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Subsocles and direct sum of uniserial modules

Research Article

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Abstract: Suppose M is a $QTAG$-module with a subsocle S such that M/S is a direct sum of uniserial modules. Our global aim here is to investigate an interesting connection between the structure of M/S and the $QTAG$-module M. Specifically, the condition $S = Soc(N)$ for some h-pure submodules N of M allows M to inherit the structure of M/S.

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1. Introduction and background material

Modules are the natural generalizations of abelian groups. The results which hold good for abelian groups need not be true for modules. By putting some restrictions on rings/modules these results hold good for modules too. In 1976 Singh [16] started the study of TAG-modules satisfying the following two conditions while the rings were associative with unity.

(I) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

In 1987 Singh made an improvement and studied the modules satisfying only the condition (I) and called them $QTAG$-modules. Other authors also have considered the problem of detecting finite direct sums of uniserial modules [2, Theorem 4]. The study of $QTAG$-modules and their structure began with work of Singh in [17]. The structure theory of such modules has been developed by various authors. Different notions and structures of $QTAG$-modules have been studied, and a theory was developed,

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introducing several notions, interesting properties, and different characterizations of submodules. Many interesting results have been surfaced, but there is still a lot to explore.

All rings below are assumed to be associative and with nonzero identity element; all modules are assumed to be unital $QTAG$-modules. A uniserial module M is a module over a ring R, whose submodules are totally ordered by inclusion. This means simply that for any two submodules N_1 and N_2 of M, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. A module M is called a serial module if it is a direct sum of uniserial modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\left\{ d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. Let us denote by M^1, the submodule of M, containing elements of infinite height. The module M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. The module M is said to be bounded [16], if there exists an integer k such that $H_M(x) \leq k$ for every uniform element $x \in M$. Moreover, a module M is called Σ-uniserial [5], if it is isomorphic to a direct sum of uniserial modules.

The sum of all simple submodules of M is called the socle of M, denoted by $Soc(M)$ and a submodule S of $Soc(M)$ is called a subsocle of M. A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. For an ordinal σ, a submodule N of M is said to be σ-pure [14], if $H_\beta(M) \cap N = H_\beta(N)$ for all $\beta \leq \sigma$. A submodule $B \subseteq M$ is a basic submodule [13] of M, if B is h-pure in M, $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible. A submodule $N \subset M$ is nice [11] in M, if $H_\sigma(M/N) = (H_\sigma(M) + N)/N$ for all ordinals σ, i.e. every coset of M modulo N may be represented by an element of the same height.

Imitating [12], the submodules $H_k(M), k \geq 0$ form a neighborhood system of zero, thus a topology known as h-topology arises. Closed modules are also closed with respect to this topology. Thus, the closure of $N \subseteq M$ is defined as $\overline{N} = \bigcap_{k=0}^{\infty} (N + H_k(M))$. Therefore the submodule $N \subseteq M$ is closed with respect to h-topology if $\overline{N} = N$.

For $QTAG$-modules M and M', a homomorphism $f : M \to M'$ is said to be small if $ker f$ contains a large submodule of M. The set of all small homomorphisms from M to M', denoted by $Hom_s(M, M')$ is a submodule of $Hom(M, M')$.

Mehran et al. [14] proved that the results which hold for TAG-modules are also valid for $QTAG$-modules. Many results of this paper are the generalization of [7]. In what follows, all notations and notions are standard and will be in agreement with those used in [3, 4].

2. Main concepts and results

The problem with which the present article is concerned is to find those classes of $QTAG$-modules, which possess the following property: M/S is a direct sum of uniserial modules such that $S = Soc(N)$ for some h-pure submodules N of M, then M is a direct sum of uniserial modules and N is a direct summand of M.

Singh [16] proved that a $QTAG$-module M is a direct sum of uniserial modules if and only if M is the union of an ascending chain of bounded submodules. This indicates that M is a direct sum of uniserial modules if and only if $Soc(M) = \bigoplus_{k\in\omega} S_k$ and $H_M(x) = k$ for every $x \in S_k$.

In [9] it was seen that any h-pure submodule N of a $QTAG$-module M with a submodule K of N containing $Soc(N)$, and for some elementary summand T of $Soc(K)$ in $Soc(M)$, $(T \oplus K)/K = Soc(L/K)$ where L/K is an h-pure submodule of M/K which is a direct sum of uniserial modules. Then N is a summand of of M and M/N is a direct sum of uniserial modules. We also indicate the well-known generalizations to the last fact that if N is an h-pure submodule of a $QTAG$-module M such that M/K, where K is a submodule of N generated by uniform elements of exponent at most k for some positive integer k, is a direct sum of uniserial modules then M is a direct sum of uniserial modules.

As a culmination of a series of such claims, the following excellent statement-improvement of the preceding one, concerning the subsocles and a direct sum of uniserial modules, namely:

Corollary 2.1. [9, Corollary 3]. *Let S be a subsocle of a $QTAG$-module M such that $S = Soc(N)$ for some h-pure submodules N of M. If M/S is a direct sum of uniserial modules then so is M.*

From the above discussion, it is naturally to pose the following actual in this topic

Problem. Given S is a subsocle of the $QTAG$-module M such that $S = Soc(N)$ for some h-pure submodules N of M. If M/S belongs to any fixed sort of $QTAG$-modules, then does this imply that M belongs to the same module sort, whenever N is a direct summand of M?

To develop the study, we need to find certain kinds of $QTAG$-modules, and we start with the following subsection.

2.1. Elementary completeness in $QTAG$-modules

The purpose of this subsection is to explore some structural consequences of completeness in $QTAG$-modules. First, we recall the following notions from [1, 6, 8], respectively.

A $QTAG$-module M is said to be quasi-complete, if the closure $\overline{N}$ of every h-pure submodule N of M, is h-pure in M and a $QTAG$-module M is called semi-complete, if it is the direct sum of a closed module and a direct sum of uniserial modules. Moreover, a $QTAG$-module M is called h-pure-complete, if for every subsocle S of M there exists an h-pure submodule N of M such that $S = Soc(N)$.

Now we are ready to deal with the following theorem.

Theorem 2.2. *Let S be a subsocle of the $QTAG$-module M such that $S = Soc(N)$ for some h-pure submodules N of M. If M/S is h-pure-complete, then M is h-pure-complete and N is a summand of M.*

Proof. By hypothesis, $Soc(M)/S$ supports an h-pure submodule K/S for some h-pure submodules K of M. By [1, Proposition 1, Proposition 2], $M/S = N/S \oplus K/S$. Note that if $M = L \oplus T$ and M is h-pure-complete, then $M/Soc(T) \simeq L \oplus H_1(T)$ is h-pure-complete. Consequently, $(M/S)/Soc(K/S)$ is h-pure-complete. But $(M/S)/Soc(K/S) = (M/S)/(Soc(M)/S) \simeq M/Soc(M) \simeq H_1(M)$. Now M is h-pure-complete if and only if $H_k(M)$ is h-pure-complete for some integer k. Consequently, M is h-pure-complete. □

We continue with other statement, namely

Corollary 2.3. *Let S be a subsocle of the $QTAG$-module M such that $S = Soc(N)$ for some h-pure submodules N of M. If M/S is h-pure-complete such that M/S has an unbounded direct sum of uniserial modules summand, then M has an unbounded direct sum of uniserial modules summand and N is a summand of M.*

Proof. Note that if $M = N \oplus K$ for some h-pure submodules K of M, and M has an unbounded direct sum of uniserial modules summand, then either N or K has such a summand. If $H_1(N)$ has an unbounded direct sum of uniserial modules summand, then N has such a summand. □

Now, we proceed by proving

Corollary 2.4. *Let S be a subsocle of the $QTAG$-module M such that $S = Soc(N)$ for some h-pure submodules N of M. If M/S is quasi-complete, then M is quasi-complete and N is a summand of M.*

Proof. Clearly, quasi-complete modules are h-pure-complete and summands of quasi-complete modules are quasi-complete. Also, $H_1(M)$ quasi-complete implies that M is quasi-complete, and we are done. □

Nevertheless, in certain specific cases, the following direct summand property holds.

Corollary 2.5. *Let S be a subsocle of the $QTAG$-module M such that $S = Soc(N)$ for some h-pure submodules N of M. If M/S is a direct sum of closed modules, then M is a direct sum of closed modules and N is a summand of M which is a direct sum of closed modules.*

Proof. If M is a direct sum of closed modules and $Soc(M) = S \oplus K$, where $H_M(x + y) = \min\{H_S(x), H_K(y)\}$ for all $x \in S$ and $y \in K$, then S and K support summands of M which are direct sums of closed modules. Then $Soc(M)/S$ supports a summand L/S in M/S which is a direct sum of closed modules.

Observe that a summand of a direct sum of closed modules is a direct sum of closed modules. Note that if $H_1(N)$ is a direct sum of closed modules, then N is such a direct sum. Consequently, we get that M is a direct sum of closed modules and N is a summand of M. The proof is over. □

As immediate consequence, we yield the following.

Corollary 2.6. *Let S be a subsocle of the $QTAG$-module M such that $S = Soc(N)$ for some h-pure submodules N of M. If M/S is semi-complete, then M is semi-complete and N is a summand of M.*

Analysis. The condition $S = Soc(N)$ for some h-pure submodules N of M is essential. It has been seen that M/S is a direct sum of uniserial modules, but M is not such a direct sum of uniserial modules. It is also easy to see that $Soc(M)/S$ is not a subsocle of M/S with $S = Soc(N)$. For some submodules L of M, consider the h-pure resolution $L \rightarrowtail M \twoheadrightarrow N$, where N is a module which is not a direct sum of uniserial modules and M is a direct sum of uniserial modules. Let $S = Soc(L)$. If $Soc(M)/S$ is a subsocle of M/S with $S = Soc(N)$, then L is a summand of M. But this contradicts the fact that N is not a direct of uniserial modules.

2.2. Large submodules and HT-modules

The study of large submodules and its numerous characterizations makes the theory of $QTAG$-modules more enlightening. A fully invariant submodule $L \subseteq M$ is large [13], if $L + B = M$, for every basic submodule B in M. It is well-known that any large submodule L of M, $L^1 = M^1$. Likewise, it is evident that L/M^1 is large in M/M^1 if and only if L is large in M. We also discuss a significant class of $QTAG$-modules which properly contains the closed module. The members of such a class are termed as HT-modules. A $QTAG$-module M is said to be a HT-module (see [5]) if for any Σ-uniserial module M', a homomorphism $f : M \to M'$ is small. In fact, a $QTAG$-module M is a HT-module if there exists $k \in Z^+$ with $N \supset Soc(H_k(M))$ whenever M/N is a direct sum of uniserial modules. Nontrivial examples of HT-modules are the quasi-complete modules and, in particular, the closed modules.

We come now to a significant characterization of large submodule.

Lemma 2.7. *Let S be a subsocle of the $QTAG$-module M. If L is a large submodule of M, then $(L+S)/S$ contains a large submodule of M/S.*

Proof. As we have noted earlier, a submodule N of M contains a large submodule L of M if and only if for each integer k there is an integer t_k such that $Soc^k(H_{t_k}(M)) \subseteq N$. Let k and t_k be the appropriate integers for L in M. For $(L + S)/S$ in M/S, let $n_k = t_{k+1}$ for each integer k. It is easy to see that $Soc^k(H_{n_k}(M/S)) \subseteq (L + S)/S$. Consequently, $(L + S)/S$ contains a large submodule of M/S. □

Next, we concentrate on the following theorem.

Theorem 2.8. *Let S be a subsocle of the $QTAG$-module M. Then M is a HT-module if and only if M/S is a HT-module.*

Proof. Suppose K is a submodule of M, and let $f : M/S \to M'$ be a map such that $\ker(f) = K/S$. Thus, the composition map

$$M \xrightarrow{g} M/S \xrightarrow{f} M'.$$

Since M is a HT-module, it follows that $K \supseteq L$, where L is large in M. The submodule K/S contains $(L+S)/S$ which contains a large submodule of M/S. Consequently, M/S is a HT-module. The converse follows from Lemma 2.7 and the following relation

$$Soc(M)/S \rightarrowtail M/S \twoheadrightarrow M/Soc(M) \simeq H_1(M).$$

□

2.3. Totally projective modules

This brief subsection is devoted to the exploration of so-called totally projective modules [10].

An h-reduced $QTAG$-module M is said to be totally projective if it possesses a collection $\mathcal{N}$ consisting of nice submodules of M which satisfies the following three conditions:

(i) $0 \in \mathcal{N}$;

(ii) if $\{N_i\}_{i\in I}$ is any subset of $\mathcal{N}$, then $\sum_{i\in I} N_i \in \mathcal{N}$;

(iii) given any $N \in \mathcal{N}$ and any countable subset X of M, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated.

Totally projective modules have a significance in the theory of $QTAG$-modules, which may be characterize in several ways. We require only for information a few more comprehensive characterizations (see [15]): Let σ be a limit ordinal such that $\sigma = \omega + \beta$. A $QTAG$-module M is called σ-projective, if there exists a submodule $N \subset H^\beta(M)$ such that M/N is a direct sum of uniserial modules. A $QTAG$-module M is totally projective, if and only if $M/H_\sigma(M)$ is σ-projective for every ordinal σ.

And so, we prepare to prove the following.

Lemma 2.9. *Let S be a subsocle of the $QTAG$-module M such that $S = Soc(N)$ for some $(\sigma+1)$-pure submodules N of M, where σ is the length of M/S. If M/S is totally projective, then M is totally projective and N is a summand of M.*

Proof. Since N/S is $(\sigma+1)$-pure in M/S which is σ-projective and so N/S is a summand of M/S. Consequently, M/N is totally projective and since $N/S \simeq H_1(N)$, N is totally projective. Consider the exact sequence $N \rightarrowtail M \twoheadrightarrow M/N$. Now N is σ-pure in M and M/N is σ-projective. Thus, the preceding exact sequence splits and N is a summand of M and M is totally projective. □

As a direct consequence of the preceding lemma we have the following corollary.

Corollary 2.10. *Let S be a subsocle of the $QTAG$-module M such that $S = Soc(N)$ for some $(\sigma+1)$-pure submodules N of M, where σ is the length of M/S. If M/S is a direct sum of countably generated h-reduced modules, then M is a direct sum of countably generated h-reduced modules and N is a summand of M.*

3. Open problems

In closing, we pose the following left-open questions:

Problem 3.1. *Let $\mathfrak{F}$ be a concrete class of QTAG-modules. Does it follow that $M \in \mathfrak{F} \Leftrightarrow S \in \mathfrak{F}$, whenever S is a subsocle of the QTAG-module M and M/S is a direct sum of uniserial modules.*

Problem 3.2. *Does the class $\mathfrak{F}$ defined above indeed contains all h-reduced QTAG-modules?*

Problem 3.3. *Is it true that if one large submodule of a QTAG-module is a direct sum of uniserial module, then all large submodules are also direct sum of uniserial modules?*

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