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Research Article

Existence and global exponential stability of periodic solution for Cohen-Grossberg neural networks model with piecewise constant argument

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Abstract

In this paper, we introduce a Cohen-Grossberg neural networks model with piecewise alternately advanced and retarded argument. Some sufficient conditions are established for the existence and global exponential stability of periodic solutions. The approaches are based on employing Brouwer's fixed-point theorem and an integral inequality of Gronwall type with deviating argument. The criteria given are easily verifiable, possess many adjustable parameters, and depend on piecewise constant argument deviations, which provide flexibility for the design and analysis of Cohen-Grossberg neural networks model. Several numerical examples and simulations are also given to show the feasibility and effectiveness of our results.

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1. Introduction

Neural networks with their various generalizations can be represented as differential equations that describe the evolution of the model as functions of time. During the past few decades, much attention has been dedicated to the studies of neural networks partially due to the fact that neural networks can be applied to pattern recognition, signal processing, system control, associative memory, parallel computing and solving optimization problems.

Among the various models of neural networks which have been studied and applied, in 1983, M. Cohen and S. Grossberg [21] proposed a new type of an artificial feedback neural network which is called the Cohen-Grossberg neural networks model. It can be described as follows:

$$\frac{dx_i(t)}{dt} = -\alpha_i(x_i(t))\left\{\beta_i(x_i(t)) - \sum_{j=1}^{m} a_{ij} f_j(x_j(t))\right\}, \qquad i = 1, ..., m,$$

where m is the number of neurons in the network, $x_i(t)$ denotes the state variable of the i-th neuron at time t, $f_j(x_j(t))$ denotes the activation function of the j-th neuron at time

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t; the feedback matrix $C = (a_{ij})_{m\times m}$ indicates the strength of the neuron interconnections within the network; $\alpha_i(\cdot)$ represents an amplification function and $\beta_i(\cdot)$ is the rate with which the unit self-regulates or resets its potential when isolated from other units and inputs.

As we know, the Cohen-Grossberg neural networks model can be used to describe the general neural networks model, cellular neural networks model, Hopfield neural networks model and bi-directional associative memory neural networks model. There are many interesting phenomena in the dynamical behaviors of the Cohen-Grossberg neural networks model. For this reason, stability analysis and existence of periodic solutions have been widely researched for the nonautonomous Cohen-Grossberg neural networks model with and without delays in the literature. See, for instance, Refs. [22, 24–26, 30, 33, 34, 40] and the references cited therein.

Some recent results on the dynamics of the Cohen-Grossberg neural networks model with impulses, have been obtained [4, 28, 31, 32, 35, 36]. Such results require suitable assumptions making the jumps sufficiently small. As a consequence, the impulsive effects on the convergence dynamics of the Cohen-Grossberg neural networks model with impulses becomes less significant.

Recently, a new type of neural networks display a combination of characteristics of both the continuous-time and discrete-time systems, which is an appropriate description of the phenomena of abrupt qualitative dynamical changes of essentially continuous-time systems; see [2, 3, 7, 13–15, 18, 20] for more details. These kinds of equations which involving piecewise constant arguments (in short DEPCAs) usually describe hybrid dynamical systems (a combination of continuous and discrete) and so combine properties of both differential and difference equations.

DEPCAs are first considered by Shah and Wiener [39] in the 80's and have been developed by many authors; see [1, 3, 8, 9, 11, 12, 16, 37, 38]. Applications of DEPCA are discussed in [2, 6, 23, 29]. Over the years, great attention has been paid to the study of the existence of periodic solutions of this type of equations. For specific references (see [2, 5, 10, 15, 17, 19]).

Motivated by the above discussion, in this paper, our main aim is to establish some sufficient conditions for the existence and global exponential stability of the periodic solution of the following DEPCA system:

$$\begin{aligned} x_i'(t) = -d_i(x_i(t))\Bigg\{ a_i(x_i(t)) &- \sum_{j=1}^{n} b_{ij}(t) f_j(x_j(t)) \\ &- \sum_{j=1}^{n} c_{ij}(t) g_j\left(x_j\left(m\left[\left[\frac{t+l}{m}\right]\right]\right)\right) - J_i(t)\Bigg\}, \end{aligned} \tag{1.1}$$

with $1 \le i \le n$, where:

- $[\![\cdot]\!]$ signifies the greatest integer function, l and m are positive real numbers such that $0 < l < m$.
- The function $d_i(\cdot)$ represents an amplification function.
- $a_i(\cdot)$ is the rate with which the unit self-regulates or resets its potential when isolated from other units and inputs.
- The measure of activation of continuous type (resp. piecewise constant type) of the j-th neuron to its incoming potentials is given at any time by the function $f_j(x_j(\cdot))$ $\left(\text{resp. } g_j\left(x_j\left(m\left[\left[\frac{\cdot+l}{m}\right]\right]\right)\right)\right)$.
- The function $b_{ij}(\cdot)$ (resp. $c_{ij}(\cdot)$) denotes the strengths of connection weight of continuous type (resp. piecewise type) of the unit i on the unit j.

- For each neuron, there is an activation flow from outside the system. It is represented by the function $J_i(\cdot)$ for the i-th one.

Let us clarify why the system (1.1) is of alternately advanced and retarded type, that is, the argument can change its deviation character during the motion. The argument is deviated if it is advanced or retarded. Fix $k \in \mathbb{N}$, and consider the system (1.1) on the interval $I_k = [mk - l, mk + m - l)$. Then, the identification function $m \left[\left[\frac{t+l}{m}\right]\right]$ is equal to mk. If $t \in I_k^+ = [mk - l, mk)$, then $m \left[\left[\frac{t+l}{m}\right]\right] \geq t$ and the system (1.1) is an equation with advanced argument. Similarly, if $t \in I_k^- = (mk, mk + m - l)$ then $m \left[\left[\frac{t+l}{m}\right]\right] < t$ and the system (1.1) is an equation with retarded argument. Consequently, the system (1.1) changes the type of deviation of the argument during the process. In other words, the system (1.1) is of alternately advanced and retarded type.

To the best of our knowledge, this paper is the first to study an ω-periodic solution for the Cohen-Grossberg neural networks model with piecewise alternately advanced and retarded argument. The approaches are based on employing Brouwer's fixed-point theorem and a DEPCA's Gronwall-type inequality.

This paper is organized as follows. In Section 2, we focus on some preliminary results which will be used in the existence and stability of an ω-periodic solution of the system (1.1). In Section 3, we derive some sufficient conditions for the global exponential stability of an ω-periodic solution of the system (1.1). In Section 4, two examples and the numerical simulations are given to demonstrate the validity of our results. The conclusions are drawn in Section 5.

2. Preliminaries

In this section, we show some preliminary concepts and results that will be utilized in the proofs of the existence of an ω-periodic solution of the Cohen-Grossberg neural networks model with DEPCA system (1.1).

For reasons of convenience, certain assumptions and the definition are formulated below, which will be convened when necessary.

($\mathcal{B}$) The *amplification functions* $d_i(u)$, $i = 1, 2, ..., n$, are continuous and bounded, and there exist positive constants $\underline{d}_i$ and $\overline{d}_i$ such that

$$0 < \underline{d}_i \leq d_i(u) \leq \overline{d}_i, \quad \forall u \in \mathbb{R}.$$

($\mathcal{I}$) The functions a_i with $a_i(0) = 0$, $i = 1, 2, ..., n$, are continuous and there exist positive constants $\mathcal{N}_i^a$ and $\mathcal{L}_i^a$ such that

$$0 < \mathcal{N}_i^a \leq \frac{a_i(u) - a_i(v)}{u - v} \leq \mathcal{L}_i^a, \qquad \forall u, v \in \mathbb{R},\ u \neq v.$$

($\mathcal{L}$) The *activation functions* f_j, g_j with $f_j(0) = 0$, $g_j(0) = 0$, $1 \leq i, j \leq n$, satisfy

$$|f_j(u) - f_j(v)| \leq \mathcal{L}_j^f |u - v|, \quad |g_j(u) - g_j(v)| \leq \mathcal{L}_j^g |u - v|,$$

for some positive constants $\mathcal{L}_j^f$, $\mathcal{L}_j^g > 0$ and for all u, $v \in \mathbb{R}$.

Definition 2.1. A function $x : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^n$ is a solution of the system (1.1), if (i) $x(t)$ is continuous on $\mathbb{R}^+$, (ii) the derivative $x'(t)$ exists at each point $t \in \mathbb{R}^+$, with the possible exception of the points $mk - l \in \mathbb{R}^+$, $k \in \mathbb{N}$, where the one-side derivatives exist, (iii) the system (1.1) is satisfied for x on each interval $\big(mk - l, mk + m - l\big)$, $k \in \mathbb{N}$, and it holds for the right derivative at the points $mk - l$, $k \in \mathbb{N}$.

To study nonlinear DEPCA, we use the approach proposed by M. U. Akhmet in [1], based on the construction of an equivalent integral equation. Let us give the following proposition.

Proposition 2.2 (Integral Representation). *Given a pair $(\tau, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$, a function $x = (x_1(\cdot), \cdots, x_n(\cdot)) : \mathbb{R}^+ \to \mathbb{R}^n$ such that $x(\tau) = x_0$ is a solution of the system* (1.1) *in the sense of Definition 2.1, if and only if their coordinates satisfy on $\mathbb{R}^+$ the following set of integral equations*

$$x_i(t) = x_i(\tau) + \int_\tau^t d_i(x_i(s)) \times \Big(- a_i(x_i(s)) + \sum_{j=1}^n b_{ij}(s) f_j(x_j(s)) + \sum_{j=1}^n c_{ij}(s) g_j\left(x_j\left(m\left[\left[\frac{s+l}{m}\right]\right]\right)\right) + J_i(s)\Big)\, ds, \tag{2.1}$$

for $i \in \{1, \cdots, n\}$.

We omit the proof of this assertion, since it can be proved in the almost identical way as Lemma 4.3 in [2] and Proposition 1 in [20].

In the next, we give the following lemma about DEPCA integral inequality of Gronwall type, which is one of the most important auxiliary results of the present paper.

Lemma 2.3 (DEPCA's Gronwall Inequality, [13, Lemma 2.1]). *Let $u : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying*

$$u(t) \le \alpha + \left| \int_\tau^t \left[\eta_1(s) u(s) + \eta_2(s) u\left(m\left[\left[\frac{s+l}{m}\right]\right]\right)\right] ds \right|, \tag{2.2}$$

where $\alpha \ge 0$ and $\eta_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i \in \{1, 2\}$, is a piecewise continuous function. Then:

- *If $t \ge \tau$,*

$$u(t) \le \alpha \exp\left(\int_\tau^t \left[\eta_1(s) + \frac{\eta_2(s)}{1-\kappa^+}\right] ds \right). \tag{2.3}$$

- *If $0 \le t \le \tau$,*

$$u(t) \le \alpha \exp\left(\int_t^\tau \left[\eta_1(s) + \frac{\eta_2(s)}{1-\kappa^-}\right] ds \right), \tag{2.4}$$

where

$$\kappa^+ := \max_{\left[\left[\frac{\tau+l}{m}\right]\right] \le k} \int_{mk-l}^{mk} [\eta_1(s) + \eta_2(s)]\, ds \le \kappa_1 < 1,$$

and

$$\kappa^- := \max_{1 \le k < \left[\left[\frac{\tau+l}{m}\right]\right]} \int_{mk}^{mk+m-l} [\eta_1(s) + \eta_2(s)]\, ds \le \kappa_2 < 1.$$

3. The existence and global exponential stability of the periodic solution

In this section, we give the sufficient conditions for the existence of global exponential stability of a unique ω-periodic solution of the Cohen-Grossberg neural networks model with DEPCA system (1.1).

3.1. Existence of periodic solutions

For $\omega > 0$, let $\mathbb{P}_\omega$ be the set of all n-vector continuous functions $x(t)$, periodic in t of period ω. Then $(\mathbb{P}_\omega, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|x\| = \max_{1\le i\le n} \|x_i\| = \max_{1\le i\le n}\left[\sup_{t\in\mathbb{R}^+} |x_i(t)|\right] = \max_{1\le i\le n}\left[\sup_{t\in[\tau,\tau+\omega]} |x_i(t)|\right].$$

In this part, we study the existence of an ω-periodic solution of the system (1.1). Here we assume the **Periodicity Condition** $(\mathcal{P})$.

$(\mathcal{P})$ There exists $\omega > 0$ such that the functions $d_i(\cdot)$, $a_i(\cdot)$, $b_{ij}(\cdot)$, $c_{ij}(\cdot)$ and $J_i(\cdot)$ are all continuous ω-periodic functions. Moreover, there exists $p \in \mathbb{N}$, for which the sequences $\{mk-l\}_{k\in\mathbb{N}}$ and $\{mk\}_{k\in\mathbb{N}}$, satisfy the (ω, p) condition, that is

$$\omega = mp.$$

Remark 3.1. Note that (ω, p) condition is a discrete relation, which moves the interval $I_k = [mk-l, mk+m-l)$ into $I_{k+p} = [m(k+p)-l, m(k+p)+m-l)$. Then we have the following consequences:

(i) For $t \in I_k$, we have

$$\text{a) } t+\omega \in I_{k+p}, \qquad \text{b) } m\left[\left[\frac{t+l}{m}\right]\right] + \omega \in I_{k+p}.$$

Then,

$$m\left[\left[\frac{(t+\omega)+l}{m}\right]\right] = m\cdot(k+p) = m\cdot k + \omega = m\left[\left[\frac{t+l}{m}\right]\right] + \omega.$$

(ii) For any $\tau \in \mathbb{R}^+$, the interval $[\tau, \tau+\omega]$ can be decomposed as follows:

$$[\tau, m\cdot i(\tau)+m-l] \cup \bigcup_{j=i(\tau)+1}^{i(\tau)+p-1} I_j \cup [m\cdot(i(\tau)+p)-l, \tau+\omega],$$

where $i(\cdot)$ be an indexer defined by $i(t) = k$ if $t \in I_k = [mk-l, mk+m-l)$.

In this section, we use the Brouwer's fixed-point theorem to obtain the existence of an ω-periodic solution for the system (1.1).

Lemma 3.2 (Brouwer's fixed-point theorem (See [27])). *Let $\mathcal{T}$ be a continuous operator that maps a closed bounded convex subset $\Omega \subset \mathbb{R}^n$ into itself. Then Ω contains at least one fixed point of the operator $\mathcal{T}$, i.e. there exists $z^* \in \Omega$, such that $\mathcal{T}(z^*) = z^*$.*

In order to discuss the existence of an ω-periodic solution, we first introduce the following lemma.

Lemma 3.3. *Let the conditions $(\mathcal{B})$, $(\mathcal{L})$, $(\mathcal{I})$ and $(\mathcal{P})$ hold. Suppose that there exist n positive constants $\alpha_1, \alpha_2, \dots, \alpha_n$ such that*

$$\mathcal{N}_i^a \alpha_i - \sum_{j=1}^{n}\left[\alpha_j\left(\mathcal{L}_j^f |b_{ij}(t)| + \mathcal{L}_j^g |c_{ij}(t)|\right)\right] - J_i(t) > 0, \quad t \in [\tau, \tau+\omega], \tag{3.1}$$

for each $i = 1, ..., n$. Then any solution of the system (1.1) with initial condition

$$|x_i(\tau)| \le \alpha_i, \quad i = 1, ..., n$$

verifies

$$|x_i(t)| \le \alpha_i, \quad \textit{for all} \quad t \in [\tau, \tau+\omega], \quad i = 1, ..., n.$$

Proof. Suppose that the initial condition satisfies

$$|x_i(\tau)| \le \alpha_i, \quad i = 1, ..., n$$

and, by Remark 3.1 (ii), we can split $[\tau, \tau+\omega]$ as

$$[\tau,\tau+\omega] = [\tau, mi(\tau)+m-l) \cup [mi(\tau)+m-l, mi(\tau)+2m-l) \cup \ldots \cup [mi(\tau)+mp-l, \tau+\omega].$$

We affirm that, for each $i = 1, ..., n$,

$$|x_i(t)| \le \alpha_i, \quad t \in (\tau, mi(\tau)+m-l).$$

Indeed, if it is not true, then there exist some i and time μ such that

$$|x_i(\mu)| = \alpha_i, \quad \frac{d}{dt}|x_i(t)||_{t=\mu} \ge 0 \quad \text{and} \quad |x_j(\mu)| \le \alpha_j, \quad j \ne i.$$

Therefore, using $(\mathcal{B})$, $(\mathcal{J})$, $(\mathcal{L})$ we have

$$\begin{aligned} \frac{d}{dt}|x_i(t)| \le d_i(x_i(t))\Bigg\{ &-\mathcal{N}_i^a |x_i(t)| + \sum_{j=1}^n \mathcal{L}_j^f |b_{ij}(t)| |x_j(t)| \\ &+ \sum_{j=1}^n \mathcal{L}_j^g |c_{ij}(t)| \left| x_j\left(m\left[\left[\frac{t+l}{m}\right]\right]\right)\right| + J_i(t)\Bigg\}. \end{aligned} \tag{3.2}$$

By (3.2), we get

$$\begin{aligned} 0 \le \Bigg\{ &-\mathcal{N}_i^a |x_i(t)| + \sum_{j=1}^n \mathcal{L}_j^f |b_{ij}(t)| |x_j(t)| \\ &+ \sum_{j=1}^n \mathcal{L}_j^g |c_{ij}(t)| \left| x_j\left(m\left[\left[\frac{t+l}{m}\right]\right]\right)\right| + J_i(t)\Bigg\} \\ \le &-\mathcal{N}_i^a \alpha_i + \sum_{j=1}^n \left[\left(\mathcal{L}_j^f |b_{ij}(t)| + \mathcal{L}_j^g |c_{ij}(t)|\right)\alpha_j\right] + J_i(t). \end{aligned} \tag{3.3}$$

Moreover, using (3.1) for $t = \mu$, we have

$$\mathcal{N}_i^a \alpha_i - \sum_{j=1}^n \left[\alpha_j\left(\mathcal{L}_j^f |b_{ij}(\mu)| + \mathcal{L}_j^g |c_{ij}(\mu)|\right)\right] - J_i(\mu) > 0$$

which contradicts (3.3). By using equality $x(mi(\tau)+m-l^-) = x(mi(\tau)+m-l)$, we can rewrite the proved claim in the form

$$|x_i(t)| \le \alpha_i, \quad t \in [\tau, mi(\tau)+m-l], \quad i = 1, ..., n,$$

the previous argument applied in $[mi(\tau)+m-l, mi(\tau)+2m-l]$ with initial condition $x_i(mi(\tau)+m-l)$, gives $|x_i(\tau)| \le \alpha_i$, $t \in [mi(\tau)+m-l, mi(\tau)+2m-l]$, $i = 1, ..., n$, and so on, till interval $[mi(\tau)+mp-l, \tau+\omega]$. □

Theorem 3.4. *Let the conditions* $(\mathcal{B})$, $(\mathcal{L})$, $(\mathcal{J})$ *and* $(\mathcal{P})$ *hold. Suppose that there exist* $n+1$ *positive constants* $p_1, p_2, \ldots, p_n$, σ *such that*

$$\mathcal{N}_i^a p_i - \sum_{j=1}^n \left[p_j\left(\mathcal{L}_j^f |b_{ij}(t)| + \mathcal{L}_j^g |c_{ij}(t)|\right)\right] > \sigma, \quad t \in [\tau, \tau+\omega], \tag{3.4}$$

for each $i = 1, ..., n$. *Then the system* (1.1) *admits at least one* ω*-periodic solution.*

Proof. Consider the n-dimensional rectangle

$$\Re = [-\alpha_1, \alpha_1] \times [-\alpha_2, \alpha_2] \times \cdots \times [-\alpha_n, \alpha_n],$$

and define the operator $\mathcal{T}$ in $\mathbb{P}_\omega$ by

$$\begin{aligned}&\mathcal{T} : \Re \to \Re\\ &\mathcal{T}(v_1, v_2, ..., v_n) = (x_1(\tau+\omega), x_2(\tau+\omega), ..., x_n(\tau+\omega)),\end{aligned}$$

where $(x_1(t), x_2(t), ..., x_n(t))$ is a solution of the system (1.1) satisfying the initial condition $x_i(\tau) = v_i$, $i = 1, ..., n$.
Note that if we take ϱ great enough to verify

$$\varrho\sigma > \max_{t\in[\tau,\tau+\omega]} |J_i(t)|, \quad i = 1, ..., n,$$

we obtain (3.1) with $\alpha_i = \varrho p_i$. Then (3.4) implies (3.1). Therefore, under our assumptions, Lemma 3.3 holds and we can conclude that $\mathcal{T}$ is well defined.
In fact the inequalities

$$|x_i(\tau)| = |v_i| \leq \alpha_i,$$

guarantee that

$$|x_i(t)| \leq \alpha_i, \quad t \geq \tau.$$

In particular $|x_i(\tau+\omega)| \leq \alpha_i$, that is $\mathcal{T}(v_1, v_2, ..., v_n) \in \Re$. Since $\mathcal{T}$ is continuous, applying the Brouwer's fixed point theorem, there exists

$$(\bar{v}_1, \bar{v}_2, ..., \bar{v}_n) \in \Re, \quad \text{for which} \quad \mathcal{T}(\bar{v}_1, \bar{v}_2, ..., \bar{v}_n) = (\bar{v}_1, \bar{v}_2, ..., \bar{v}_n).$$

The solution $\bar{x}(t)$ with initial condition $\bar{x}_i(\tau) = \bar{v}_i$ is ω-periodic, because

$$\bar{x}(\tau+\omega) = \mathcal{T}(\bar{v}_1, \bar{v}_2, ..., \bar{v}_n) = (\bar{v}_1, \bar{v}_2, ..., \bar{v}_n) = \bar{x}(\tau).$$

The proof is complete. □

Remark 3.5. The previous Theorem 3.4 is a version of B. Lisena's result [34] for the corresponding DEPCA system.

The next results are particular cases of Theorem 3.4.

Corollary 3.6. *Let the conditions* $(\mathcal{B})$, $(\mathcal{L})$, $(\mathcal{I})$ *and* $(\mathcal{P})$ *hold. Suppose that there exist* $n+1$ *positive constants* $p_1, p_2, \ldots, p_n$, σ *such that*

$$\mathcal{N}_i^a p_i - \sum_{j=1}^{n} \left[p_j \left(\mathcal{L}_j^f \left|\bar{b}_{ij}\right| + \mathcal{L}_j^g \left|\bar{c}_{ij}\right| \right) \right] > \sigma, \quad t \in [\tau, \tau+\omega], \quad i = 1, ..., n, \tag{3.5}$$

where $\bar{b}_{ij} = \sup_{t\in\mathbb{R}^+} |b_{ij}(t)|$ *and* $\bar{c}_{ij} = \sup_{t\in\mathbb{R}^+} |c_{ij}(t)|$. *Then the system* (1.1) *has at least one* ω*-periodic solution.*

Corollary 3.7. *For* $a_i(t) \equiv a_i > 0$, $b_{ij}(t) \equiv b_{ij}$, $c_{ij}(t) \equiv c_{ij}$ *and* $J_i(t) \equiv J_i > 0$ *constants, if* $(\mathcal{L})$, $(\mathcal{P})$ *and suppose that there exist* n *positive constants* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *such that*

$$a_i\alpha_i - \sum_{j=1}^{n} \left[\alpha_j \left(\mathcal{L}_j^f |b_{ij}| + \mathcal{L}_j^g |c_{ij}| \right) \right] > J_i, \quad i = 1, ..., n, \tag{3.6}$$

are satisfied. Then the system (1.1) *with constant coefficients admits at least one* ω*-periodic solution.*

Remark 3.8. If the conditions $(\mathcal{L})$, $(\mathcal{P})$ are satisfied and suppose that there exist $n+1$ positive constants $p_1, p_2, \dots, p_n$, σ such that

$$\underline{a}_i p_i - \sum_{j=1}^{n} \left[p_j \left(\mathcal{L}_j^f \left| \bar{b}_{ij} \right| + \mathcal{L}_j^g \left| \bar{c}_{ij} \right| \right) \right] > \sigma, \quad t \in [\tau, \tau+\omega], \tag{3.7}$$

where $\underline{a}_i = \inf_{t\in\mathbb{R}^+} |a_i(t)|$, $\bar{b}_{ij} = \sup_{t\in\mathbb{R}^+} |b_{ij}(t)|$ and $\bar{c}_{ij} = \sup_{t\in\mathbb{R}^+} |c_{ij}(t)|$. Then the Hopfield neural networks model with DEPCA system

$$\begin{aligned} x_i'(t) = -a_i(t)x_i(t) &+ \sum_{j=1}^{n} b_{ij}(t) f_j(x_j(t)) \\ &+ \sum_{j=1}^{n} c_{ij}(t) g_j \left(x_j \left(m \left[\left[\frac{t+l}{m} \right]\right] \right) \right) + J_i(t), \quad i = 1, ..., n, \end{aligned} \tag{3.8}$$

admits at least one ω-periodic solution.

Note that the periodic result for the Hopfield neural networks model with DEPCA system is new and this result cannot be found in any of the previous applied studies.

3.2. Global exponential stability of the periodic solution

In this subsection, we will derive a simple criterion ensuring that the Cohen-Grossberg neural networks model with DEPCA system (1.1) has a unique ω-periodic solution which is globally exponentially stable.

First of all, we give the following definition and **Stability Condition** $(\mathcal{S})$, which will be used in the proof of the stability of the ω-periodic solution for the system (1.1).

Definition 3.9. The solution $x^*(t)$ of the system (1.1) is said to be globally exponentially stable if there exist positive constants α and λ such that the estimation

$$|x(t) - x^*(t)| \le \alpha \left| x(\tau) - x^*(\tau) \right| e^{-\lambda(t-\tau)}, \quad t \ge \tau,$$

is valid for any solution $x(t)$ of the system (1.1).

$(\mathcal{S})$ There exists $\mu \in \mathbb{R}^+$ such that for $t \in \mathbb{R}^+$,

$$\underline{d}\,\underline{\mathcal{N}}^a - \max_{1\le i\le n} \left[\sum_{j=1}^{n} \mathcal{L}_j^f \tilde{b}_{ij}(t) + \frac{e^{\underline{d}\,\underline{\mathcal{N}}^a \cdot \left(t - m\left[\left[\frac{t+l}{m}\right]\right]\right)}}{1-v} \sum_{j=1}^{n} \mathcal{L}_j^g \tilde{c}_{ij}(t) \right] \ge \mu > 0,$$

where $\underline{d} = \min_{1\le i\le n} \{d_i\}$, $\underline{\mathcal{N}}^a = \min_{1\le i\le n} \{\mathcal{N}_i^a\}$, $\tilde{b}_{ij}(t) = \bar{d}_i |b_{ij}(t)|$, $\tilde{c}_{ij}(t) = \bar{d}_i |c_{ij}(t)|$ and

$$v := \max_{1\le i\le n} \left\{ \max_{k\in\mathbb{N}} \int_{mk-l}^{mk} \left[\sum_{j=1}^{n} \mathcal{L}_j^f \tilde{b}_{ij}(s) + e^{\underline{d}\,\underline{\mathcal{N}}^a \cdot (s-mk)} \sum_{j=1}^{n} \mathcal{L}_j^g \tilde{c}_{ij}(s) \right] ds \right\} < \bar{v} < 1. \tag{3.9}$$

For analytical convenience, we shall transform the system (1.1).

Using Hypothesis $(\mathcal{B})$, the antiderivative of $\frac{1}{d_i(x_i(t))}$ exists. Then we choose an antiderivative $F_i(x_i)$ of $\frac{1}{d_i(x_i(t))}$ such that $F_i(0) = 0$.

Evidently, $F_i'(x_i) = \frac{1}{d_i(x_i(t))}$. By $d_i(x_i(t)) > 0$, we are getting $F_i(x_i)$ strictly monotone increasing on x_i.

Using the inverse function theorem, there exists an inverse function $F_i^{-1}(x_i)$ of $F_i(x_i)$ which is continuous and differential.

Furthermore, we have $(F_i^{-1}(x_i))' = d_i(x_i(t))$. Indicating $F_i'(x_i)x_i'(t) = \frac{x_i'(t)}{d_i(x_i(t))} = u_i'(t)$, we get $x_i(t) = F_i^{-1}(u_i(t))$. As a result, we can write the system (1.1) in the following form:

$$
\begin{aligned}
u_i'(t) = -a_i(F_i^{-1}(u_i(t))) &+ \sum_{j=1}^{n} b_{ij}(t) f_j(F_j^{-1}(u_j(t))) \\
&+ \sum_{j=1}^{n} c_{ij}(t) g_j \left(F_j^{-1} \left(u_j \left(m \left[\left[\frac{t+l}{m} \right]\right] \right) \right) \right) + J_i(t).
\end{aligned} \tag{3.10}
$$

Using Hypothesis $(\mathcal{I})$, and the mean value theorem, there exist a constant $\bar{\varpi}_i \in [0,1]$ such that

$$
a_i \left(F_i^{-1}(u_i(t)) \right) = \left[a_i \left(F_i^{-1}(\bar{\varpi}_i u_i(t)) \right) \right]' u_i(t) = \tilde{a}_i(u_i(t)) u_i(t).
$$

Then the system (3.10) can be rewritten as

$$
\begin{aligned}
u_i'(t) = -\tilde{a}_i(u_i(t)) u_i(t) &+ \sum_{j=1}^{n} b_{ij}(t) f_j(F_j^{-1}(u_j(t))) \\
&+ \sum_{j=1}^{n} c_{ij}(t) g_j \left(F_j^{-1} \left(u_j \left(m \left[\left[\frac{t+l}{m} \right]\right] \right) \right) \right) + J_i(t).
\end{aligned} \tag{3.11}
$$

Now, by the Lagrange theorem we have

$$
\left| F_i^{-1}(u) - F_i^{-1}(v) \right| = \left| \left[F_i^{-1}(v + \bar{\varpi}_i(u-v)) \right]' (u-v) \right| = \left| d_i(v + \bar{\varpi}_i(u-v)) \right| |u-v| .
$$

Thanks to Hypothesis $(\mathcal{B})$ again, we have

$$
\underline{d}_i |u-v| \leq \left| F_i^{-1}(u) - F_i^{-1}(v) \right| \leq \bar{d}_i |u-v| .
$$

Combined with Hypothesis $(\mathcal{I})$, we obtain

$$
\underline{d}_i \mathcal{N}_i^a \leq \left[a_i \left(F_i^{-1}(\cdot) \right) \right]' \leq \bar{d}_i \mathcal{L}_i^a . \tag{3.12}
$$

Remark 3.10. It is clear that the system (1.1) has a unique globally exponentially stable ω-periodic solution if and only if the system (3.11) has a uniqueb globally exponentially stable ω-periodic solution. To facilitate our analysis, we only consider the system (3.11).

The following result shows sufficient conditions for the global exponential stability of the unique ω-periodic solution of the system (1.1).

Theorem 3.11. *Suppose that the assumptions of Theorem 3.4 and* $(\mathcal{S})$ *are satisfied. Then system* (1.1) *has a unique* ω*-periodic solution and all other solutions converge exponentially to it as* $t \to \infty$.

Proof. According to Theorem 3.4, we know that the system (1.1) has at least one ω-periodic solution $x^*(t) = (x_1^*(t), ..., x_n^*(t))^T$ with initial value $x^*(\tau) = (x_1^*(\tau), ..., x_n^*(\tau))^T$. Suppose that $x(t) = (x_1(t), ..., x_n(t))^T$ is an arbitrary solution of the system (1.1) with initial value $x(\tau) = (x_1(\tau), ..., x_n(\tau))^T$.

For $i=1,...,n$, let $y_i(t)=x_i(t)-x^*(t)$ and $y_i(\tau)=x_i(\tau)-x_i^*(\tau)$, we obtain

$$\begin{aligned}y_i(t) =& y_i(\tau)e^{-\int_\tau^t \tilde{a}_i(u_i(s))ds}\\ &+\int_\tau^t e^{-\int_s^t \tilde{a}_i(u_i(k))dk}\Bigg\{\sum_{j=1}^n b_{ij}(s)\left[f_j\left\langle F_j^{-1}(y_j(s))+F_j^{-1}\left(x_j^*(s)\right)\right\rangle\right.\\ &-\left.f_j\left(F_j^{-1}\left(x_j^*(s)\right)\right)\right]+\sum_{j=1}^n c_{ij}(s)\left[g_j\left\langle F_j^{-1}\left(y_j\left(m\left[\left[\frac{s+l}{m}\right]\right]\right)\right)\right.\right.\\ &\left.\left.+F_j^{-1}\left(x_j^*\left(m\left[\left[\frac{s+l}{m}\right]\right]\right)\right)\right\rangle-g_j\left(F_j^{-1}\left(x_j^*\left(m\left[\left[\frac{s+l}{m}\right]\right]\right)\right)\right)\right]\Bigg\}ds.\end{aligned}$$

Now, by $(\mathcal{B})$, $(\mathcal{L})$ and (3.12) it can be proved that

$$\begin{aligned}|y_i(t)| \le& |y_i(\tau)|\,e^{-\underline{d}_i\mathcal{N}_i^a\cdot(t-\tau)}+\int_\tau^t e^{-\underline{d}_i\mathcal{N}_i^a\cdot(t-s)}\Bigg\{\sum_{j=1}^n \bar{d}_i\mathcal{L}_j^f\,|b_{ij}(s)|\,|y_i(s)|\\ &+\sum_{j=1}^n \bar{d}_i\mathcal{L}_j^g\,|c_{ij}(s)|\left|y_j\left(m\left[\left[\frac{s+l}{m}\right]\right]\right)\right|\Bigg\}ds\\ \le& |y_i(\tau)|\,e^{-\underline{d}\,\underline{\mathcal{N}}^a\cdot(t-\tau)}+\int_\tau^t e^{-\underline{d}\,\underline{\mathcal{N}}^a\cdot(t-s)}\Bigg\{\sum_{j=1}^n \bar{d}_i\mathcal{L}_j^f\,|b_{ij}(s)|\,|y_i(s)|\\ &+\sum_{j=1}^n \bar{d}_i\mathcal{L}_j^g\,|c_{ij}(s)|\left|y_j\left(m\left[\left[\frac{s+l}{m}\right]\right]\right)\right|\Bigg\}ds.\end{aligned}$$

Let us define

$$\upsilon_i(t)=|y_i(t)|\,e^{\underline{d}\,\underline{\mathcal{N}}^a\cdot(t-\tau)},\quad t\in[\tau,\infty),\quad i=1,...,n. \tag{3.13}$$

Then

$$\begin{aligned}\upsilon_i(t)\le& \upsilon_i(\tau)+\int_\tau^t\Bigg\{\sum_{j=1}^n \bar{d}_i\mathcal{L}_j^f\,|b_{ij}(s)|\,\upsilon_j(s)\\ &+e^{\underline{d}\,\underline{\mathcal{N}}^a\cdot\left(s-m\left[\left[\frac{s+l}{m}\right]\right]\right)}\sum_{j=1}^n \bar{d}_i\mathcal{L}_j^g\,|c_{ij}(s)|\upsilon_j\left(m\left[\left[\frac{s+l}{m}\right]\right]\right)\Bigg\}ds\end{aligned}$$

or

$$\begin{aligned}\upsilon(t)\le& \upsilon(\tau)+\int_\tau^t\max_{1\le i\le n}\Bigg[\bar{d}_i\sum_{j=1}^n\mathcal{L}_j^f\,|b_{ij}(s)|\,\upsilon(s)\\ &+\bar{d}_i e^{\underline{d}\,\underline{\mathcal{N}}^a\cdot\left(s-m\left[\left[\frac{s+l}{m}\right]\right]\right)}\sum_{j=1}^n\mathcal{L}_j^g\,|c_{ij}(s)|\,\upsilon\left(m\left[\left[\frac{s+l}{m}\right]\right]\right)\Bigg\}ds.\end{aligned}$$

Hence, by (3.9) and Lemma 2.3, we arrive at

$$\begin{aligned}\upsilon(t)\le& \upsilon(\tau)\exp\Bigg\{\max_{1\le i\le n}\int_\tau^t\Bigg[\bar{d}_i\sum_{j=1}^n\mathcal{L}_j^f\,|b_{ij}(s)|\\ &+\bar{d}_i\frac{\exp\left(\underline{d}\,\underline{\mathcal{N}}^a\cdot\left(s-m\left[\left[\frac{s+l}{m}\right]\right]\right)\right)}{1-v}\sum_{j=1}^n\mathcal{L}_j^g\,|c_{ij}(s)|\Bigg]ds\Bigg\}.\end{aligned}$$

For (3.13), we have

$$\max_{1\le i\le n}|y_i(t)| \le \max_{1\le i\le n}|y_i(\tau)|\exp\int_\tau^t\Bigg\{-\underline{d}\,\underline{\mathcal{N}}^a+\max_{1\le i\le n}\Bigg[\bar{d}_i\sum_{j=1}^n\mathcal{L}_j^f\,|b_{ij}(s)| + \bar{d}_i\frac{\exp\left(\underline{d}\,\underline{\mathcal{N}}^a\cdot\left(s-m\left[\left[\frac{s+l}{m}\right]\right]\right)\right)}{1-v}\sum_{j=1}^n\mathcal{L}_j^g\,|c_{ij}(s)|\Bigg]ds\Bigg\}.$$

Then, for any solution $x(t)$ of (1.1), we easily get

$$\begin{aligned}&\max_{1\le i\le n}|x_i(t)-x_i^*(t)|\\ &\le \max_{1\le i\le n}|x_i(\tau)-x_i^*(\tau)|\exp\int_\tau^t\Bigg\{-\underline{d}\,\underline{\mathcal{N}}^a+\max_{1\le i\le n}\Bigg[\sum_{j=1}^n\mathcal{L}_j^f\tilde{b}_{ij}(s)\\ &\qquad+\frac{\exp\left(\underline{d}\,\underline{\mathcal{N}}^a\cdot\left(s-m\left[\left[\frac{s+l}{m}\right]\right]\right)\right)}{1-v}\sum_{j=1}^n\mathcal{L}_j^g\tilde{c}_{ij}(s)\Bigg]\Bigg\}ds.\end{aligned}\tag{3.14}$$

The uniqueness of the ω-periodic solution of system (1.1) follows from $(\mathcal{S})$ and (3.14). Moreover, the ω-periodic solution of the system (1.1) is globally exponentially stable and this completes the proof of the theorem. □

As immediate corollaries of Theorem 3.11, the following results are true.

Corollary 3.12. *Suppose that the assumptions of Corollary 3.6 and*

$$\underline{d}\,\underline{\mathcal{N}}^a-\max_{1\le i\le n}\left[\bar{d}_i\sum_{j=1}^n\mathcal{L}_j^f|\bar{b}_{ij}|+\frac{\exp\left(\underline{d}\,\underline{\mathcal{N}}^a\cdot(m-l)\right)}{1-v}\bar{d}_i\sum_{j=1}^n\mathcal{L}_j^g|\bar{c}_{ij}|\right]\ge\mu>0$$

are satisfied, where

$$v:=\max_{1\le i\le n}\left\{l\cdot\bar{d}_i\left[\sum_{j=1}^n\mathcal{L}_j^f\left|\bar{b}_{ij}\right|+\frac{1-\exp\left(-\underline{d}\,\underline{\mathcal{N}}^a l\right)}{\underline{d}\,\underline{\mathcal{N}}^a}\sum_{j=1}^n\mathcal{L}_j^g\left|\bar{c}_{ij}\right|\right]\right\}<\bar{v}<1.$$

Then the ω-periodic solution of the system (1.1) *is globally exponentially stable.*

Corollary 3.13. *Suppose that the assumptions of Corollary 3.7 and*

$$\underline{d}_i\underline{a}_i-\max_{1\le i\le n}\left[\bar{d}_i\sum_{j=1}^n\mathcal{L}_j^f|b_{ij}|+\frac{\exp(\underline{d}_i\underline{a}_i\cdot(m-l))}{1-v}\bar{d}_i\sum_{j=1}^n\mathcal{L}_j^g|c_{ij}|\right]\ge\mu>0$$

are satisfied, where

$$v:=\max_{1\le i\le n}\left\{l\cdot d_i\left[\sum_{j=1}^n\mathcal{L}_j^f\left|b_{ij}\right|+\frac{1-\exp\left(-\underline{d}\,\underline{\mathcal{N}}^a l\right)}{\underline{d}\,\underline{\mathcal{N}}^a}\sum_{j=1}^n\mathcal{L}_j^g\left|c_{ij}\right|\right]\right\}<\bar{v}<1.$$

Then the ω-periodic solution of the system (1.1) *with constant coefficients is globally exponentially stable.*

Remark 3.14. Suppose that the assumptions of Remark 3.8 and

$$\underline{a}_i-\max_{1\le i\le n}\left[\sum_{j=1}^n\mathcal{L}_j^f|\bar{b}_{ij}|+\frac{\exp(\underline{a}_i\cdot(m-l))}{1-v}\sum_{j=1}^n\mathcal{L}_j^g|\bar{c}_{ij}|\right]\ge\mu>0\tag{3.15}$$

are satisfied, where

$$v := \max_{1\le i\le n}\left\{ l\cdot\left[\sum_{j=1}^{n}\mathcal{L}_j^f\,|b_{ij}| + \frac{1-\exp(-\underline{a}_i l)}{\underline{a}_i}\sum_{j=1}^{n}\mathcal{L}_j^g\,|c_{ij}|\right]\right\} < \bar{v} < 1.$$

Then the ω-periodic solution of Hopfield neural networks model with DEPCA system (3.8) is globally exponentially stable.

4. Examples and simulations

In this section, we give two examples with numerical simulations to illustrate the effectiveness of the proposed methods and results.

Example 1. Consider the following nonautonomous 2-dimensional Cohen-Grossberg neural networks model with DEPCA system:

$$\begin{aligned} x_1' = -d_1(x_1)\Bigg\{ & a_1(x_1) - \sum_{j=1}^{n} b_{1j}(t)f_j(x_j) \\ & - \sum_{j=1}^{n} c_{1j}(t)g_j\left(x_j\left(2\left[\left[\frac{t+1}{2}\right]\right]\right)\right) - J_1(t)\Bigg\}, \\ x_2' = -d_2(x_2)\Bigg\{ & a_2(x_2) - \sum_{j=1}^{n} b_{2j}(t)f_j(x_j) \\ & - \sum_{j=1}^{n} c_{2j}(t)g_j\left(x_j\left(2\left[\left[\frac{t+1}{2}\right]\right]\right)\right) - J_2(t)\Bigg\}, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} a_1(u) &= 0.25u, & d_1(u) &= 1.7 - 0.1\sin(\pi u), \\ a_2(u) &= 0.6u, & d_2(u) &= 1.2 + 0.1\cos(\pi u), \\ b_{11}(t) &= 0.2\sin(\pi t), & b_{21}(t) &= 0.15\cos(\pi t), \\ b_{12}(t) &= 0.15\cos(\pi t), & b_{22}(t) &= 0.2\sin(\pi t), \\ c_{11}(t) &= 0.125\sin(\pi t), & c_{21}(t) &= 0.25\cos(\pi t), \\ c_{12}(t) &= 0.25\sin(\pi t), & c_{22}(t) &= 0.125\cos(\pi t), \\ f_1(u) &= \tfrac{|u+1|}{16}, & g_1(u) &= \tanh(0.125u), \\ f_2(u) &= \tfrac{|u-1|}{16}, & g_2(u) &= \tanh(0.25u), \\ J_1(t) &= 0.8 + 0.7\sin(\pi t), & J_2(t) &= 0.8 - 0.7\cos(\pi t). \end{aligned}$$

Through simple computation, we have the distances $m - l = 1$, $l = 1$, $\mathcal{L}_1^d = \mathcal{L}_2^d = 0.1$, $\mathcal{L}_1^a = \mathcal{N}_1^a = 0.25$, $\mathcal{L}_2^a = \mathcal{N}_2^a = 0.6$, $\mathcal{L}_1^f = \mathcal{L}_2^f = 1/16$, $\mathcal{L}_1^g = 0.125$, $\mathcal{L}_2^g = 0.25$, $\underline{d}_1 = 1.6$, $\overline{d}_1 = 1.8$, $\underline{d}_2 = 1.1$, $\overline{d}_2 = 1.3$ and $\{2k-1\}_{k\in\mathbb{N}}$, $\{2k\}_{k\in\mathbb{N}}$ satisfy the $(2,1)$ condition.
It follows that:

(a) Let $p_1 = 9$, $p_2 = 5$ and $\sigma = 1.5$,

$$1.6609375 = \mathcal{N}_1^a p_1 - \sum_{j=1}^{2}\left[p_j\left(\mathcal{L}_j^f\left|\bar{b}_{1j}\right| + \mathcal{L}_j^g\,|\bar{c}_{1j}|\right)\right] > \sigma = 1.5,$$

$$1.9875 = \mathcal{N}_2^a p_2 - \sum_{j=1}^{2}\left[p_j\left(\mathcal{L}_j^f\left|\bar{b}_{2j}\right| + \mathcal{L}_j^g\,|\bar{c}_{2j}|\right)\right] > \sigma = 1.5.$$

(b)

$$\underline{d}\,\mathcal{N}^a - \max_{1\le i\le 2}\left[\bar{d}_i\sum_{j=1}^{2}\mathcal{L}_j^f|\bar{b}_{ij}| + \frac{\exp(\underline{d}\,\mathcal{N}^a\cdot(m-l))}{1-v}\bar{d}_i\sum_{j=1}^{2}\mathcal{L}_j^g|\bar{c}_{ij}|\right]$$

$$\approx 0.64667 > 0,$$

where

$$v := \max_{1\le i\le n}\left\{l\cdot d_i\left[\sum_{j=1}^{n}\mathcal{L}_j^f\,|b_{ij}| + \frac{1-\exp\left(-\underline{d}\,\mathcal{N}^a l\right)}{\underline{d}\,\mathcal{N}^a}\mathcal{L}_j^g\,|c_{ij}|\right]\right\}$$
$$\approx 0.14889 < 1.$$

One can see that all conditions $(\mathcal{B})$, $(\mathcal{L})$, $(\mathcal{I})$, $(\mathcal{P})$ and (3.5) in Corollary 3.12 are satisfied. Therefore, the system (4.1) has a 2-periodic solution and all other solution of the system (4.1) converge exponentially to it as $t\to\infty$. The numerical simulations are given in Figs. 1-3.

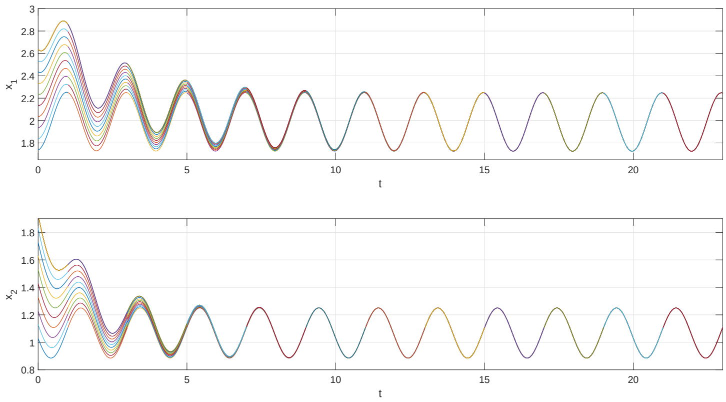

Fig. 1. Exponential convergence of two trajectories towards a 2-periodic solution of the system (4.1).

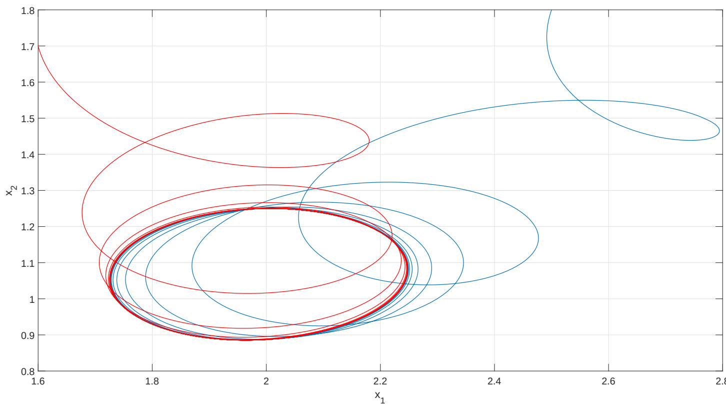

Fig. 2. Phase plane behavior of the state variables x_1 and x_2 for the system (4.1). Initial conditions: (a) $(1.6, 1.7)$ in red and (b) $(2.5, 1.8)$ in blue.

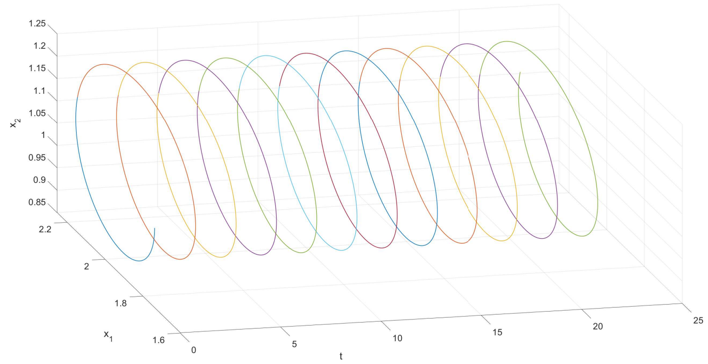

Fig. 3. Phase response behavior of state variables $x(t)$ for the system (4.1) with the initial condition $(1.736, 1.028)$.

Example 2. Let $d_1(x_1) = 1.05 + 0.15\cos\left(\frac{2\pi}{3}x_1\right)$, $d_2(x_2) = 1.05 - 0.15\sin\left(\frac{2\pi}{3}x_2\right)$, $a(x_1) = 0.25x_1$, $a(x_2) = 0.3x_2$, $b_{11}(t) \equiv 0.25$, $b_{12}(t) = b_{21}(t) \equiv 0.3$, $b_{22}(t) = c_{12}(t) = c_{21}(t) \equiv 0.125$, $c_{11}(t) \equiv 0.2$, $c_{22}(t) \equiv 0.3$, $J_1(t) = 0.2 + 0.2\sin\left(\frac{2\pi}{3}t\right)$ and $J_2(t) = 0.2 + 0.2\cos\left(\frac{2\pi}{3}t\right)$. Then we have the following Cohen-Grossberg neural networks model with DEPCA system:

$$\begin{aligned}
x_1'(t) &= \left(1.05 + 0.15\cos\left(\frac{2\pi}{3}x_1(t)\right)\right) \\
&\quad\times\Bigg[-0.25(x_1(t)) + 0.25\tanh\left(\frac{(x_1(t))}{10}\right) + 0.3\tanh\left(\frac{(x_2(t))}{10}\right) \\
&\quad + 0.2\tanh\left(\frac{x_1\left(3\left[\left[\frac{t+1}{3}\right]\right]\right)}{10}\right) + 0.125\tanh\left(\frac{x_2\left(3\left[\left[\frac{t+1}{3}\right]\right]\right)}{10}\right) \\
&\quad + 0.2 + 0.2\sin\left(\frac{2\pi}{3}t\right)\Bigg], \\
x_2'(t) &= \left(1.05 - 0.15\sin\left(\frac{2\pi}{3}x_2(t)\right)\right) \\
&\quad\times\Bigg[-0.3(x_2(t)) + 0.3\tanh\left(\frac{(x_1(t))}{10}\right) + 0.125\tanh\left(\frac{(x_2(t))}{10}\right) \\
&\quad + 0.125\tanh\left(\frac{x_1\left(3\left[\left[\frac{t+1}{3}\right]\right]\right)}{10}\right) + 0.3\tanh\left(\frac{x_2\left(3\left[\left[\frac{t+1}{3}\right]\right]\right)}{10}\right) \\
&\quad + 0.2 + 0.2\cos\left(\frac{2\pi}{3}t\right)\Bigg].
\end{aligned} \tag{4.2}$$

Through simple computation, we have the distances $m - l = 1$, $l = 2$, $\mathcal{L}_1^d = \mathcal{L}_2^d = 0.15$, $\mathcal{L}_1^a = \mathcal{N}_1^a = 0.25$, $\mathcal{L}_2^a = \mathcal{N}_2^a = 0.15$, $\mathcal{L}_i^f = \mathcal{L}_i^g = 0.1$, $\underline{d_i} = 0.9$, $\overline{d_i} = 1.2$, $i = 1, 2$ and $\{3k-1\}_{k\in\mathbb{N}}$, $\{3k\}_{k\in\mathbb{N}}$ satisfy the $(3,1)$ condition. It follows that:

(a) Let $p_1 = 5$, $p_2 = 6$ and $\sigma = 0.4$,

$$0.77 = \mathcal{N}_1^a p_1 - \sum_{j=1}^{2}\left[p_j\left(\mathcal{L}_j^f\left|\bar{b}_{1j}\right| + \mathcal{L}_j^g\left|\bar{c}_{1j}\right|\right)\right] > \sigma = 0.4,$$

$$0.4325 = \mathcal{N}_2^a p_2 - \sum_{j=1}^{2} \left[p_j \left(\mathcal{L}_j^f \left| \bar{b}_{2j} \right| + \mathcal{L}_j^g \left| \bar{c}_{2j} \right| \right) \right] > \sigma = 0.4.$$

(b)

$$\underline{d}\,\underline{\mathcal{N}}^a - \max_{1\le i\le 2} \left[\bar{d}_i \sum_{j=1}^{2} \mathcal{L}_j^f |\bar{b}_{ij}| + \frac{\exp(\underline{d}\,\underline{\mathcal{N}}^a \cdot (m-l))}{1-v} \bar{d}_i \sum_{j=1}^{2} \mathcal{L}_j^g |\bar{c}_{ij}| \right]$$

$$\approx 0.00987 > 0,$$

where

$$v := \max_{1\le i\le n} \left\{ l \cdot d_i \left[\sum_{j=1}^{n} \mathcal{L}_j^f \, |b_{ij}| + \frac{1-\exp\left(-\underline{d}\,\underline{\mathcal{N}}^a l\right)}{\underline{d}\,\underline{\mathcal{N}}^a} \mathcal{L}_j^g \, |c_{ij}| \right] \right\}$$
$$\approx 0.098707 < 1.$$

In this case, we can easily verify that all conditions of Corollary 3.12 are satisfied. Thus, according to Corollary 3.12, the system (4.2) has a 3–periodic solution and all other solution of the system (4.2) converge exponentially to it as $t \to \infty$. The numerical simulations are given in Figs. 4-6.

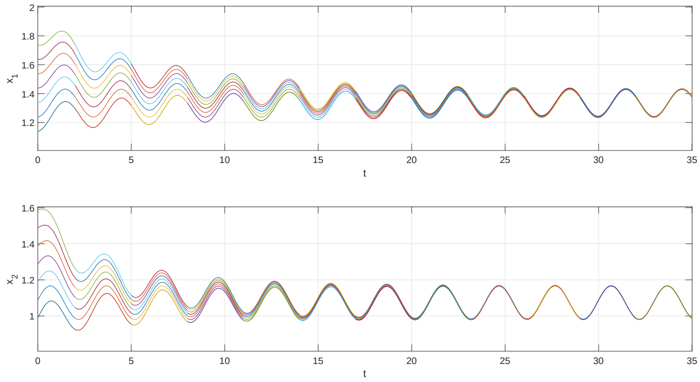

Fig. 4. Exponential convergence of two trajectories towards a 3–periodic solution of the system (4.2).

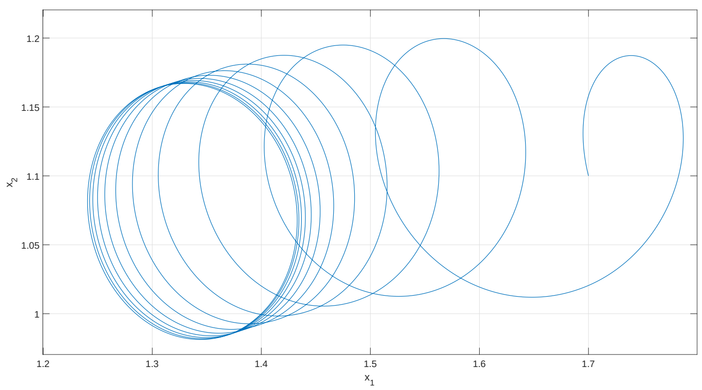

Fig. 5. Phase plane behavior of the state variables x_1 and x_2 for the system (4.2) with the initial value: $(1.7, 1.1)$.

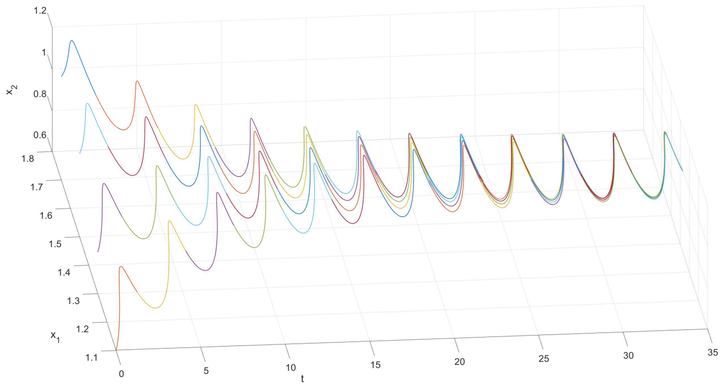

Fig. 6. Phase response behavior of state variables $x(t)$ for the system (4.2).

5. Conclusions

This is the first time that differential equations with alternately advanced and retarded argument have been applied to the model of the Cohen-Grossberg neural networks model, and this paper has provided sufficient conditions guaranteeing the existence and global exponential stability of periodic solutions of the Cohen-Grossberg neural networks model for the considered system based on a DEPCA integral inequality of Gronwall type and Brouwer's fixed point theorem. In addition, our method gives new ideas not only from the modeling point of view, but also from that of theoretical opportunities since the Cohen-Grossberg neural networks model involves piecewise constant argument of both advanced and delayed types. The obtained results could be useful in the design and applications of the Cohen-Grossberg neural networks model. Furthermore, the examples with numerical simulations are given to show the effectiveness of the proposed method and results.

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