}-3}{2}$, *where* j *is odd,*
   $\omega_j = -3u_{2j-5} + 3u_{2j-3} - 3u_{2j-1} - 3u_{2j+1}$;
   $\omega_{j+1} = 6u_{2j-1}$;
3. *For* $j = \frac{3^{r-1}+1}{2}$,
   $\omega_j = -u_{2j-7} - 2u_{2j-6} - 5u_{2j-5} - 4u_{2j-4} + 4u_{2j-3} + 2u_{2j-2}$;
   $\omega_{j+1} = -u_{2j-9} - 2u_{2j-8} - u_{2j-7} - 2u_{2j-6} + 3u_{2j-5} + 6u_{2j-4} - u_{2j-3} - 2u_{2j-2}$;
   $\omega_{j+2} = -u_{2j-9} - 2u_{2j-8} + 3u_{2j-7} + 6u_{2j-6} - u_{2j-5} - 2u_{2j-4} + u_{2j-3} + 2u_{2j-2}$;
   $\omega_{j+3} = 3u_{2j-9} + 6u_{2j-8} - u_{2j-7} - 2u_{2j-6} + u_{2j-5} + 2u_{2j-4} + u_{2j-3} + 2u_{2j-2}$;
4. *For* $j = 1, 2, \dots, \frac{3^{r-1}-9}{8}$, *with* $r > 3$,
   $\omega_{3^{r-1}-4j} = -u_{8j-5} - 2u_{8j-4} - 2u_{8j-3} - 4u_{8j-2} + u_{8j-1} + 2u_{8j} - u_{8j+1} - 2u_{8j+2} - u_{8j+5} - 2u_{8j+6} - 2u_{8j+7} - 4u_{8j+8}$;
   $\omega_{3^{r-1}+1-4j} = -u_{8j-7} - 2u_{8j-6} - u_{8j-5} - 2u_{8j-4} + 3u_{8j-3} + 6u_{8j-2} - u_{8j-1} - 2u_{8j}$;
   $\omega_{3^{r-1}+2-4j} = -u_{8j-7} - 2u_{8j-6} + 3u_{8j-5} + 6u_{8j-4} - u_{8j-3} - 2u_{8j-2} + u_{8j-1} + 2u_{8j}$;
   $\omega_{3^{r-1}+3-4j} = 3u_{8j-7} + 6u_{8j-6} - u_{8j-5} - 2u_{8j-4} + u_{8j-3} + 2u_{8j-2} + u_{8j-1} + 2u_{8j}$.

Therefore, $\Lambda = \frac{1}{6\sqrt{3^{r-1}}}\sigma(\mathcal{I}) \subseteq \mathbb{R}^{3^{r-1}}$ *is a rotated version of a* $D_{3^{r-1}}$*-lattice.*

Proof. From Proposition 3.2, it follows that

$$\begin{aligned} q(\omega_0, \omega_0) &= Tr_{\mathbb{K}/\mathbb{Q}}(\omega_0\omega_0) = Tr_{\mathbb{K}/\mathbb{Q}}((-6u_0 - 3u_1 - 3u_3)(-6u_0 \\ &\quad - 3u_1 - 3u_3) = Tr_{\mathbb{K}/\mathbb{Q}}(36u_0u_0 + 36u_0u_1 + 36u_0u_3 \\ &\quad + 9u_1u_1 + 18u_1u_3 + 9u_3u_3) = 36q(u_0, u_0) + 36q(u_0, u_1) \end{aligned}$$

$$\begin{aligned}&+36q(u_0,u_3)+9q(u_1,u_1)+18q(u_1,u_3)+9q(u_3,u_3)\\&=72\cdot 3^{r-1}.\end{aligned}$$

$$\begin{aligned}q(\omega_1,\omega_1)&=36q(u_0,u_0)+9q(u_1,u_1)+9q(u_3,u_3)=72\cdot 3^{r-1}.\\q(\omega_2,\omega_2)&=36q(u_1,u_1)=72\cdot 3^{r-1}.\\q(\omega_0,\omega_1)&=q(\omega_0,\omega_3)=q(\omega_1,\omega_3)=0.\\q(\omega_0,\omega_2)&=q(\omega_1,\omega_2)=q(\omega_2,\omega_3)=-36\cdot 3^{r-1}.\end{aligned}$$

Let $3\le j\le \frac{3^{r-1}-3}{2}$, with j odd. Since $q(u_i,u_j)\neq 0$ if and only if $i+j\equiv 3\ (mod\ 4)$ and $|i-j|=1$, it follows that

$$\begin{aligned}&q(\omega_j,\omega_j)=9q(u_{2j-5},u_{2j-5})+9q(u_{2j-3},u_{2j-3})+9q(u_{2j-1},u_{2j-1})\\&\qquad+9q(u_{2j+1},u_{2j+1})=72\cdot 3^{r-1}.\\&q(\omega_{j+1},\omega_{j+1})=36q(u_{2j-1},u_{2j-1})=72\cdot 3^{r-1}.\end{aligned}$$

Furthermore,

$$\begin{aligned}&q(\omega_j,\omega_{j+1})=-18q(u_{2j-1},u_{2j-1})=-36\cdot 3^{r-1},\text{ and for } j<\tfrac{3^{r-1}-3}{2},\\&q(\omega_{j+1},\omega_{j+2})=q(6u_{2j-1},-3u_{2(j+2)-5}+3u_{2(j+2)-3}-3u_{2(j+2)-1}\\&\qquad-3u_{2(j+2)+1})=q(6u_{2j-1},-3u_{2j-1}+3u_{2j+1}\\&\qquad-3u_{2j+3}-3u_{2j+5})=-18q(u_{2j-1},u_{2j-1})=\\&\qquad=-36\cdot 3^{r-1}.\end{aligned}$$

For $j=\frac{3^{r-1}+1}{2}$, it follows that

$$\begin{aligned}q(\omega_j,\omega_j)&=q(u_{2j-7},u_{2j-7})+4q(u_{2j-7},u_{2j-6})+4q(u_{2j-6},u_{2j-6})\\&+25q(u_{2j-5},u_{2j-5})+40q(u_{2j-5},u_{2j-4})\\&+16q(u_{2j-4},u_{2j-4})+16q(u_{2j-3},u_{2j-3})\\&+16q(u_{2j-3},u_{2j-2})+4q(u_{2j-2},u_{2j-2})=72\cdot 3^{r-1}.\end{aligned}$$

In the same way, it follows that

$$q(\omega_{j+1},\omega_{j+1})=q(\omega_{j+2},\omega_{j+2})=q(\omega_{j+3},\omega_{j+3})=72\cdot 3^{r-1}.$$

Also,

$$\begin{aligned}q(\omega_j,\omega_{j+1})&=q(\omega_{2j-7},\omega_{2j-7})+4q(\omega_{2j-7},\omega_{2j-6})+4q(\omega_{2j-6},\omega_{2j-6})\\&-15q(\omega_{2j-5},\omega_{2j-5})-42q(\omega_{2j-5},\omega_{2j-4})\\&-24q(\omega_{2j-4},\omega_{2j-4})-4q(\omega_{2j-3},\omega_{2j-3})\\&-10q(\omega_{2j-3},\omega_{2j-2})-4q(\omega_{2j-2},\omega_{2j-2})=-36\cdot 3^{r-1}.\end{aligned}$$

In the same way, it follows that

$$\begin{aligned}&q(\omega_{j+1},\omega_{j+2})=q(\omega_{j+2},\omega_{j+3})=-36\cdot 3^{r-1}.\text{ and for } k=\tfrac{3^{r-1}-9}{8},\\&q(\omega_{j+3},\omega_{3^{r-1}-4k})=q(\omega_{\frac{3^{r-1}+7}{2}},\omega_{\frac{3^{r-1}+9}{2}})=-36\cdot 3^{r-1}.\end{aligned}$$

For $j=1,2,\ldots,\frac{3^{r-1}-9}{8}$, with $r>3$,

$$\begin{aligned}q(\omega_{3^{r-1}-4j},\omega_{3^{r-1}-4j})&=q(\omega_{8j-5},\omega_{8j-5})+4q(\omega_{8j-5},\omega_{8j-4})\\&+4q(\omega_{8j-4},\omega_{8j-4})+4q(\omega_{8j-3},\omega_{8j-3})+16q(\omega_{8j-3},\omega_{8j-2})\\&+16q(\omega_{8j-2},\omega_{8j-2})+q(\omega_{8j-1},\omega_{8j-1})+4q(\omega_{8j-1},\omega_{8j})\\&+4q(\omega_{8j},\omega_{8j})+q(\omega_{8j+1},\omega_{8j+1})+4q(\omega_{8j+1},\omega_{8j+2})\\&+4q(\omega_{8j+2},\omega_{8j+2})+q(\omega_{8j+5},\omega_{8j+5})+4q(\omega_{8j+5},\omega_{8j+6})\\&+4q(\omega_{8j+6},\omega_{8j+6})+4q(\omega_{8j+7},\omega_{8j+7})+16q(\omega_{8j+7},\omega_{8j+8})\\&+16q(\omega_{8j+8},\omega_{8j+8})=72\cdot 3^{r-1}.\end{aligned}$$

In the same way, it follows that

$$\begin{aligned}q(\omega_{3^{r-1}+1-4j},\omega_{3^{r-1}+1-4j})&=q(\omega_{3^{r-1}+2-4j},\omega_{3^{r-1}+2-4j})\\&=q(\omega_{3^{r-1}+3-4j},\omega_{3^{r-1}+3-4j})=72\cdot 3^{r-1}.\end{aligned}$$

Also,

$$\begin{aligned}q(\omega_{3^{r-1}-4j},\omega_{3^{r-1}+1-4j})&=q(u_{8j-5},u_{8j-5})+4q(u_{8j-5},u_{8j-4})\\&+4q(u_{8j-4},u_{8j-4})-6q(u_{8j-3},u_{8j-3})-24q(u_{8j-3},u_{8j-2})\\&-24q(u_{8j-2},u_{8j-2})-q(u_{8j-1},u_{8j-1})-4q(u_{8j-1},u_{8j})\\&-4q(u_{8j},u_{8j})=-36\cdot 3^{r-1}.\end{aligned}$$

In the same way, it follows that

$$q(\omega_{3^{r-1}+1-4j},\omega_{3^{r-1}+2-4j})=q(\omega_{3^{r-1}+2-4j},\omega_{3^{r-1}+3-4j})=-36\cdot 3^{r-1}.$$

Finally, for $k,l=1,2,\ldots,3^{r-1}-2$, with $l>k+1$, it follows that $q(\omega_k,\omega_l)=0$. Now, n $\mathcal{C}=\{\omega_0,\omega_1,\ldots,\omega_{3^{r-1}-1}\}$ is a basis of a free $\mathbb{Z}$-module $\mathcal{I}$. A generator matrix of the algebraic lattice $\frac{1}{6\sqrt{3^{r-1}}}\sigma(\mathcal{I})$ is given by $M = \frac{1}{6\sqrt{3^{r-1}}}N$, where $N = (\sigma_i(\omega_{j-1}))_{i,j=1}^{3^{r-1}}$, and the associated Gram matrix is

$$G = MM^t = \frac{1}{36 \cdot 3^{r-1}} \left(q(\omega_i, \omega_j)\right)_{i,j=0}^{3^{r-1}-1} =$$

$$= \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Therefore, G is the Gram matrix of a $D_{3^{r-1}}$-lattice. □

4. Conclusions

In this paper, we construct full diversity rotated versions of $D_{3^{r-1}}$-lattices via the canonical embedding and two families of $\mathbb{Z}$-modules of the ring of the integers $\mathbb{Z}[\zeta_{3^r} + \zeta_{3^r}^{-1}]$, for $r \geq 3$ a positive integer, since it is impossible to construct rotated D_n-lattices via fractional ideals of $\mathbb{Z}[\zeta_{3^r} + \zeta_{3^r}^{-1}]$ [13]. The lattices obtained here are sublattices of the family of rotated $\mathbb{Z} \oplus \mathcal{A}_2^k$-lattices, where $\mathcal{A}_2^k$ is a direct sum of $k = \frac{3^{r-1}-1}{2}$ copies of the A_2-lattice.

In [1] and [4] families of rotated $\mathbb{Z}^{2^{r-2}}$-lattices were obtained via the ring of integers $\mathbb{Z}[\zeta_{2^r} + \zeta_{2^r}^{-1}]$. In [5] a family of rotated $\mathbb{Z}^{(p-1)/2}$-lattices was obtained via the ring of integers $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$, with p prime. In [9] two families of rotated $\mathbb{Z}^{3^{r-1}}$-lattices were obtained via free $\mathbb{Z}$-modules of $\mathbb{Z}[\zeta_{3^r} + \zeta_{3^r}^{-1}]$, one for r odd and one for r even. In [12] two families of rotated $D_{2^{r-2}}$-lattices were obtained, one via the ring of integers $\mathbb{Z}[\zeta_{2^r} + \zeta_{2^r}^{-1}]$ and one via a principal ideal of $\mathbb{Z}[\zeta_{2^r} + \zeta_{2^r}^{-1}]$. Also in [12] a family of rotated $D_{(p-1)/2}$-lattices was presented via free $\mathbb{Z}$-modules in $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$, with p prime, that are not ideals. In [13] considering the compositum of $\mathbb{Q}(\zeta_{2^r} + \zeta_{2^r}^{-1})$ and $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$ and the compositum of $\mathbb{Q}(\zeta_{p_1} + \zeta_{p_1}^{-1})$ and $\mathbb{Q}(\zeta_{p_2} + \zeta_{p_2}^{-1})$, where p, p_1 and p_2 are prime numbers with $p_1 \neq p_2$, were constructed families of rotated D_n-lattices via free $\mathbb{Z}$-modules of rank n that are not ideals. In Table 1, we list the number fields considered in [1, 4, 5, 9, 12, 13] and here for constructing rotated $\mathbb{Z}^n$ and D_n-lattices for some values of n. Let $\mathbb{K}_1 = \mathbb{Q}(\zeta_{2^r} + \zeta_{2^r}^{-1})$, $\mathbb{K}_2 = \mathbb{Q}(\zeta_p + \zeta_p^{-1})$, where p is a prime, $\mathbb{K}_3 = \mathbb{Q}(\zeta_{2^r} + \zeta_{2^r}^{-1})\mathbb{Q}(\zeta_p + \zeta_p^{-1})$, $\mathbb{K}_4 = \mathbb{Q}(\zeta_{p_1} + \zeta_{p_1}^{-1})\mathbb{Q}(\zeta_{p_2} + \zeta_{p_2}^{-1})$, with $p_1 \neq p_2$, and $\mathbb{K}_5 = \mathbb{Q}(\zeta_{3^r} + \zeta_{3^r}^{-1})$. We observe that for $r = 14, 21, 25, 26, 28, 29$ and 30 there are not p, p_1, p_2 prime numbers with $p_1 \neq p_2$ such that the degree of $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$ and $\mathbb{Q}(\zeta_{p_1} + \zeta_{p_1}^{-1})\mathbb{Q}(\zeta_{p_2} + \zeta_{p_2}^{-1})$ be 3^{r-2}.

n	$\mathbb{Z}^n$			D_n				
	$\mathbb{K}_1$	$\mathbb{K}_2$	$\mathbb{K}_5$	$\mathbb{K}_1$	$\mathbb{K}_2$	$\mathbb{K}_3$	$\mathbb{K}_4$	$\mathbb{K}_5$
2	$r=3$	$p=5$	–	–	–	–	–	–
3	–	$p=7$	$r=3$	–	$p=7$	–	–	$r=3$
4	$r=4$	–	–	$r=4$	–	$r=3,\ p=5$	–	–
8	$r=5$	$p=17$	–	$r=5$	$p=17$	$r=4,\ p=5$	–	–
9	–	$p=19$	$r=4$	–	$p=19$	–	–	$r=4$
16	$r=6$	–	–	$r=6$	–	$r=p=5$	$p_1,p_2 \in \{5,17\}$	–
27	–	–	$r=5$	–	–	–	$p_1,p_2 \in \{7,19\}$	$r=5$
32	$r=7$	–	–	$r=7$	–	$r=4,\ p=17$	–	–
64	$r=8$	–	–	$r=8$	–	$r=7,\ p=5$	–	–
81	–	$p=163$	$r=6$	–	$p=163$	–	–	$r=6$

n	$\mathbb{Z}^n$			D_n				
	$\mathbb{K}_1$	$\mathbb{K}_2$	$\mathbb{K}_5$	$\mathbb{K}_1$	$\mathbb{K}_2$	$\mathbb{K}_3$	$\mathbb{K}_4$	$\mathbb{K}_5$
128	$r=9$	$p=257$	–	$r=9$	$p=257$	$r=8,\ p=5$	–	–
243	–	$p=487$	$r=7$	–	$p=487$	–	$p_1,p_2\in\{7,163\}$	$r=7$
256	$r=10$	–	–	$r=10$	–	$r=7,\ p=17$	$p_1,p_2\in\{5,257\}$	–
512	$r=11$	–	–	$r=11$	–	$r=10,\ p=5$	–	–
729	–	$p=1459$	$r=8$	–	$p=1459$	–	$p_1,p_2\in\{7,487\}$ $p_1,p_2\in\{19,163\}$	$r=8$
1024	$r=12$	–	–	$r=12$	–	$r=9,\ p=17$	$p_1,p_2\in\{17,257\}$	–
2048	$r=13$	–	–	$r=13$	–	$r=10,\ p=17$	–	–
2187	–	–	$r=9$	–	–	–	$p_1,p_2\in\{7,1459\}$ $p_1,p_2\in\{19,487\}$	$r=9$
4096	$r=14$	–	–	$r=14$	–	$r=13,\ p=5$	–	–
6561	–	–	$r=10$	–	–	–	$p_1,p_2\in\{19,1459\}$	$r=10$
8192	$r=15$	–	–	$r=15$	–	$r=12,\ p=17$	–	–
16384	$r=16$	–	–	$r=16$	–	$r=13,\ p=5$	–	–
19683	–	$p=39367$	$r=11$	–	$p=39367$	–	$p_1,p_2\in\{163,487\}$	$r=11$
32768	$r=17$	$p=65537$	–	$r=17$	$p=65537$	$r=14,\ p=17$	–	–
59049	–	–	$r=12$	–	–	–	$p_1,p_2\in\{7,39367\}$ $p_1,p_2\in\{163,1459\}$	$r=12$
65536	$r=18$	–	–	$r=18$	–	$r=17,\ p=5$	–	–
131072	$r=19$	–	–	$r=19$	–	$r=16,\ p=17$	–	–
177147	–	–	$r=13$	–	–	–	$p_1,p_2\in\{19,39367\}$ $p_1,p_2\in\{487,1459\}$	$r=13$
262144	$r=20$	–	–	$r=20$	–	$r=13,\ p=257$	–	–
524288	$r=21$	–	–	$r=21$	–	$r=20,\ p=5$	–	–
531441	–	–	$r=14$	–	–	–	–	$r=14$
1048576	$r=22$	–	–	$r=22$	–	$r=19,\ p=17$	–	–
1594323	–	–	$r=15$	–	–	–	$p_1,p_2\in\{163,39367\}$	$r=15$

Table 2. Rotated $\mathbb{Z}^n$ and D_n-lattices for n powers of 2 and 3.

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