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On metric dimension of plane graphs $\mathfrak{J}_n$, $\mathfrak{K}_n$ and $\mathfrak{L}_n$

Research Article

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Abstract: Let $\Gamma = \Gamma(\mathbb{V}, \mathbb{E})$ be a simple (i.e., multiple edges and loops and are not allowed), connected (i.e., there exists a path between every pair of vertices), and an undirected (i.e., all the edges are bidirectional) graph. Let $d_\Gamma(\varrho_i, \varrho_j)$ denotes the geodesic distance between two nodes $\varrho_i, \varrho_j \in \mathbb{V}$. The problem of characterizing the classes of plane graphs with constant metric dimensions is of great interest nowadays. In this article, we characterize three classes of plane graphs (viz., $\mathfrak{J}_n$, $\mathfrak{K}_n$, and $\mathfrak{L}_n$) which are generated by taking n-copies of the complete bipartite graph (or a star) $K_{1,5}$, and all of these plane graphs are radially symmetrical with the constant metric dimension. We show that three vertices is a minimal requirement for the unique identification of all vertices of these three classes of plane graphs.

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Keywords: Resolving set, Metric basis, Independent set, Metric dimension, Planar graph

1. Introduction

Let Γ be a simple connected graph with the vertex set $\mathbb{V}$ and the edge set $\mathbb{E}$, and let $d_\Gamma(\varrho_i, \varrho_j)$ denotes the geodesic distance between two vertices $\varrho_i, \varrho_j \in \mathbb{V}$. A subset of vertices $\mathfrak{R} \subseteq \mathbb{V}$ is said to be a resolving set (metric generator or locating set) if for every pair of distinct vertices $\varsigma, \varrho \in \mathbb{V}$ there exists at least one $\alpha \in \mathfrak{R}$ such that $d_\Gamma(\alpha, \varsigma) \neq d_\Gamma(\alpha, \varrho)$. In other words, for an ordered subset of vertices $\mathfrak{R} = \{\varrho_1, \varrho_2, \varrho_3, ..., \varrho_k\}$ of Γ, any vertex $\alpha \in \mathbb{V}$ may be represented uniquely in the form of the vector $\gamma(\alpha|\mathfrak{R}) = (d_\Gamma(\alpha, \varrho_1), d_\Gamma(\alpha, \varrho_2), ..., d_\Gamma(\alpha, \varrho_k))$. Then $\mathfrak{R}$ is the metric generator of Γ if $\gamma(\delta|\mathfrak{R}) = \gamma(\beta|\mathfrak{R})$ implies that $\delta = \beta$, $\forall\ \beta, \delta \in \mathbb{V}$. The metric generator $\mathfrak{R}$ with the minimum possible cardinality is the metric basis for Γ, and this minimum cardinality is known as the *metric dimension* of Γ, denoted by $\beta(\Gamma)$. A set $\mathbb{S}$ consisting of vertices of the graph Γ is said to be an independent metric generator for Γ, if $\mathbb{S}$ is both metric generator and independent.

For the given ordered subset of vertices, $\mathfrak{R} = \{\rho_1, \rho_2, \rho_3, ..., \rho_k\}$ of Γ, the f^{th} component (or distance coordinate) of the code $\gamma(\rho|\mathfrak{R})$ is zero iff $\rho = \rho_f$. Subsequently, to see that the set $\mathfrak{R}$ is the metric generator, it is sufficient to prove that $\zeta(\rho|\mathfrak{R}) \neq \zeta(\varrho|\mathfrak{R})$ for any pair of distinguishable nodes $\varrho, \rho \in \mathbb{V}(\Gamma) \setminus \mathfrak{R}$.

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The notions of locating or resolving set and that of the metric dimension go back to 1950s. These were characterized by Blumenthal [2] with regard to metric space, and were acquainted with graph networks independently by Melter and Harary in 1976 [6], and Slater in 1975 [11]. Graph theory has applications in numerous zones of figuring, social, and normal sciences and is likewise an affable play area for the investigation of the verification procedure in discrete science. Utilizations of this invariant to the problem of picture preparing (or image processing) and design acknowledgment (or pattern recognition) are given in [9], to the route of exploring specialist (navigating agent or robots) in systems (or networks) are examined in [8], applications to science are given in [5], application to combinatorial enhancement (or optimization) is yielded in [10], and to problems of network discovery and verification in [1].

Let $\aleph$ represent a family of connected graphs. We say that the family $\aleph$ has constant metric dimension if $dim(\Gamma)$ does not depend upon the choice of the graph Γ in $\aleph$ and is finite. In other words, if all the graphs in $\aleph$ have an indistinguishable metric dimension, at that point $\aleph$ is known as a family with a constant (steady) metric dimension [12]. Chartrand et al. in [5], demonstrated that graphs on n vertices have metric dimension one iff it is a path $\wp_n$. Additionally, cycle $\mathcal{C}_n$ has metric dimension two for each positive integer n; $n \geq 3$. With this, $\mathcal{C}_n$ $(n \geq 3)$ and $\wp_n$ $(n \geq 2)$ establish a family of graphs with a steady metric dimension. Additionally, Harary graphs $H_{4,n}$ and generalized Petersen graphs $P(n, 2)$, are also the families of graphs with constant metric dimension [7].

By joining of two graphs $\Gamma_1 = \Gamma_1(\mathbb{V}_1, \mathbb{E}_1)$ and $\Gamma_2 = \Gamma_2(\mathbb{V}_2, \mathbb{E}_2)$, denoted by $\Gamma = \Gamma_1 + \Gamma_2$, we mean a graph $\Gamma = \Gamma(\mathbb{V}, \mathbb{E})$ such that $\mathbb{V} = \mathbb{V}_1 \cup \mathbb{V}_2$ and $\mathbb{E} = \mathbb{E}_1 \cup \mathbb{E}_2 \cup \{\varrho\varsigma : \varrho \in \mathbb{V}_1 \textit{ and } \varsigma \in \mathbb{V}_2\}$. Then a Fan graph F_m is characterized as $F_m = K_1 + \wp_m$ for $m \geq 1$, a Wheel graph W_m is characterized as $W_m = K_1 + \mathcal{C}_m$, for $m \geq 3$, and the Jahangir graph J_{2m} $(m \geq 2)$ is obtained from the Wheel graph W_{2m} by alternately deleting m spokes of the Wheel graph (which is otherwise known as the Gear graph).

In [4], Caceres et al. decided the location number of the Fan graph F_m $(m \geq 1)$ which is $\lfloor \frac{2m+2}{5} \rfloor$ for $m \notin \{1, 2, 3, 6\}$. Tomescu and Javaid [13] acquired the location number of the Jahangir graph J_{2m} $(m \geq 4)$ which is $\lfloor \frac{2m}{3} \rfloor$, and in [3] Chartrand et al. decided the location number of the Wheel graph W_m $(m \geq 3)$ which is $\lfloor \frac{2m+2}{5} \rfloor$ for $m \notin \{3, 6\}$. It is important to note that the metric dimension of these three graphs depend upon the number of vertices in the graph and thus these three families of graphs do not constitute the families with constant metric dimensions.

For the simple connected graphs with the metric dimension 2, Khuller et al. [8] proved the following important result:

Theorem 1.1. *[8] Let $\mathbb{A} \subseteq \mathbb{V}(\Gamma)$ be the basis set of the connected graph $\Gamma = \Gamma(\mathbb{V}, \mathbb{E})$ of cardinality two i.e., $|\mathbb{A}| = dim(\Gamma) = \beta(\Gamma) = 2$, and say $\mathbb{A} = \{\varpi, \xi\}$. Then, the following are true:*

1. *Between the vertices ϖ and ξ, there exists a unique shortest path $\wp$.*
2. *The valencies (or degrees) of the nodes ϖ and ξ can never exceed 3.*
3. *The valency of any other node on $\wp$ can never exceed 5.*

The main motivation in characterizing the families of plane graphs with constant metric dimension (or with non-constant metric dimension) is towards making metric dimension of possibly all plane graphs known. In this article, we determine the metric dimension of three classes of plane graphs (viz., $\mathfrak{J}_n$, $\mathfrak{K}_n$, and $\mathfrak{L}_n$) which are generated by taking n-copies of the complete bipartite graph (or a star) $K_{1,5}$ (see Figure 1). These classes of plane graphs are radially symmetric and possess an independent minimum resolving set with cardinality three i.e., three vertices is a minimal requirement for the unique identification of all vertices of these three classes of plane graphs. Throughout this article, all vertex indices are taken to be modulo n.

In the accompanying section, we acquire the exact metric dimension of the radially symmetrical plane graph $\mathfrak{J}_n$ (see Figure 2), and for each positive integer n: $n \geq 6$ we prove that $\beta(\mathfrak{J}_n) = 3$.

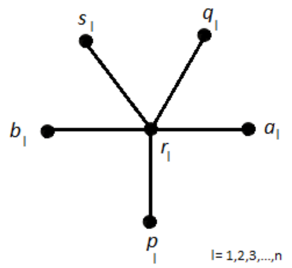

Figure 1. n-copies of the star $K_{1,5}$

2. Metric dimension of the planar graph $\mathfrak{J}_n$

The construction of the plane graph $\mathfrak{J}_n$ can be done in the following four steps:

1. Construct n-copies of the complete bipartite graph (or the star) $K_{1,5}$. Denote the central node of each star by r_l and the outer nodes of the star $K_{1,5}$ by p_l, q_l, s_l, a_l, and b_l $(1 \leq l \leq n)$. This results in a disconnected graph on $6n$ nodes with $5n$ edges (r_lp_l, r_lq_l, r_ls_l, r_la_l, and r_lb_l for $1 \leq l \leq n$).
2. Placing new edges between these stars as b_la_{l+1} and s_lq_{l+1} for $1 \leq l \leq n$. This adds $2n$ new edges.
3. Adding n new edges in each star as s_lq_l for $1 \leq l \leq n$.
4. Finally, adding n new nodes $\{c_l : 1 \leqslant l \leqslant n\}$, and $2n$ new edges as p_lc_l and c_lp_{l+1} for $1 \leq l \leq n$.

Thus, the radially symmetrical plane graph $\mathfrak{J}_n$ comprises of $7n$ nodes and $10n$ edges. It has n 7-sided cycles, n 6-sided cycles, n 3-sided cycles, and a pair of $2n$-sided faces (see Figure 2).

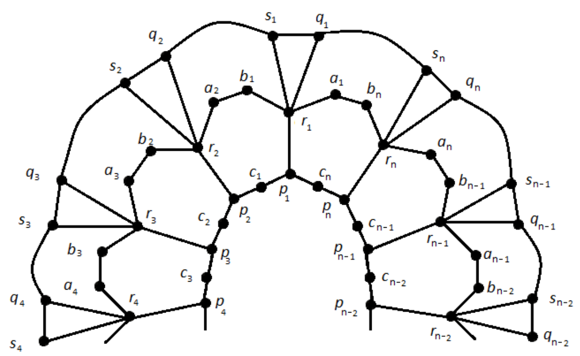

Figure 2. The radially symmetrical graph $\mathfrak{J}_n$

For our purpose, we name the cycle generated by the set of vertices $\{p_l : 1 \leqslant l \leqslant n\} \cup \{c_l : 1 \leqslant l \leqslant n\}$ in the graph, $\mathfrak{J}_n$ as the inner cycle, the cycle generated by the set of vertices $\{r_l : 1 \leqslant l \leqslant n\} \cup \{a_l : 1 \leqslant l \leqslant n\} \cup \{b_l : 1 \leqslant l \leqslant n\}$ in the graph, $\mathfrak{J}_n$ as the middle cycle, and the cycle generated by the set of vertices $\{q_l : 1 \leqslant l \leqslant n\} \cup \{s_l : 1 \leqslant l \leqslant n\}$ in the graph, $\mathfrak{J}_n$ as the outer cycle. In the following theorem, we obtain that the minimum cardinality of resolving set for the plane graph, $\mathfrak{J}_n$ is 3 i.e., three vertices is a minimal requirement for the unique identification of all vertices in the graph $\mathfrak{J}_n$.

Theorem 2.1. *Let $\mathfrak{J}_n$ be the planar graph on $7n$ vertices as defined above. Then for each $n \geq 6$, we have $\beta(\mathfrak{J}_n) = 3$ i.e., it has metric dimension 3.*

Proof. To establish this, we study the following two cases relying upon the positive integer n i.e., when n is even and when it is odd.
Case(1) When the integer n is even.
In this case, the positive integer n can be written as $n = 2w$, where $w \in \mathbb{N}$ and $w \geq 3$. Let $\mathfrak{R} = \{q_1, q_{w+1}, p_1\} \subset \mathbb{V}(\mathfrak{J}_n)$. Now, to unveil that $\mathfrak{R}$ is the resolving set for the graph $\mathfrak{J}_n$, we consign the metric codes for each vertex of the graph $\mathfrak{J}_n \setminus \mathfrak{R}$ regarding the set $\mathfrak{R}$.

Now, the metric codes for the vertices of inner cycle $\{p_l : 1 \leqslant l \leqslant n\} \cup \{c_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(p_l|\mathfrak{R}) = \begin{cases} (2l, 2w-2l+3, 2l-2), & 2 \leq l \leq w \\ (4w-2l+3, 2l-2w, 4w-2l+2), & w+1 \leq l \leq 2w. \end{cases}$$

and

$$\gamma(c_l|\mathfrak{R}) = \begin{cases} (2l+1, 2w-2l+2, 2l-1), & 1 \leq l \leq w-1 \\ (2w+1, 3, 2w-1), & l = w; \\ (4w-2l+2, 2l-2w+1, 4w-2l+1), & w+1 \leq l \leq 2w-1; \\ (3, 2l-2w+1, 4w-2l+1), & l = 2w. \end{cases}$$

The metric codes for the vertices of the middle cycle $\{r_l : 1 \leqslant l \leqslant n\} \cup \{a_l : 1 \leqslant l \leqslant n\} \cup \{b_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(r_l|\mathfrak{R}) = \begin{cases} (1, 2w, 1), & l = 1; \\ (2l-1, 2w-2l+2, 2l-1), & 2 \leq l \leq w; \\ (2w, 1, 2w+1), & l = w+1; \\ (4w-2l+2, 2l-2w-1, 4w-2l+3), & w+2 \leq l \leq 2w. \end{cases}$$

$$\gamma(a_l|\mathfrak{R}) = \begin{cases} (2, 2w+1, 2), & l = 1; \\ (2l-1, 2w-2l+3, 2l-1), & 2 \leq l \leq w; \\ (2w+1, 2, 2w+1), & l = w+1; \\ (4w-2l+3, 2l-2w-1, 4w-2l+4), & w+2 \leq l \leq 2w. \end{cases}$$

and

$$\gamma(b_l|\mathfrak{R}) = \begin{cases} (2, 2w, 2), & l = 1; \\ (2l, 2w-2l+2, 2l), & 2 \leq l \leq w-1; \\ (2w, 3, 2w), & l = w; \\ (2w, 2, 2w+1), & l = w+1; \\ (4w-2l+2, 2l-2w, 4w-2l+3), & w+2 \leq l \leq 2w-1; \\ (3, 2l-2w, 4w-2l+3), & l = 2w. \end{cases}$$

At last, the metric codes for the vertices of outer cycle $\{s_l : 1 \leqslant l \leqslant n\} \cup \{q_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(s_l|\mathfrak{R}) = \begin{cases} (2l-1, 2w-2l+1, 2l), & 1 \leq l \leq w; \\ (4w-2l+1, 2l-2w-1, 4w-2l+3), & w+1 \leq l \leq 2w. \end{cases}$$

and

$$\gamma(q_l|\mathfrak{R}) = \begin{cases} (2l-2, 2w-2l+2, 2l-1), & 2 \leq l \leq w; \\ (4w-2l+2, 2l-2w-2, 4w-2l+4), & w+2 \leq l \leq 2w. \end{cases}$$

From above, we find that there do not exist two vertices with the same metric codes, which suggest that $\beta(\mathfrak{J}_n) \leq 3$ i.e., the location number of the plane graph $\mathfrak{J}_n$ is less than or equal to 3. Now, to finish the evidence for this case, we show that $\beta(\mathfrak{J}_n) \geq 3$ by working out that there does not exist a resolving set $\mathfrak{R}$ such that $|\mathfrak{R}| = 2$. On contrary, suppose that $\beta(\mathfrak{J}_n) = 2$. Now, from Theorem 1.1, we find that the degree of basis vertices can be at most 3. But except the vertices p_l, q_l, s_l, a_l, b_l, and c_l $(1 \leq l \leq n)$, all other vertices of the radially symmetrical plane graph $\mathfrak{J}_n$ have a degree 5. Then, without loss of generality, we suppose that the first resolving vertex is one of the vertices p_1, q_1, s_1, a_1, b_l, or c_1 and other nodes lie in the inner cycle, the middle cycle, and in the outer cycle. Therefore, we have the following possibilities to be discussed.

Resolving sets	Contradictions
$\{p_1, p_h\}$, p_h $(2 \leq h \leq n)$	For $2 \leq h \leq w$, we have $\gamma(r_1\|\{p_1, p_h\}) = \gamma(c_n\|\{p_1, p_h\})$; and for $h = w+1$, we have $\gamma(c_1\|\{p_1, p_{w+1}\}) = \gamma(c_n\|\{p_1, p_{w+1}\})$, a contradiction.
$\{c_1, c_h\}$, c_h $(2 \leq h \leq n)$	For $2 \leq h \leq w$, we have $\gamma(r_1\|\{c_1, c_h\}) = \gamma(c_n\|\{c_1, c_h\})$; and for $h = w+1$, we have $\gamma(p_1\|\{c_1, c_{w+1}\}) = \gamma(p_2\|\{c_1, c_{w+1}\})$, a contradiction.
$\{a_1, a_h\}$, a_h $(2 \leq h \leq n)$	For $2 \leq h \leq w+1$, we have $\gamma(p_1\|\{a_1, a_h\}) = \gamma(q_1\|\{a_1, a_h\})$, a contradiction.
$\{b_1, b_h\}$, b_h $(2 \leq h \leq n)$	For $2 \leq h \leq w$, we have $\gamma(p_1\|\{b_1, b_h\}) = \gamma(q_1\|\{b_1, b_h\})$; and for $h = w+1$, we have $\gamma(q_1\|\{b_1, b_{w+1}\}) = \gamma(s_1\|\{b_1, b_{w+1}\})$, a contradiction.
$\{q_1, q_h\}$, q_h $(2 \leq h \leq n)$	For $2 \leq h \leq w$, we have $\gamma(b_n\|\{q_1, q_h\}) = \gamma(c_n\|\{q_1, q_h\})$; and for $h = w+1$, we have $\gamma(s_n\|\{q_1, q_{w+1}\}) = \gamma(s_1\|\{q_1, q_{w+1}\})$, a contradiction.
$\{s_1, s_h\}$, s_h $(2 \leq h \leq n)$	For $2 \leq h \leq w$, we have $\gamma(b_n\|\{s_1, s_h\}) = \gamma(c_n\|\{s_1, s_h\})$; and for $h = w+1$, we have $\gamma(q_2\|\{s_1, s_{w+1}\}) = \gamma(q_1\|\{s_1, s_{w+1}\})$, a contradiction.
$\{p_1, c_h\}$, c_h $(1 \leq h \leq n)$	For $1 \leq h \leq w$, we have $\gamma(c_n\|\{p_1, c_h\}) = \gamma(r_1\|\{p_1, c_h\})$; and for $h = w+1$, we have $\gamma(q_2\|\{p_1, c_{w+1}\}) = \gamma(a_2\|\{p_1, c_{w+1}\})$, a contradiction.
$\{p_1, a_h\}$, a_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(q_1\|\{p_1, a_1\}) = \gamma(b_1\|\{p_1, a_1\})$; when $h = 2$, we have $\gamma(s_2\|\{p_1, a_2\}) = \gamma(b_2\|\{p_1, a_2\})$; when $h = 3$, we have $\gamma(s_3\|\{p_1, a_3\}) = \gamma(b_3\|\{p_1, a_3\})$; and for $4 \leq h \leq w+1$, we have $\gamma(q_1\|\{p_1, a_h\}) = \gamma(b_1\|\{p_1, a_h\})$, a contradiction.
$\{p_1, b_h\}$, b_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(q_1\|\{p_1, b_1\}) = \gamma(s_1\|\{p_1, b_1\})$; when $h = 2$, we have $\gamma(a_2\|\{p_1, b_2\}) = \gamma(q_2\|\{p_1, b_2\})$; when $3 \leq h \leq w$, we have $\gamma(q_1\|\{p_1, b_h\}) = \gamma(b_1\|\{p_1, b_h\})$; and for $h = w+1$, we have $\gamma(r_2\|\{p_1, b_{w+1}\}) = \gamma(b_n\|\{p_1, b_{w+1}\})$, a contradiction.
$\{p_1, q_h\}$, q_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(a_1\|\{p_1, q_1\}) = \gamma(b_1\|\{p_1, q_1\})$; when $h = 2$, we have $\gamma(p_2\|\{p_1, q_2\}) = \gamma(q_1\|\{p_1, q_2\})$; and for $3 \leq h \leq w+1$, we have $\gamma(q_1\|\{p_1, q_h\}) = \gamma(b_1\|\{p_1, q_h\})$, a contradiction.
$\{p_1, s_h\}$, s_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(a_1\|\{p_1, s_1\}) = \gamma(b_1\|\{p_1, s_1\})$; when $2 \leq h \leq w$, we have $\gamma(q_1\|\{p_1, s_h\}) = \gamma(b_1\|\{p_1, s_h\})$; and for $h = w+1$, we have $\gamma(r_2\|\{p_1, s_{w+1}\}) = \gamma(b_n\|\{p_1, s_{w+1}\})$, a contradiction.
$\{c_1, a_h\}$, a_h $(1 \leq h \leq n)$	For $1 \leq h \leq 2$, we have $\gamma(s_1\|\{c_1, a_h\}) = \gamma(q_1\|\{c_1, a_h\})$; when $h = 3$, we have $\gamma(a_2\|\{c_1, a_3\}) = \gamma(q_2\|\{c_1, a_3\})$; and for $4 \leq h \leq w+1$, we have $\gamma(a_2\|\{c_1, a_h\}) = \gamma(s_1\|\{c_1, a_h\})$, a contradiction.
$\{c_1, b_h\}$, b_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(s_1\|\{c_1, b_1\}) = \gamma(q_1\|\{c_1, b_1\})$; when $h = 2$, we have $\gamma(a_2\|\{c_1, a_2\}) = \gamma(q_2\|\{c_1, a_2\})$; and for $3 \leq h \leq w+1$, we have $\gamma(a_2\|\{c_1, b_h\}) = \gamma(s_1\|\{c_1, b_h\})$, a contradiction.
$\{c_1, s_h\}$, s_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(a_1\|\{c_1, s_1\}) = \gamma(b_1\|\{c_1, s_1\})$; and for $2 \leq h \leq w+1$, we have $\gamma(a_2\|\{c_1, s_h\}) = \gamma(s_1\|\{c_1, s_h\})$, a contradiction.
$\{c_1, t_h\}$, t_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(a_1\|\{c_1, t_1\}) = \gamma(b_1\|\{c_1, t_1\})$; when $h = 2$, we have $\gamma(a_2\|\{c_1, t_2\}) = \gamma(b_2\|\{c_1, t_2\})$; and for $3 \leq h \leq w+1$, we have $\gamma(a_2\|\{c_1, t_h\}) = \gamma(s_1\|\{c_1, t_h\})$, a contradiction.
$\{a_1, b_h\}$, b_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(s_1\|\{a_1, b_1\}) = \gamma(q_1\|\{a_1, b_1\})$; when $h = 2$, we have $\gamma(a_2\|\{a_1, b_2\}) = \gamma(q_2\|\{a_1, b_2\})$; when $h = 3$, we have $\gamma(a_3\|\{a_1, b_3\}) = \gamma(q_2\|\{a_1, b_3\})$; when $4 \leq h \leq w$, we have $\gamma(q_1\|\{a_1, b_h\}) = \gamma(b_1\|\{a_1, b_h\})$; and for $h = w+1$, we have $\gamma(a_n\|\{a_1, b_{w+1}\}) = \gamma(s_n\|\{a_1, b_{w+1}\})$, a contradiction.
$\{a_1, s_h\}$, s_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(b_1\|\{a_1, s_1\}) = \gamma(p_1\|\{a_1, s_1\})$; when $2 \leq h \leq w$, we have $\gamma(q_1\|\{a_1, s_h\}) = \gamma(b_1\|\{a_1, s_h\})$; and for $h = w+1$, we have $\gamma(a_n\|\{a_1, s_{w+1}\}) = \gamma(s_n\|\{a_1, s_{w+1}\})$, a contradiction.

Resolving sets	Contradictions
$\{a_1, q_h\}$, q_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(b_1\|\{a_1, q_1\}) = \gamma(p_1\|\{a_1, q_1\})$; when $h = 2$, we have $\gamma(s_n\|\{a_1, q_2\}) = \gamma(c_1\|\{a_1, q_2\})$; and for $3 \leq h \leq w+1$, we have $\gamma(q_1\|\{a_1, q_h\}) = \gamma(b_1\|\{a_1, q_h\})$, a contradiction.
$\{b_1, s_h\}$, s_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(a_1\|\{b_1, s_1\}) = \gamma(p_1\|\{b_1, s_1\})$; when $h = 2$, we have $\gamma(b_2\|\{b_1, s_2\}) = \gamma(p_2\|\{b_1, s_2\})$; and for $3 \leq h \leq w+1$, we have $\gamma(q_2\|\{b_1, s_h\}) = \gamma(b_2\|\{b_1, s_h\})$, a contradiction.
$\{b_1, q_h\}$, q_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(a_1\|\{b_1, q_1\}) = \gamma(p_1\|\{b_1, q_1\})$; when $2 \leq h \leq 3$, we have $\gamma(b_2\|\{b_1, q_2\}) = \gamma(p_2\|\{b_1, q_2\})$; and for $4 \leq h \leq w+1$, we have $\gamma(q_2\|\{b_1, q_h\}) = \gamma(b_2\|\{b_1, q_h\})$, a contradiction.
$\{s_1, q_h\}$, q_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(a_1\|\{s_1, q_1\}) = \gamma(b_1\|\{s_1, q_1\})$; and for $2 \leq h \leq w+1$, we have $\gamma(r_1\|\{s_1, q_h\}) = \gamma(q_1\|\{s_1, q_h\})$, a contradiction.

Thus, from the above table, we obtain that there does not exist a resolving set consisting of two vertices for $\mathbb{V}(\mathfrak{J}_n)$, suggesting that $\beta(\mathfrak{J}_n) = 3$ in this case.

Case(2) When the integer n is odd.

In this case, the positive integer n can be written as $n = 2w + 1$, where $w \in \mathbb{N}$ and $w \geq 3$. Let $\mathfrak{R} = \{q_1, q_{w+1}, p_1\} \subset \mathbb{V}(\mathfrak{J}_n)$. Now, to unveil that $\mathfrak{R}$ is the resolving set for the graph $\mathfrak{J}_n$, we consign the metric codes for each vertex of the graph $\mathfrak{J}_n \setminus \mathfrak{R}$ regarding the set $\mathfrak{R}$.

Now, the metric codes for the vertices of inner cycle $\{p_l : 1 \leqslant l \leqslant n\} \cup \{c_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(p_l|\mathfrak{R}) = \begin{cases} (2l, 2w-2l+3, 2l-2), & 2 \leq l \leq w \\ (2w+2, 2, 2w), & l = w+1; \\ (4w-2l+5, 2l-2w, 4w-2l+4), & w+2 \leq l \leq 2w+1. \end{cases}$$

and

$$\gamma(c_l|\mathfrak{R}) = \begin{cases} (2l+1, 2w-2l+2, 2l-1), & 1 \leq l \leq w-1 \\ (2w+1, 3, 2w-1), & l = w; \\ (4w-2l+4, 2l-2w+1, 4w-2l+3), & w+1 \leq l \leq 2w; \\ (3, 2l-2w, 1), & l = 2w+1. \end{cases}$$

The metric codes for the vertices of the middle cycle $\{r_l : 1 \leqslant l \leqslant n\} \cup \{a_l : 1 \leqslant l \leqslant n\} \cup \{b_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(r_l|\mathfrak{R}) = \begin{cases} (1, 2w, 1), & l = 1; \\ (2l-1, 2w-2l+2, 2l-1), & 2 \leq l \leq w; \\ (2w+1, 1, 2w+1), & l = w+1; \\ (4w-2l+4, 2l-2w-1, 4w-2l+5), & w+2 \leq l \leq 2w+1. \end{cases}$$

$$\gamma(a_l|\mathfrak{R}) = \begin{cases} (2, 2w+1, 2), & l = 1; \\ (2l-1, 2w-2l+3, 2l-1), & 2 \leq l \leq w; \\ (2w+1, 2, 2w+1), & l = w+1; \\ (4w-2l+5, 2l-2w-1, 4w-2l+6), & w+2 \leq l \leq 2w+1. \end{cases}$$

and

$$\gamma(b_l|\mathfrak{R}) = \begin{cases} (2, 2w, 2), & l = 1; \\ (2l, 2w-2l+2, 2l), & 2 \leq l \leq w-1; \\ (2w, 3, 2w), & l = w; \\ (2w+2, 2, 2w+2), & l = w+1; \\ (4w-2l+4, 2l-2w, 4w-2l+5), & w+2 \leq l \leq 2w; \\ (3, 2l-2w, 4w-2l+5), & l = 2w+1. \end{cases}$$

At last, the metric codes for the vertices of outer cycle $\{s_l : 1 \leqslant l \leqslant n\} \cup \{q_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(s_l|\mathfrak{R}) = \begin{cases} (2l-1, 2w-2l+1, 2l), & 1 \leq l \leq w; \\ (2w+1, 1, 2w+2), & l = w+1; \\ (4w-2l+3, 2l-2w-1, 4w-2l+5), & w+2 \leq l \leq 2w+1. \end{cases}$$

and

$$\gamma(q_l|\mathfrak{R}) = \begin{cases} (2l-2, 2w-2l+2, 2l-1), & 2 \leq l \leq w; \\ (4w-2l+4, 2l-2w-2, 4w-2l+6), & w+2 \leq l \leq 2w+1. \end{cases}$$

Again, we find that there do not exist two vertices with the same metric codes, which suggest that $\beta(\mathfrak{J}_n) \leq 3$ i.e., the location number of the plane graph $\mathfrak{J}_n$ is less than or equal to 3. Now, on assuming that $\beta(\mathfrak{J}_n) = 2$, we get the same eventualities as in Case(1), and similarly, the contradiction can be obtained. So, in this case, we have $\beta(\mathfrak{J}_n) = 3$ as well and hence the theorem. □

Now, in terms of independent resolving set, we have the following result:

Theorem 2.2. *Let $\mathfrak{J}_n$ be the planar graph on $7n$ vertices as defined above. Then for every positive integer n; $n \geq 6$, its independent resolving number is 3.*

Proof. For proof, refer to Theorem 2.1. □

In the accompanying section, we acquire the exact metric dimension of the radially symmetrical plane graph $\mathfrak{K}_n$ (see Figure 3), and for each positive integer n: $n \geq 6$ we prove that $\beta(\mathfrak{K}_n) = 3$.

3. Metric dimension of the planar graph $\mathfrak{K}_n$

The construction of the plane graph $\mathfrak{K}_n$ can be done in the following four steps:

1. Construct n-copies of the complete bipartite graph (or the star) $K_{1,5}$. Denote the central node of each star by r_l and the outer nodes of the star $K_{1,5}$ by p_l, q_l, s_l, a_l, and b_l $(1 \leq l \leq n)$. This results in a disconnected graph on $6n$ nodes with $5n$ edges (r_lp_l, r_lq_l, r_ls_l, r_la_l, and r_lb_l for $1 \leq l \leq n$).
2. Placing new edges between these stars as q_lp_{l+1} and b_la_{l+1} for $1 \leq l \leq n$. This adds $2n$ new edges.
3. Adding n new edges in each of these stars as p_lq_l for $1 \leq l \leq n$.
4. Finally, adding n new nodes $\{t_l : 1 \leqslant l \leqslant n\}$, and $2n$ new edges as s_lt_l and t_ls_{l+1} for $1 \leq l \leq n$.

Thus, the radially symmetrical plane graph $\mathfrak{K}_n$ comprises of $7n$ nodes and $10n$ edges. It has n 7-sided cycles, n 6-sided cycles, n 3-sided cycles, and a pair of $2n$-sided faces (see Figure 3).

For our purpose, we name the cycle generated by the set of vertices $\{p_l : 1 \leqslant l \leqslant n\} \cup \{q_l : 1 \leqslant l \leqslant n\}$ in the graph, $\mathfrak{K}_n$ as the inner cycle, the cycle generated by the set of vertices $\{r_l : 1 \leqslant l \leqslant n\} \cup \{a_l : 1 \leqslant l \leqslant n\} \cup \{b_l : 1 \leqslant l \leqslant n\}$ in the graph, $\mathfrak{K}_n$ as the middle cycle, and the cycle generated by the set of vertices $\{t_l : 1 \leqslant l \leqslant n\} \cup \{s_l : 1 \leqslant l \leqslant n\}$ in the graph, $\mathfrak{K}_n$ as the outer cycle. In the following theorem, we obtain that the minimum cardinality of resolving set for the plane graph, $\mathfrak{K}_n$ is 3 i.e., three vertices is a minimal requirement for the unique identification of all vertices in the graph $\mathfrak{K}_n$.

Theorem 3.1. *Let $\mathfrak{K}_n$ be the planar graph on $7n$ vertices as defined above. Then for each $n \geq 6$, we have $\beta(\mathfrak{K}_n) = 3$ i.e., it has metric dimension 3.*

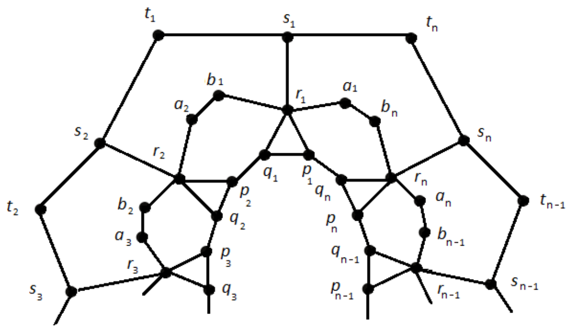

Figure 3. The radially symmetrical graph $\mathfrak{K}_n$

Proof. To establish this, we study the following two cases relying upon the positive integer n i.e., when n is even and when it is odd.
Case(1) When the integer n is even.
In this case, the positive integer n can be written as $n = 2w$, where $w \in \mathbb{N}$ and $w \geq 3$. Let $\mathfrak{R} = \{p_1, p_{w+1}, s_2\} \subset \mathbb{V}(\mathfrak{K}_n)$. Now, to unveil that $\mathfrak{R}$ is the resolving set for the graph $\mathfrak{K}_n$, we consign the metric codes for each vertex of the graph $\mathfrak{K}_n \setminus \mathfrak{R}$ regarding $\mathfrak{R}$.

Now, the metric codes for the vertices of inner cycle $\{p_l : 1 \leqslant l \leqslant n\} \cup \{q_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(p_l|\mathfrak{R}) = \begin{cases} (2, 2w-2, 2), & l = 2; \\ (2l-2, 2w-2l+2, 2l-3), & 3 \leq l \leq w \\ (4w-2l+2, 2, 2w+1), & l = w+2; \\ (4w-2l+2, 2l-2w-2, 4w-2l+6), & w+3 \leq l \leq 2w. \end{cases}$$

and

$$\gamma(q_l|\mathfrak{R}) = \begin{cases} (1, 2w-1, 3), & l = 1; \\ (2l-1, 2w-2l+1, 2l-2), & 2 \leq l \leq w \\ (4w-2l+1, 1, 2w), & l = w+1; \\ (4w-2l+1, 2l-2w-1, 4w-2l+5), & w+2 \leq l \leq 2w. \end{cases}$$

The metric codes for the vertices of the middle cycle $\{r_l : 1 \leqslant l \leqslant n\} \cup \{a_l : 1 \leqslant l \leqslant n\} \cup \{b_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(r_l|\mathfrak{R}) = \begin{cases} (1, 2w, 3), & l = 1; \\ (2l-1, 2w-2l+2, 2l-3), & 2 \leq l \leq w; \\ (2w, 1, 2w-1), & l = w+1; \\ (4w-2l+2, 2l-2w-1, 4w-2l+5), & w+2 \leq l \leq 2w. \end{cases}$$

$$\gamma(a_l|\mathfrak{R}) = \begin{cases} (2, 2w+1, 4), & l = 1; \\ (3, 2w-1, 2), & l = 2; \\ (2l-1, 2w-2l+3, 2l-3), & 3 \leq l \leq w; \\ (2w+1, 2, 2w-1), & l = w+1; \\ (4w-2l+3, 2l-2w-1, 2w+1), & l = w+2; \\ (4w-2l+3, 2l-2w-1, 4w-2l+6), & w+3 \leq l \leq 2w. \end{cases}$$

and

$$\gamma(b_l|\mathfrak{R}) = \begin{cases} (2,2w,3), & l=1; \\ (2l,2w-2l+2,2l-2), & 2\le l\le w-1; \\ (2w,3,2w-2), & l=w; \\ (2w,2,2w), & l=w+1; \\ (4w-2l+2,2l-2w,4w-2l+5), & w+2\le l\le 2w-1; \\ (3,2l-2w,4w-2l+5), & l=2w. \end{cases}$$

At last, metric codes for the vertices of outer cycle $\{s_l : 1\leqslant l\leqslant n\}\cup\{t_l : 1\leqslant l\leqslant n\}$ are

$$\gamma(s_l|\mathfrak{R}) = \begin{cases} (2l,2w-2l+3,2), & l=1; \\ (2l,2w-2l+3,2l-4), & 3\le l\le w; \\ (2w+1,2,2w-2), & l=w+1; \\ (4w-2l+3,2l-2w,4w-2l+4), & w+2\le l\le 2w. \end{cases}$$

and

$$\gamma(t_l|\mathfrak{R}) = \begin{cases} (3,2w,1), & l=1; \\ (2l+1,2w-2l+2,2l-3), & 2\le l\le w-1; \\ (2w+1,3,2w-3), & l=w; \\ (4w-2l+2,3,2w-1), & l=w+1; \\ (4w-2l+2,2l-2w+1,4w-2l+3), & w+2\le l\le 2w-1; \\ (3,2l-2w+1,4w-2l+3), & l=2w. \end{cases}$$

From above, we find that there do not exist two vertices with the same metric codes, which suggest that $\beta(\mathfrak{K}_n)\le 3$ i.e., the location number of the plane graph $\mathfrak{K}_n$ is less than or equal to 3. Now, so as to finish the evidence for this case, we show that $\beta(\mathfrak{K}_n)\ge 3$ by working out that there does not exist a resolving set $\mathfrak{R}$ such that $|\mathfrak{R}|=2$. On contrary, suppose that $\beta(\mathfrak{K}_n)=2$. Now, from Theorem 1.1, we find that the degree of basis vertices can be at most 3. But except the vertices p_l, q_l, s_l, a_l, b_l, and t_l $(1\le l\le n)$, all other vertices of the radially symmetrical plane graph $\mathfrak{K}_n$ have a degree 5. Then, without loss of generality, we suppose that the first resolving vertex is one of the vertices p_1, q_1, s_1, a_1, b_l or t_1 and other nodes lie in the inner cycle, the middle cycle, and in the outer cycle. Therefore, we have the following possibilities to be discussed.

Resolving sets	Contradictions
$\{p_1,p_h\}$, p_h $(2\le h\le n)$	For $2\le h\le w+1$, we have $\gamma(a_1\|\{p_1,p_h\})=\gamma(s_1\|\{p_1,p_h\})$, a contradiction.
$\{q_1,q_h\}$, q_h $(2\le h\le n)$	For $2\le h\le w$, we have $\gamma(a_1\|\{q_1,q_h\})=\gamma(s_1\|\{q_1,q_h\})$; and for $h=w+1$, we have $\gamma(p_2\|\{q_1,q_{w+1}\})=\gamma(p_1\|\{q_1,q_{w+1}\})$, a contradiction.
$\{a_1,a_h\}$, a_h $(2\le h\le n)$	For $2\le h\le w+1$, we have $\gamma(p_2\|\{a_1,a_h\})=\gamma(s_2\|\{a_1,a_h\})$, a contradiction.
$\{b_1,b_h\}$, b_h $(2\le h\le n)$	For $2\le h\le w+1$, we have $\gamma(p_2\|\{b_1,b_h\})=\gamma(s_2\|\{b_1,b_h\})$, a contradiction.
$\{s_1,s_h\}$, s_h $(2\le h\le n)$	For $2\le h\le w$, we have $\gamma(r_1\|\{s_1,s_h\})=\gamma(t_n\|\{s_1,s_h\})$; and for $h=w+1$, we have $\gamma(t_1\|\{s_1,s_{w+1}\})=\gamma(t_n\|\{s_1,s_{w+1}\})$, a contradiction.
$\{t_1,t_h\}$, t_h $(2\le h\le n)$	For $2\le h\le w$, we have $\gamma(r_1\|\{t_1,t_h\})=\gamma(t_n\|\{t_1,t_h\})$; and for $h=w+1$, we have $\gamma(s_1\|\{t_1,t_{w+1}\})=\gamma(s_2\|\{t_1,t_{w+1}\})$, a contradiction.
$\{p_1,q_h\}$, q_h $(1\le h\le n)$	For $1\le h\le w$, we have $\gamma(s_1\|\{p_1,q_h\})=\gamma(a_1\|\{p_1,q_h\})$; and for $h=w+1$, we have $\gamma(b_n\|\{p_1,q_{w+1}\})=\gamma(r_2\|\{p_1,q_{w+1}\})$, a contradiction.
$\{p_1,a_h\}$, a_h $(1\le h\le n)$	For $h=1$, we have $\gamma(s_1\|\{p_1,a_1\})=\gamma(b_1\|\{p_1,a_1\})$; when $h=2$, we have $\gamma(s_2\|\{p_1,a_2\})=\gamma(b_2\|\{p_1,a_2\})$; when $h=3$, we have $\gamma(s_3\|\{p_1,a_3\})=\gamma(b_3\|\{p_1,a_3\})$; and for $4\le h\le w+1$, we have $\gamma(b_1\|\{p_1,a_h\})=\gamma(s_1\|\{p_1,a_h\})$, a contradiction.
$\{p_1,b_h\}$, b_h $(1\le h\le n)$	For $h=1$, we have $\gamma(s_1\|\{p_1,b_1\})=\gamma(a_1\|\{p_1,b_1\})$; when $h=2$, we have $\gamma(s_n\|\{p_1,b_2\})=\gamma(b_n\|\{p_1,b_2\})$; when $3\le h\le w$, we have $\gamma(s_1\|\{p_1,b_h\})=\gamma(b_1\|\{p_1,b_h\})$; and for $h=w+1$, we have $\gamma(b_1\|\{p_1,b_h\})=\gamma(s_1\|\{p_1,b_h\})$, a contradiction.

Resolving sets	Contradictions
$\{p_1, s_h\}$, s_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(b_1\|\{p_1, s_1\}) = \gamma(a_1\|\{p_1, s_1\})$; and for $2 \leq h \leq w+1$, we have $\gamma(t_1\|\{p_1, s_h\}) = \gamma(r_2\|\{p_1, s_h\})$, a contradiction.
$\{p_1, t_h\}$, t_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(b_1\|\{p_1, t_1\}) = \gamma(a_1\|\{p_1, t_1\})$; and for $2 \leq h \leq w+1$, we have $\gamma(t_1\|\{p_1, t_h\}) = \gamma(r_2\|\{p_1, t_h\})$, a contradiction.
$\{q_1, a_h\}$, a_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(b_1\|\{q_1, a_1\}) = \gamma(s_1\|\{q_1, a_1\})$; when $h = 2$, we have $\gamma(a_1\|\{q_1, a_2\}) = \gamma(s_1\|\{q_1, a_2\})$; when $h = 3$, we have $\gamma(b_3\|\{q_1, a_3\}) = \gamma(s_3\|\{q_1, a_3\})$; and for $4 \leq h \leq w+1$, we have $\gamma(s_1\|\{q_1, a_h\}) = \gamma(b_1\|\{q_1, a_h\})$, a contradiction.
$\{q_1, b_h\}$, b_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(a_1\|\{q_1, b_1\}) = \gamma(s_1\|\{q_1, b_1\})$; when $h = 2$, we have $\gamma(a_2\|\{q_1, b_2\}) = \gamma(s_2\|\{q_1, b_2\})$; when $3 \leq h \leq w$, we have $\gamma(b_1\|\{q_1, b_h\}) = \gamma(s_1\|\{q_1, b_h\})$; and for $h = w+1$, we have $\gamma(r_n\|\{q_1, b_{w+1}\}) = \gamma(a_2\|\{q_1, b_{w+1}\})$, a contradiction.
$\{q_1, s_h\}$, s_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(b_1\|\{q_1, s_1\}) = \gamma(a_1\|\{q_1, s_1\})$, when $h = 2$, we have $\gamma(a_2\|\{q_1, s_2\}) = \gamma(b_2\|\{q_1, s_2\})$; and for $3 \leq h \leq w+1$, we have $\gamma(r_2\|\{q_1, s_h\}) = \gamma(q_2\|\{q_1, s_h\})$, a contradiction.
$\{q_1, t_h\}$, t_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(b_1\|\{q_1, t_1\}) = \gamma(a_1\|\{q_1, t_1\})$; when $h = 2$, we have $\gamma(a_2\|\{q_1, t_2\}) = \gamma(b_2\|\{q_1, t_2\})$; and for $3 \leq h \leq w+1$, we have $\gamma(r_2\|\{q_1, t_h\}) = \gamma(q_2\|\{q_1, t_h\})$, a contradiction.
$\{a_1, b_h\}$, b_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(p_1\|\{a_1, b_1\}) = \gamma(q_1\|\{a_1, b_1\})$; when $h = 2$, we have $\gamma(s_2\|\{a_1, b_2\}) = \gamma(q_2\|\{a_1, b_2\})$; and for $3 \leq h \leq w+1$, we have $\gamma(s_1\|\{a_1, b_h\}) = \gamma(b_1\|\{a_1, b_h\})$, a contradiction.
$\{a_1, s_h\}$, s_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(p_1\|\{a_1, s_1\}) = \gamma(q_1\|\{a_1, s_1\})$, when $h = 2$, we have $\gamma(a_2\|\{a_1, s_2\}) = \gamma(p_2\|\{a_1, s_2\})$, and for $3 \leq h \leq w+1$, we have $\gamma(r_2\|\{a_1, s_h\}) = \gamma(q_2\|\{a_1, s_h\})$, a contradiction.
$\{a_1, t_h\}$, t_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(p_1\|\{a_1, t_1\}) = \gamma(q_1\|\{a_1, t_1\})$; when $h = 2$, we have $\gamma(a_2\|\{a_1, t_2\}) = \gamma(p_2\|\{a_1, t_2\})$; and for $3 \leq h \leq w+1$, we have $\gamma(r_2\|\{a_1, t_h\}) = \gamma(q_2\|\{a_1, t_h\})$, a contradiction.
$\{b_1, s_h\}$, s_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(p_1\|\{b_1, s_1\}) = \gamma(q_1\|\{b_1, s_1\})$; when $h = 2$, we have $\gamma(q_2\|\{b_1, s_2\}) = \gamma(p_2\|\{b_1, s_2\})$; and for $3 \leq h \leq w+1$, we have $\gamma(b_2\|\{b_1, s_h\}) = \gamma(t_1\|\{b_1, s_h\})$, a contradiction.
$\{b_1, t_h\}$, t_h $(1 \leq h \leq n)$	For $h = 1$, we have $\gamma(p_1\|\{b_1, t_1\}) = \gamma(q_1\|\{b_1, t_1\})$; when $h = 2$, we have $\gamma(q_2\|\{b_1, t_2\}) = \gamma(p_2\|\{b_1, t_2\})$; and for $3 \leq h \leq w+1$, we have $\gamma(b_2\|\{b_1, t_h\}) = \gamma(t_1\|\{b_1, t_h\})$, a contradiction.
$\{s_1, t_h\}$, t_h $(1 \leq h \leq n)$	For $1 \leq h \leq w$, we have $\gamma(r_1\|\{s_1, t_h\}) = \gamma(t_n\|\{s_1, t_h\})$; and for $h = w+1$, we have $\gamma(b_2\|\{s_1, t_{w+1}\}) = \gamma(r_2\|\{s_1, t_{w+1}\})$, a contradiction.

Thus, from the above table, we obtain that there does not exist a resolving set consisting of two vertices for $\mathbb{V}(\mathfrak{K}_n)$, suggesting that $\beta(\mathfrak{K}_n) = 3$ in this case.

Case(2) When the integer n is odd.

In this case, the positive integer n can be written as $n = 2w + 1$, where $w \in \mathbb{N}$ and $w \geq 3$. Let $\mathfrak{R} = \{p_1, p_{w+1}, s_2\} \subset \mathbb{V}(\mathfrak{K}_n)$. Now, to unveil that $\mathfrak{R}$ is the resolving set for the graph $\mathfrak{K}_n$, we consign the metric codes for each vertex of the graph $\mathfrak{K}_n \setminus \mathfrak{R}$ regarding the set $\mathfrak{R}$.

Now, the metric codes for the vertices of inner cycle $\{p_l : 1 \leqslant l \leqslant n\} \cup \{q_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(p_l|\mathfrak{R}) = \begin{cases} (2, 2w-2, 2), & l = 2; \\ (2l-2, 2w-2l+2, 2l-3), & 3 \leq l \leq w \\ (4w-2l+4, 2, 2w+1), & l = w+2; \\ (4w-2l+4, 2l-2w-2, 4w-2l+8), & w+3 \leq l \leq 2w+1. \end{cases}$$

and

$$\gamma(q_l|\mathfrak{R}) = \begin{cases} (1, 2w-1, 3), & l = 1; \\ (2l-1, 2w-2l+1, 2l-2), & 2 \leq l \leq w \\ (4w-2l+3, 1, 2w), & l = w+1; \\ (4w-2l+3, 2l-2w-1, 2w+2), & l = w+2; \\ (4w-2l+3, 2l-2w-1, 4w-2l+7), & w+3 \leq l \leq 2w+1. \end{cases}$$

The metric codes for the vertices of the middle cycle $\{r_l : 1 \leqslant l \leqslant n\} \cup \{a_l : 1 \leqslant l \leqslant n\} \cup \{b_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(r_l|\mathfrak{R}) = \begin{cases} (1, 2w, 3), & l = 1; \\ (2l-1, 2w-2l+2, 2l-3), & 2 \leq l \leq w; \\ (2w+1, 1, 2w-1), & l = w+1; \\ (4w-2l+4, 3, 2w+1), & l = w+2; \\ (4w-2l+4, 2l-2w-1, 4w-2l+7), & w+3 \leq l \leq 2w+1. \end{cases}$$

$$\gamma(a_l|\mathfrak{R}) = \begin{cases} (2, 2w+1, 4), & l = 1; \\ (3, 2w-1, 2), & l = 2; \\ (2l-1, 2w-2l+3, 2l-3), & 3 \leq l \leq w; \\ (2w+1, 2, 2w-1), & l = w+1; \\ (4w-2l+5, 2l-2w-1, 2w+1), & l = w+2; \\ (4w-2l+5, 2l-2w-1, 4w-2l+8), & w+3 \leq l \leq 2w+1. \end{cases}$$

and

$$\gamma(b_l|\mathfrak{R}) = \begin{cases} (2, 2w, 3), & l = 1; \\ (2l, 2w-2l+2, 2l-2), & 2 \leq l \leq w-1; \\ (2w, 3, 2w-2), & l = w; \\ (2w+2, 2, 2w), & l = w+1; \\ (4w-2l+4, 2l-2w, 2w+2), & l = w+2; \\ (4w-2l+4, 2l-2w, 4w-2l+7), & w+3 \leq l \leq 2w; \\ (3, 2l-2w, 4w-2l+7), & l = 2w+1. \end{cases}$$

At last, the metric codes for the vertices of outer cycle $\{s_l : 1 \leqslant l \leqslant n\} \cup \{t_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(s_l|\mathfrak{R}) = \begin{cases} (2l, 2w-2l+3, 2), & l = 1; \\ (2l, 2w-2l+3, 2l-4), & 3 \leq l \leq w; \\ (2w+2, 2, 2w-2), & l = w+1; \\ (4w-2l+5, 2l-2w, 2w), & l = w+2; \\ (4w-2l+5, 2l-2w, 4w-2l+6), & w+3 \leq l \leq 2w+1. \end{cases}$$

and

$$\gamma(t_l|\mathfrak{R}) = \begin{cases} (3, 2w, 1), & l = 1; \\ (2l+1, 2w-2l+2, 2l-3), & 2 \leq l \leq w-1; \\ (2w+1, 3, 2w-3), & l = w; \\ (4w-2l+4, 3, 2w-1), & l = w+1; \\ (4w-2l+4, 2l-2w+1, 4w-2l+5), & w+2 \leq l \leq 2w; \\ (3, 2w+2, 4w-2l+5), & l = 2w+1. \end{cases}$$

Again, we find that there do not exist two vertices with the same metric codes, which suggest that $\beta(\mathfrak{K}_n) \leq 3$ i.e., the location number of the plane graph $\mathfrak{K}_n$ is less than or equal to 3. Now, on assuming that $\beta(\mathfrak{K}_n) = 2$, we get the same eventualities as in Case(1), and similarly, the contradiction can be obtained. So, in this case, we have $\beta(\mathfrak{K}_n) = 3$ as well and hence the theorem. □

Now, in terms of independent resolving set, we have the following result:

Theorem 3.2. *Let $\mathfrak{K}_n$ be the planar graph on $7n$ vertices as defined above. Then for every positive integer n; $n \geq 6$, its independent resolving number is 3.*

Proof. For proof, refer to Theorem 3.1. □

In the accompanying section, we acquire the exact metric dimension of the radially symmetrical plane graph $\mathfrak{L}_n$ (see Figure 4), and for each positive integer n: $n \geq 6$ we prove that $\beta(\mathfrak{L}_n) = 3$.

4. Metric dimension of the planar graph $\mathfrak{L}_n$

The construction of the plane graph $\mathfrak{L}_n$ can be done in the following three steps:

1. Construct n-copies of the complete bipartite graph (or the star) $K_{1,5}$. Denote the central node of each star by r_l and the outer nodes of the star $K_{1,5}$ by p_l, q_l, s_l, a_l, and b_l $(1 \leq l \leq n)$. This results in a disconnected graph on $6n$ nodes with $5n$ edges (r_lp_l, r_lq_l, r_ls_l, r_la_l, and r_lb_l for $1 \leq l \leq n$).
2. Placing new edges between these stars as q_lp_{l+1} and b_la_{l+1} for $1 \leq l \leq n$. This adds $2n$ new edges.
3. Finally adding $2n$ new edges in each star as p_lq_l and a_lb_l for $1 \leq l \leq n$.

Thus, the radially symmetrical plane graph $\mathfrak{L}_n$ comprises of $6n$ nodes and $9n$ edges. It has n 6-sided cycles, $2n$ 3-sided cycles, and a pair of $2n$-sided faces (see Figure 4).

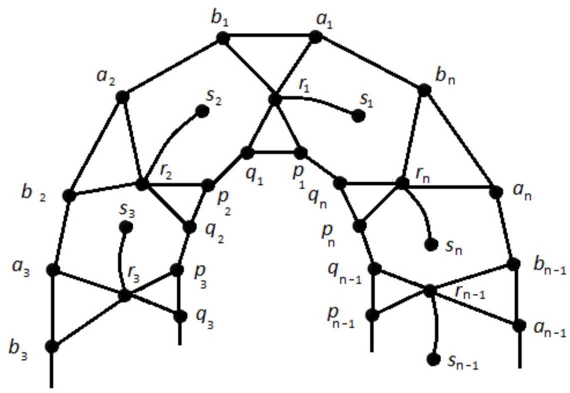

Figure 4. The radially symmetrical graph $\mathfrak{L}_n$

For our purpose, we name the cycle generated by the set of vertices $\{p_l : 1 \leqslant l \leqslant n\} \cup \{q_l : 1 \leqslant l \leqslant n\}$ in the graph, $\mathfrak{L}_n$ as the inner cycle, the set of vertices $\{r_l : 1 \leqslant l \leqslant n\} \cup \{s_l : 1 \leqslant l \leqslant n\}$ in the graph, $\mathfrak{L}_n$ as the set of middle vertices, and the cycle generated by the set of vertices $\{a_l : 1 \leqslant l \leqslant n\} \cup \{b_l : 1 \leqslant l \leqslant n\}$ in the graph, $\mathfrak{L}_n$ as the outer cycle. In the following theorem, we obtain that the minimum cardinality of resolving set for the plane graph, $\mathfrak{L}_n$ is 3 i.e., three vertices is a minimal requirement for the unique identification of all vertices in the graph $\mathfrak{L}_n$.

Theorem 4.1. *Let $\mathfrak{L}_n$ be the planar graph on $6n$ vertices as defined above. Then for each $n \geqslant 6$, we have $\beta(\mathfrak{L}_n) = 3$ i.e., it has metric dimension 3.*

Proof. In order to establish this, we study the following two cases relying upon the positive integer n i.e., when n is even natural and when it is odd.
Case(1) When the integer n is even.
In this case, the positive integer n can be written as $n = 2w$, where $w \in \mathbb{N}$ and $w \geq 3$. Let $\mathfrak{R} = \{p_1, p_{w+1}, s_1\} \subset \mathbb{V}(\mathfrak{L}_n)$. Now, to unveil that $\mathfrak{R}$ is the resolving set for the graph $\mathfrak{L}_n$, we consign the metric codes for each vertex of the graph $\mathfrak{L}_n \setminus \mathfrak{R}$ regarding the set $\mathfrak{R}$.

Now, the metric codes for the vertices of inner cycle $\{p_l : 1 \leqslant l \leqslant n\} \cup \{q_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(p_l|\mathfrak{R}) = \begin{cases} (2l-2, 2w-2l+2, 2l-1), & 2 \leq l \leq w \\ (4w-2l+2, 2l-2w-2, 4w-2l+4), & w+2 \leq l \leq 2w. \end{cases}$$

and

$$\gamma(q_l|\mathfrak{R}) = \begin{cases} (2l-1, 2w-2l+1, 2l), & 1 \leq l \leq w \\ (4w-2l+1, 2l-2w-1, 4w-2l+3), & w+1 \leq l \leq 2w. \end{cases}$$

The metric codes for the set of middle vertices $\{r_l : 1 \leqslant l \leqslant n\} \cup \{s_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(r_l|\mathfrak{R}) = \begin{cases} (1, 2w, 1), & l = 1; \\ (2l-1, 2w-2l+2, 2l), & 2 \leq l \leq w; \\ (4w-2l+2, 2l-2w-1, 4w-2l+4), & w+1 \leq l \leq 2w. \end{cases}$$

, and

$$\gamma(s_l|\mathfrak{R}) = \begin{cases} (2l, 2w-2l+3, 2l+1), & 2 \leq l \leq w; \\ (4w-2l+3, 2l-2w, 4w-2l+5), & w+1 \leq l \leq 2w. \end{cases}$$

At last, the metric codes for the vertices of outer cycle $\{a_l : 1 \leqslant l \leqslant n\} \cup \{b_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(a_l|\mathfrak{R}) = \begin{cases} (2, 2w+1, 2), & l = 1; \\ (2l-1, 2w-2l+3, 2l-1), & 2 \leq l \leq w; \\ (2w+1, 2, 2w+1), & l = w+1; \\ (4w-2l+3, 2l-2w-1, 4w-2l+4), & w+2 \leq l \leq 2w. \end{cases}$$

and

$$\gamma(b_l|\mathfrak{R}) = \begin{cases} (2, 2w, 2), & l = 1; \\ (2l, 2w-2l+2, 2l), & 2 \leq l \leq w; \\ (4w-2l+2, 2l-2w, 4w-2l+3), & w+1 \leq l \leq 2w-1; \\ (3, 2l-2w, 4w-2l+3), & l = 2w. \end{cases}$$

From above, we find that there do not exist two vertices with the same metric codes, which suggest that $\beta(\mathfrak{L}_n) \leq 3$ i.e., the location number of the plane graph $\mathfrak{L}_n$ is less than or equal to 3. Now, so as to finish the evidence for this case, we show that $\beta(\mathfrak{L}_n) \geq 3$ by working out that there does not exist a resolving set $\mathfrak{R}$ such that $|\mathfrak{R}| = 2$. On contrary, suppose that $\beta(\mathfrak{L}_n) = 2$. Now, from Theorem 1.1, we find that the degree of basis vertices can be at most 3. But except the vertices p_l, q_l, s_l, a_l, and b_l $(1 \leq l \leq n)$, all other vertices of the radially symmetrical plane graph $\mathfrak{L}_n$ have a degree 5. Then, without loss of generality, we suppose that the first resolving vertex is one of the vertices p_1, q_1, s_1, a_1 or b_l, and other nodes lie in the inner cycle, the set of middle nodes, and in the outer cycle. Therefore, we have the following possibilities to be discussed.

Resolving sets	Contradictions
$\{p_1, p_h\}$, p_h $(2 \le h \le n)$	For $2 \le h \le w+1$, we have $\gamma(a_1\|\{p_1, p_h\}) = \gamma(s_1\|\{p_1, p_h\})$, a contradiction.
$\{q_1, q_h\}$, q_h $(2 \le h \le n)$	For $2 \le h \le w$, we have $\gamma(a_1\|\{q_1, q_h\}) = \gamma(s_1\|\{q_1, q_h\})$; and for $h = w+1$, we have $\gamma(p_2\|\{q_1, q_{w+1}\}) = \gamma(p_1\|\{q_1, q_{w+1}\})$, a contradiction.
$\{s_1, s_h\}$, s_h $(2 \le h \le n)$	For $2 \le h \le w+1$, we have $\gamma(q_1\|\{s_1, s_h\}) = \gamma(b_1\|\{s_1, s_h\})$, a contradiction.
$\{a_1, a_h\}$, a_h $(2 \le h \le n)$	For $2 \le h \le w$, we have $\gamma(a_n\|\{a_1, a_h\}) = \gamma(r_n\|\{a_1, a_h\})$; and for $h = w+1$, we have $\gamma(b_1\|\{a_1, a_{w+1}\}) = \gamma(b_n\|\{a_1, a_{w+1}\})$, a contradiction.
$\{b_1, b_h\}$, b_h $(2 \le h \le n)$	For $2 \le h \le w-1$, we have $\gamma(a_n\|\{b_1, b_h\}) = \gamma(r_n\|\{b_1, b_h\})$; when $h = w$, we have $\gamma(s_2\|\{b_1, b_w\}) = \gamma(p_2\|\{b_1, b_w\})$; and for $h = w+1$, we have $\gamma(a_1\|\{b_1, b_{w+1}\}) = \gamma(a_2\|\{b_1, b_{w+1}\})$, a contradiction.
$\{p_1, q_h\}$, q_h $(1 \le h \le n)$	For $1 \le h \le w$, we have $\gamma(a_1\|\{p_1, q_h\}) = \gamma(s_1\|\{p_1, q_h\})$; and for $h = w+1$, we have $\gamma(b_n\|\{p_1, q_{w+1}\}) = \gamma(s_n\|\{p_1, q_{w+1}\})$, a contradiction.
$\{p_1, s_h\}$, s_h $(1 \le h \le n)$	For $1 \le h \le w-1$, we have $\gamma(p_n\|\{p_1, s_h\}) = \gamma(r_n\|\{p_1, s_h\})$; and for $h = w, w+1$, we have $\gamma(r_2\|\{p_1, s_h\}) = \gamma(a_2\|\{p_1, s_h\})$, a contradiction.
$\{p_1, a_h\}$, a_h $(1 \le h \le n)$	For $h = 1$, we have $\gamma(p_n\|\{p_1, a_1\}) = \gamma(p_2\|\{p_1, a_1\})$; when $h = 2$, we have $\gamma(p_2\|\{p_1, a_2\}) = \gamma(a_1\|\{p_1, a_2\})$; and for $3 \le h \le w+1$, we have $\gamma(r_1\|\{p_1, a_h\}) = \gamma(q_1\|\{p_1, a_h\})$, a contradiction.
$\{p_1, b_h\}$, b_h $(1 \le h \le n)$	For $h = 1$, we have $\gamma(r_n\|\{p_1, b_1\}) = \gamma(p_2\|\{p_1, b_1\})$; and for $2 \le h \le w+1$, we have $\gamma(r_1\|\{p_1, b_h\}) = \gamma(q_1\|\{p_1, b_h\})$, a contradiction.
$\{q_1, s_h\}$, s_h $(1 \le h \le n)$	For $1 \le h \le w-1$, we have $\gamma(p_n\|\{q_1, s_h\}) = \gamma(r_n\|\{q_1, s_h\})$; when $h = w$, we have $\gamma(p_1\|\{q_1, s_w\}) = \gamma(r_1\|\{q_1, s_w\})$; and for $h = w+1$, we have $\gamma(b_n\|\{q_1, s_{w+1}\}) = \gamma(r_n\|\{q_1, s_{w+1}\})$, a contradiction.
$\{q_1, a_h\}$, a_h $(1 \le h \le n)$	For $h = 1$, we have $\gamma(q_n\|\{q_1, a_1\}) = \gamma(r_2\|\{q_1, a_1\})$; when $h = 2$, we have $\gamma(b_1\|\{q_1, a_2\}) = \gamma(r_2\|\{q_1, a_2\})$; when $h = 3$, we have $\gamma(b_1\|\{q_1, a_3\}) = \gamma(q_2\|\{q_1, a_3\})$; and for $4 \le h \le w+1$, we have $\gamma(q_2\|\{q_1, a_h\}) = \gamma(r_2\|\{q_1, a_h\})$, a contradiction.
$\{q_1, b_h\}$, b_h $(1 \le h \le n)$	For $h = 1$, we have $\gamma(s_1\|\{q_1, b_1\}) = \gamma(r_2\|\{q_1, b_1\})$; when $h = 2$, we have $\gamma(b_1\|\{q_1, b_2\}) = \gamma(q_2\|\{q_1, b_2\})$; and for $3 \le h \le w+1$, we have $\gamma(q_2\|\{q_1, b_h\}) = \gamma(r_2\|\{q_1, b_h\})$, a contradiction.
$\{s_1, a_h\}$, a_h $(1 \le h \le n)$	For $h = 1$, we have $\gamma(p_1\|\{s_1, a_1\}) = \gamma(q_1\|\{s_1, a_1\})$; when $h = 2$, we have $\gamma(b_2\|\{s_1, a_2\}) = \gamma(r_2\|\{s_1, a_2\})$; and for $3 \le h \le w+1$, we have $\gamma(q_1\|\{s_1, a_h\}) = \gamma(a_1\|\{s_1, a_h\})$, a contradiction.
$\{s_1, b_h\}$, b_h $(1 \le h \le n)$	For $h = 1$, we have $\gamma(p_1\|\{s_1, b_1\}) = \gamma(q_1\|\{s_1, b_1\})$; and for $2 \le h \le w+1$, we have $\gamma(a_1\|\{s_1, b_h\}) = \gamma(q_1\|\{s_1, b_h\})$, a contradiction.
$\{a_1, b_h\}$, b_h $(1 \le h \le n)$	For $1 \le h \le w-1$, we have $\gamma(r_n\|\{a_1, b_h\}) = \gamma(a_n\|\{a_1, b_h\})$; when $h = w$, we have $\gamma(q_n\|\{a_1, b_w\}) = \gamma(s_n\|\{a_1, b_w\})$; and for $h = w+1$, we have $\gamma(r_2\|\{a_1, b_{w+1}\}) = \gamma(q_n\|\{a_1, b_{w+1}\})$, a contradiction.

Thus, from the above table, we obtain that there does not exist a resolving set consisting of two vertices for $\mathbb{V}(\mathfrak{L}_n)$, suggesting that $\beta(\mathfrak{L}_n) = 3$ in this case.

Case(2) When the integer n is odd.

In this case, the positive integer n can be written as $n = 2w+1$, where $w \in \mathbb{N}$ and $w \ge 3$. Let $\mathfrak{R} = \{p_1, q_{w+1}, s_1\} \subset \mathbb{V}(\mathfrak{L}_n)$. Now, to unveil that $\mathfrak{R}$ is the resolving set for the graph $\mathfrak{L}_n$, we consign the metric codes for each vertex of the graph $\mathfrak{L}_n \setminus \mathfrak{R}$ regarding the set $\mathfrak{R}$.

Now, the metric codes for the vertices of inner cycle $\{p_l : 1 \leqslant l \leqslant n\} \cup \{q_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(p_l|\mathfrak{R}) = \begin{cases} (2l-2, 2w-2l+3, 2l-1), & 2 \le l \le w; \\ (2l-2, 1, 2l-1), & l = w+1; \\ (4w-2l+4, 2l-2w-3, 4w-2l+6), & w+2 \le l \le 2w+1. \end{cases}$$

and

$$\gamma(q_l|\mathfrak{R}) = \begin{cases} (2l-1, 2w-2l+2, 2l), & 1 \le l \le w; \\ (4w-2l+3, 2l-2w-2, 4w-2l+5), & w+2 \le l \le 2w+1. \end{cases}$$

The metric codes for the set of middle vertices $\{r_l : 1 \leqslant l \leqslant n\} \cup \{s_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(r_l|\mathfrak{R}) = \begin{cases} (1, 2w+1, 1), & l = 1; \\ (2l-1, 2w-2l+3, 2l), & 2 \le l \le w; \\ (2w+1, 1, 2w+2), & l = w+1; \\ (4w-2l+4, 2l-2w-2, 4w-2l+6), & w+2 \le l \le 2w+1. \end{cases}$$

and

$$\gamma(s_l|\mathfrak{R}) = \begin{cases} (2l, 2w-2l+4, 2l+1), & 2 \le l \le w+1; \\ (4w-2l+5, 2l-2w-1, 4w-2l+7), & w+2 \le l \le 2w+1. \end{cases}$$

At last, the metric codes for the vertices of outer cycle $\{a_l : 1 \leqslant l \leqslant n\} \cup \{b_l : 1 \leqslant l \leqslant n\}$ are

$$\gamma(a_l|\mathfrak{R}) = \begin{cases} (2, 2w+2, 2), & l = 1; \\ (2l-1, 2w-2l+4, 2l-1), & 2 \le l \le w+1; \\ (4w-2l+5, 3, 4w-2l+6), & l = w+2; \\ (4w-2l+5, 2l-2w-2, 4w-2l+6), & w+3 \le l \le 2w+1. \end{cases}$$

and

$$\gamma(b_l|\mathfrak{R}) = \begin{cases} (2, 2w+1, 2), & l = 1; \\ (2l, 2w-2l+3, 2l), & 2 \le l \le w; \\ (2w+2, 2, 2w+2), & l = w+1; \\ (4w-2l+4, 2l-2w-1, 4w-2l+5), & w+2 \le l \le 2w; \\ (3, 2l-2w-1, 4w-2l+5), & l = 2w+1. \end{cases}$$

Again, we find that there do not exist two vertices with the same metric codes, which suggest that $\beta(\mathfrak{L}_n) \le 3$ i.e., the location number of the plane graph $\mathfrak{L}_n$ is less than or equal to 3. Now, on assuming that $\beta(\mathfrak{L}_n) = 2$, we get the same eventualities as in Case(1), and similarly, the contradiction can be obtained. So, in this case, we have $\beta(\mathfrak{L}_n) = 3$ as well and hence the theorem. □

Now, in terms of independent resolving set, we have the following result:

Theorem 4.2. *Let $\mathfrak{L}_n$ be the planar graph on $7n$ vertices as defined above. Then for every positive integer n; $n \ge 6$, its independent resolving number is 3.*

Proof. For proof, refer to Theorem 4.1. □

5. Conclusion

In this article, we determined the metric dimension of three classes of plane graphs (viz., $\mathfrak{J}_n$, $\mathfrak{K}_n$, and $\mathfrak{L}_n$), which are generated by taking n-copies of the complete bipartite graph (or a star) $K_{1,5}$, and are radially symmetric with the constant metric dimension. We have proved that the metric dimension of these three classes of plane graphs is finite and is independent of the number of vertices in these graphs and three vertices is a minimal requirement for the unique identification of all vertices of these three classes of plane graphs. We also observed that the basis set $\mathfrak{R}$ is independent for all of these three families of plane graphs. We now have an open problem that naturally arises from the text.

Open Problem: *Characterize those classes of radially symmetrical graphs* $\mathfrak{M}_n$ *which are generated by taking* n*-copies of the complete bipartite graph (or a star)* $K_{1,5}$ *with constant or non-constant metric dimension.*

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