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s-to-z Transformation Tool for Discretization

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Abstract

The signals/systems in nature are analog in terms of their sources. These continuous-time signals/systems need to be discretized in order to be used in digital systems (processing, storage etc.). For this purpose, different methods have been developed and continue to be developed. In the work carried out; a software tool with a user-friendly interface has been designed that performs discretization of continuous-time systems with different methods in a fast, accurate and effective manner, presents single or comparative results (parameters, responses, etc.) both numerically and graphically.

1. INTRODUCTION

Because of the physical systems are naturally analog, the signals related to these systems are represented in continuous-time. By developments of digital systems in parallel with progress of technology, various discrete-time techniques have been developed for analysis and design of continuous-time systems. In order to integrate classical analog systems into digital systems, discrete models or discrete equivalents must be obtained by discretization. Transformation of continuous-time to discrete-time is the fundamental operation in discrete signal processing and discrete circuit design. This process, which is expressed with concepts such as "$\mathrm{s-to-z}$ transform", "$\mathrm{s-to-z}$ transformation", "$\mathrm{s-z}$ transform", "$\mathrm{s-to-z}$ mapping" provides a transition from s-domain to z-domain.

There are many methods in the literature for $\mathrm{s-to-z}$ transformation: backward or forward difference, bilinear (Tustin) transform, impulse/step/ramp invariance/invariant methods, magnitude/phase invariance methods, matched z transform etc. [1-10]. The main purpose of all these methods developed using different approaches is to obtain the discrete-time equivalent that best describes the transfer function of the continuous-time system. However, as the order of the systems increases, it becomes more difficult to perform the relevant transformations manually or to use higher order $\mathrm{s-z}$ transformations.

As a result of the development in the computers, computer-aided technologies (CAx) concepts (such as computer-aided analysis (CAA), computer-aided design (CAD), computer-aided engineering (CAE), computer-aided instruction (CAI), computer-aided learning (CAL), computer-aided manufacturing (CAM), computer-aided software, computer-aided software engineering (CASE), etc.) have currently gained a significant place. Simulators, applications, web pages, etc. are developed in many areas using computer software. There are many studies in the field of system analysis with software in the literature [11-21]. However, there are not many software with a user-friendly interface that is specially designed for discretization and can perform these operations quickly, accurately and effectively with many different methods. For example, the "c2d" discretization command in MATLAB does this in just six different methods and provides numerical results [22].

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In this study, a software tool has been developed that discretizes the continuous-time transfer functions with different methods. With this developed software, very high order transfer functions in the s-domain are transferred to the z-domain easily, effectively and quickly with the selected method or methods (comparative analysis). In addition, the software provides detailed numerical and graphical results of continuous and discrete time transfer functions.

This paper is organized as follows: In Section 2, $\mathrm{s}-\mathrm{to}-\mathrm{z}$ mapping functions are summarized. In Section 3, designed software tool is explained and sample applications are given. Finally, Section 4 contains conclusions.

2. s-to-z MAPPING FUNCTIONS

Different $\mathrm{s}-\mathrm{to}-\mathrm{z}$ mapping functions are available for obtaining discrete equivalents/models of continuous time functions. From the similarities in discrete expressions of the unilateral Laplace transform and z-transform of a continuous-time function $f(t)$ (Table 1),

$$F(s) = F(z)|_{z=e^{sT}} \Longrightarrow \begin{cases} z = e^{sT} \\ s = \frac{1}{T} Ln(z) \end{cases} \tag{1}$$

is obtained, where T is the sampling period. Some fundamental mapping functions can be obtained from the Taylor series of s ve z [23] using first terms are given in Table 2.

$$\begin{cases} z = e^{sT} = 1 + \frac{sT}{1!} + \frac{(sT)^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{(sT)^k}{k!} \\ s = \frac{1}{T} Ln(z) = \frac{2}{T}\left\{\frac{z-1}{z+1} + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \cdots\right\} = \frac{2}{T}\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{z-1}{z+1}\right)^{2k-1} \end{cases} \tag{2}$$

Table 1. *Unilateral Laplace and z-transforms*

Transform	Discrete-time expression
Unilateral Laplace transform	$F(s) = \sum_{n=0}^{\infty} f[n]e^{-snT}$
Unilateral z-transform	$F(z) = \sum_{n=0}^{\infty} f[n]z^{-n}$

Table 2. *Popular $s-to-z$ mapping methods*

Method/Rule	s	z
Forward difference / Forward rectangular	$s = \frac{z-1}{T}$	$z = 1 + sT$
Backward difference / Backward rectangular	$s = \frac{z-1}{Tz}$	$z = \frac{1}{1-sT}$
Bilinear / Trapezoidal / Tustin	$s = \frac{2}{T}\frac{z-1}{z+1}$	$z = \frac{1+sT/2}{1-sT/2}$

The general classification of $\mathrm{s}-\mathrm{to}-\mathrm{z}$ mapping methods is given in Figure 1 [2]. Matching responses methods are based on matching system responses (impulse, step etc.) (Table 3).

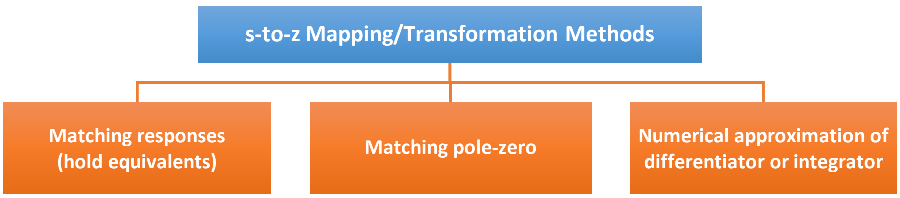


Figure 1. *The general classification of s-to-z mapping methods*

For example, the impulse-invariant discrete-time system equivalent of the continuous-time system with the transfer function $H(s)$ is obtained as follows:

- Obtain the impulse response of continuous-time system: $h(t) = \mathcal{L}^{-1}\{H(s)\}$
- Derive samples $h(k)$ from $h(t)$ with suitable sampling interval: $h(k) = h(t)|_{t=kT}$

- Obtain z-transform of $h(k)$: $H(z) = \mathcal{Z}\{h(k)\}$

This process can be represented by

$$H(z) = \mathcal{Z}\left\{\mathcal{L}^{-1}\{H(s)\}\big|_{t=kT}\right\} \tag{3}$$

and this operation commonly indicated as

$$H(z) = \mathcal{Z}\{H(s)\} \tag{4}$$

Table 3. *The some methods based on matching step and other responses*

Method	Definition [$H(z)$]
Impulse invariance	$\mathcal{Z}\{H(s)\}$
Step invariance (ZOH)	$Z\left\{\frac{1-e^{-sT}}{s}H(s)\right\} = \left(\frac{z-1}{z}\right)Z\left\{\frac{H(s)}{s}\right\}$
Ramp invariance (FOH)	$Z\left\{\frac{1-e^{-sT}}{s^2}H(s)\right\} = \frac{(z-1)^2}{Tz}Z\left\{\frac{H(s)}{s^2}\right\}$

Table 4. *Some matched pole-zero models*

	Integrator: $\frac{1}{s}$	Real poles/zeros: $s+a$	Complex poles/zeros: $(s+a)^2+b^2$
Matched pole-zero model: impulse hold	$\frac{z-1}{Tz}$	$\frac{z-e^{-aT}}{Tz}$	$\frac{z^2-2ze^{-aT}\mathrm{Cos}(bT)-e^{-aT}}{(Tz)^2}$
Matched pole-zero model: zero-order hold	$\frac{z-1}{T}$	$\frac{z-e^{-aT}}{T}$	$\frac{z^2-2ze^{-aT}\mathrm{Cos}(bT)-e^{-2aT}}{T^2}$
Matched pole-zero model: triangular hold	$\frac{2}{T}\frac{z-1}{z+1}$	$\frac{a^2}{a-\frac{1-e^{-aT}}{T}}\frac{z-e^{-aT}}{z-\frac{\left(ae^{-aT}-\frac{1-e^{-aT}}{T}\right)}{\left(a-\frac{1-e^{-aT}}{T}\right)}}$	$\frac{(a^2+b^2)T}{\alpha}\frac{z^2-2ze^{-aT}\mathrm{Cos}(bT)-e^{-2aT}}{z^2-\frac{\gamma}{\alpha}z+\frac{\beta}{\alpha}}$ $\alpha = e^{-2aT}+(a^2+b^2)T-2aT-2e^{-2aT}\mathrm{Cos}(bT)+2e^{-aT}\{\mathrm{aCos}(bT)-bSin(bT)\}$ $\beta = e^{-2aT}+2aTe^{-2aT}+(a^2+b^2)T^2e^{-2aT}-2e^{-aT}Cos(bT)-2Te^{-aT}\{\mathrm{aCos}(bT)+bSin(bT)\}$ $\gamma = 2\left\{\begin{matrix} e^{-2aT}+aTe^{-2aT}+1+(a^2+b^2)T^2e^{-aT}Cos(bT) \\ -a-2e^{-aT}Cos(bT)-2bTe^{-aT}Sin(bT)\end{matrix}\right\}$

Matching pole-zero (pole-zero matching, pole-zero mapping, matched z-transform method) is a method based on mapping all poles and zeros of continuous-time system in s-plane to z-plane locations ($z = e^{sT}$) for a sample interval (Table 4) [24]. This procedure is summarized as follow:

- Map all poles and zeros of system according to $z = e^{sT}$.
- If order of the numerator is lower than the denominator, add powers of $(z+1)$ to the numerator until the orders are equal.
- Set equivalent DC or low-frequency gain.

Numerical approximations of differentiator or integrator methods are based on the use of numerical approaches instead of derivatives or integrals in the continuous-time system equation. For example, using the finite differences for the derivative and the rectangular and trapezoidal rules for the integral, the results in Table 2 are given in Table 5 and Table 6, respectively. In many transformation methods, the discrete-time transfer function is obtained by using the substitution method: $Z\{(s^{\pm 1})^k\} = [Z\{s^{\pm 1}\}]^k$. But, it should be noted that in some substitution methods, the terms may change depending on the degree of derivative and integrator, namely: $Z\{(s^{\pm 1})^k\} \neq [Z\{s^{\pm 1}\}]^k$, as seen at Table 7. In Table 8, many methods based on numerical approximation are given [1-5, 10, 25-54].

Table 5. *Obtaining the methods in Table 2 with finite difference approximation of derivatives*

1st order differential equation	Laplace transform	Transfer function (s-domain)
$y'(t)+y(t)=x(t)$	$sY(s)+Y(s)=X(s)$	$\frac{Y(s)}{X(s)}=\frac{1}{s+1}$

Method	Approximation	Step	Result
Forward difference	$y'(t)=\left.\frac{dy(t)}{dt}\right\|_{t=kT}=\frac{y(k+1)-y(k)}{T}$	Substituting	$\frac{y(k+1)-y(k)}{T}+y(k)=x(k)$
		z-transform	$\frac{zY(z)-Y(z)}{T}+Y(z)=X(z)$
		Transfer function	$\frac{Y(z)}{X(z)}=\frac{1}{\frac{z-1}{T}+1}$
Backward difference	$y'(t)=\left.\frac{dy(t)}{dt}\right\|_{t=kT}=\frac{y(k)-y(k-1)}{T}$	Substituting	$\frac{y(k)-y(k-1)}{T}+y(k)=x(k)$
		z-transform	$\frac{Y(z)-z^{-1}Y(z)}{T}+Y(z)=X(z)$
		Transfer function	$\frac{Y(z)}{X(z)}=\frac{1}{\frac{1-z^{-1}}{T}+1}$

Table 6. *Obtaining the methods in Table 2 with rectangular and trapezoidal rules for integration*

1st order differential equation	Integration	Transfer function (s-domain)
$y'(t)+y(t)=x(t)$	$y(t)=y(0)-\int_0^t y(\tau)d\tau+\int_0^t x(\tau)d\tau$	$\frac{Y(s)}{X(s)}=\frac{1}{s+1}$

Method	Approximation	Step	Result
Forward rectangular	$\int_{(k-1)T}^{kT} y(t)dt \cong y(k-1)T$	Substituting	$y(k)=y(k-1)-y(k-1)T+x(k-1)T$
		z-transform	$Y(z)=z^{-1}Y(z)-z^{-1}Y(z)T+z^{-1}X(z)T$
		Transfer function	$\frac{Y(z)}{X(z)}=\frac{1}{\frac{z-1}{T}+1}$
Backward rectangular	$\int_{(k-1)T}^{kT} y(t)dt \cong y(k)T$	Substituting	$y(k)=y(k-1)-y(k)T+x(k)T$
		z-transform	$Y(z)=z^{-1}Y(z)-Y(z)T+X(z)T$
		Transfer function	$\frac{Y(z)}{X(z)}=\frac{1}{\frac{1-z^{-1}}{T}+1}$
Trapezoidal	$\int_{(k-1)T}^{kT} y(t)dt \cong \frac{y(k)+y(k-1)}{2}T$	Substituting	$y(k)=y(k-1)-\frac{y(k)+y(k-1)}{2}T+\frac{x(k)+x(k-1)}{2}T$
		z-transform	$Y(z)=z^{-1}Y(z)-Y(z)(1+z^{-1})\frac{T}{2}+X(z)(1+z^{-1})\frac{T}{2}$
		Transfer function	$\frac{Y(z)}{X(z)}=\frac{1}{\frac{2(z-1)}{T(z+1)}+1}$

Table 7. *Common substitution methods[25]*

Method	s^{-1}	s^{-2}	s^{-3}	s^{-4}
Backward rectangular	$\frac{Tz}{z-1}$	$\frac{T^2z^2}{(z-1)^2}$	$\frac{T^3z^3}{(z-1)^3}$	$\frac{T^4z^4}{(z-1)^4}$
Bilinear (Tustin) transform	$\frac{T}{2}\frac{z+1}{z-1}$	$\frac{T^2}{4}\frac{(z+1)^2}{(z-1)^2}$	$\frac{T^3}{8}\frac{(z+1)^3}{(z-1)^3}$	$\frac{T^4}{16}\frac{(z+1)^4}{(z-1)^4}$
z-transform	$\frac{z}{z-1}$	$\frac{Tz}{(z-1)^2}$	$\frac{T^2}{2}\frac{(z^2+z)}{(z-1)^3}$	$\frac{T^3}{6}\frac{(z^3+4z^2+z)}{(z-1)^4}$
Halijak	$\frac{Tz}{z-1}$	$\frac{T^2z}{(z-1)^2}$	$\frac{T^3}{2}\frac{(z^2+z)}{(z-1)^3}$	$\frac{T^4}{4}\frac{(z^3+2z^2+z)}{(z-1)^4}$
Boxer-Thaler	$\frac{T}{2}\frac{z+1}{z-1}$	$\frac{T^2}{12}\frac{(z^2+10z+1)}{(z-1)^2}$	$\frac{T^3}{2}\frac{(z^2+z)}{(z-1)^3}$	$\frac{T^4}{6}\frac{(z^3+4z^2+z)}{(z-1)^4}-\frac{T^4}{720}$
Madwed	$\frac{T}{2}\frac{z+1}{z-1}$	$\frac{T^2}{6}\frac{(z^2+4z+1)}{(z-1)^2}$	$\frac{T^3}{24}\frac{(z^3+11z^2+11z+1)}{(z-1)^3}$	$\frac{T^4}{120}\frac{(z^4+26z^3+66z^2+26z+1)}{(z-1)^4}$

Table 8. *The some methods based on numerical approximation*

Method	Differentiator (s)	Integrator ($1/s$)
Forward rectangular integration (Euler approximation of first order)	$\frac{z-1}{T}$	$\frac{Tz^{-1}}{1-z^{-1}}$
Backward rectangular integration (Euler approximation of second order)	$\frac{z-1}{Tz}$	$\frac{T}{1-z^{-1}}$
Trapezoidal integration (Bilinear / Tustin method)	$\frac{2}{T}\frac{z-1}{z+1}$	$\frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}$
Trapezoidal integration with prewarping (Bilinear / Tustin method with prewarping)	$\frac{\omega_0}{tan\left(\frac{\omega_0 T}{2}\right)}\frac{z-1}{z+1}$	$\frac{tan\left(\frac{\omega_0 T}{2}\right)}{\omega_0}\frac{1+z^{-1}}{1-z^{-1}}$
Compensated trapezoidal integration	$\frac{2}{T}\frac{n(z-1)}{(n+2)z+(n-2)}$	$\frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}+\frac{T}{n}$
Upward parabolic integration	$\frac{3}{T}\frac{z-1}{2z+1}$	$\frac{T}{3}\frac{2+z^{-1}}{1-z^{-1}}$
Downward parabolic integration	$\frac{3}{T}\frac{z-1}{z+2}$	$\frac{T}{3}\frac{1+2z^{-1}}{1-z^{-1}}$
Al-Alaoui minimum phase integrator-1 (for $a=3/4$). (Stabilized version of Al-Alaoui nonminimum phase integrator-1)	$\frac{8}{7T}\frac{z-1}{z+(1/7)}$	$\frac{7T}{8}\frac{1+(1/7)z^{-1}}{1-z^{-1}}$
Al-Alaoui minimum phase integrator-2 (for $a=3/4$) (combined backward rectangular and trapezoidal integration rules) $H_{BT}(z)=aH_B(z)+(1-a)H_T(z)$	$\frac{8}{7T}\frac{z-1}{z+(1/7)}$	$\frac{7T}{8}\frac{1+(1/7)z^{-1}}{1-z^{-1}}$
Al-Alaoui minimum phase integrator-3 (Stabilized version of Al-Alaoui nonminimum phase integrator-3)	$\frac{6(z^2-1)}{Tr_1(3-a)(z+r_2)}$, $(0\le a\le 1)$ $r_2=\frac{3+a-2\sqrt{3a}}{3-a}$	$\frac{Tr_1(3-a)(z+r_2)}{6(z^2-1)}$, $(0\le a\le 1)$ $r_2=\frac{3+a-2\sqrt{3a}}{3-a}$
Al-Alaoui nonminimum phase integrator-1 for $a=3/4$. (combined forward rectangular and trapezoidal integration rules) $H_{FT}(z)=aH_F(z)+(1-a)H_T(z)$	$\frac{8}{T}\frac{z-1}{z+7}$	$\frac{T}{8}\frac{1+7z^{-1}}{1-z^{-1}}$
Al-Alaoui nonminimum phase integrator-2 (combined Simpson 1/3 and trapezoidal integration rules) $H_{ST}(z)=aH_S(z)+(1-a)H_T(z)$	$\frac{6(z^2-1)}{T(3-a)[(z+r_1)(z+r_2)]}$ $r_1=\frac{3+a+2\sqrt{3a}}{3-a}$, $r_2=\frac{3+a-2\sqrt{3a}}{3-a}$ $r_1=1/r_2$, $(0\le a\le 1)$	$\frac{T(3-a)[(z+r_1)(z+r_2)]}{6(z^2-1)}$ $r_1=\frac{3+a+2\sqrt{3a}}{3-a}$, $r_2=\frac{3+a-2\sqrt{3a}}{3-a}$ $r_1=1/r_2$, $(0\le a\le 1)$
Le Bihan nonminimum phase integrator for $\chi=0{,}793$. $H_{FT}(z)=\chi H_F(z)+(1-\chi)H_T(z)$	$\frac{2}{T}\frac{z-1}{(1-\chi)z+(1+\chi)}$	$\frac{T}{2}\frac{(1-\chi)+(1+\chi)z^{-1}}{1-z^{-1}}$
Le Bihan minimum phase integrator (for $\chi=0{,}793$). (Stabilized version of Le Bihan nonminimum phase integrator)	$\frac{2}{T}\frac{z-1}{(1+\chi)z+(1-\chi)}$	$\frac{T}{2}\frac{(1+\chi)+(1-\chi)z^{-1}}{1-z^{-1}}$
Tick integration rule	$\frac{2{,}7902}{T}\frac{z^2-1}{z^2+3{,}5804z+1}$	$\frac{T}{2{,}7902}\frac{1+3{,}5804z^{-1}+z^{-2}}{1-z^{-2}}$

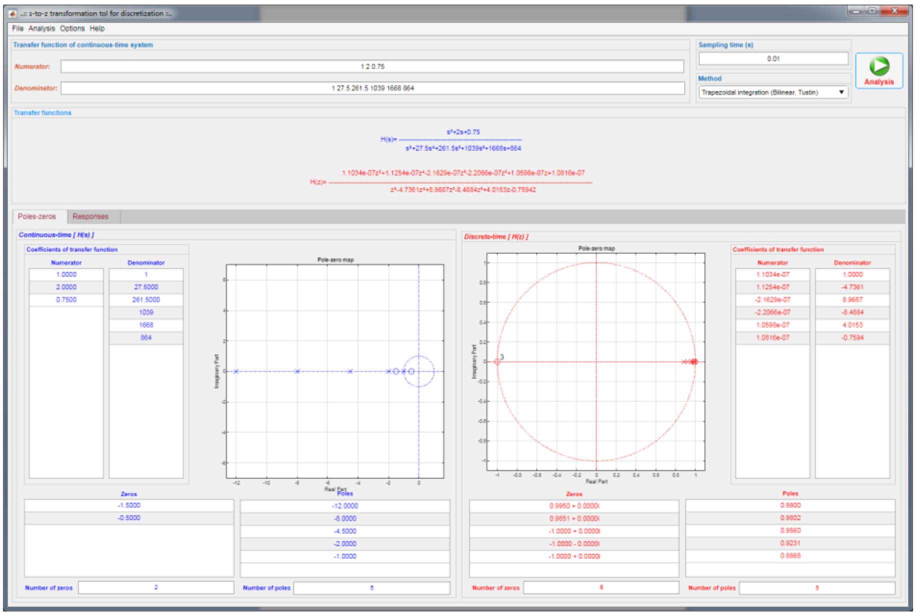
Al-Alaoui – Tick integrator (Stabilized Tick integrator)	$\frac{0,852}{T}\frac{z^2-1}{z^2+0,611z+0,0932}$	$\frac{T}{0,852}\frac{1+0,611z^{-1}+0,0932z^{-2}}{1-z^{-2}}$
Simpson 1/3 integrator	$\frac{3}{T}\frac{z^2-1}{z^2+4z+1}$	$\frac{T}{3}\frac{1+4z^{-1}+z^{-2}}{1-z^{-2}}$
Al-Alaoui – Simpson method (Stabilized Simpson 1/3 integrator)	$\frac{0,8039}{T}\frac{z^2-1}{z^2+0,5358z+0,0718}$	$\frac{T}{0,8039}\frac{1+0,5358z^{-1}+0,0718z^{-2}}{1-z^{-2}}$
Parametric BD-BL transform - Dostal parametric transform (combined backward difference and bilinear integration rules) $H_{BT}(z)=\alpha H_B(z)+(1-\alpha)H_T(z)$	$\frac{1+r}{T}\frac{z-1}{z+r}$, $(0<r<1)$	$\frac{T}{1+r}\frac{1+rz^{-1}}{1-z^{-1}}$, $(0<r<1)$
Papamarkos–Chamzas integrator	$\frac{z^2-1}{T(0,476337z^2+1,076644z+0,476337)}$	$\frac{T(0,476337+1,076644z^{-1}+0,476337z^{-2})}{1-z^{-2}}$
Gurova-Georgiev transformation	$\frac{2,9382}{T}\frac{z^2-1}{z^2+3,8765z+1}$	$\frac{T}{2,9382}\frac{1+3,8765z^{-1}+z^{-2}}{1-z^{-2}}$
Simpson 3/8 integrator	$\frac{8}{3T}\frac{z^3-1}{z^3+3z^2+3z+1}$	$\frac{3T}{8}\frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-z^{-3}}$
Boole integration	$\frac{45}{2T}\frac{z^4-1}{7z^4+32z^3+12z^2+32z+7}$	$\frac{2T}{45}\frac{7+32z^{-1}+12z^{-2}+32z^{-3}+7z^{-4}}{1-z^{-4}}$
Adams-Moulton 3rd order integration (Schneider method)	$\frac{12}{T}\frac{z(z-1)}{5z^2+8z-1}$	$\frac{T}{12}\frac{5+8z^{-1}-z^{-2}}{1-z^{-1}}$
Adams-Moulton 4th order integration (SKG method)	$\frac{24}{T}\frac{z^2(z-1)}{9z^3+19z^2-5z+1}$	$\frac{T}{24}\frac{9+19z^{-1}-5z^{-2}+z^{-3}}{1-z^{-1}}$
Adams-Moulton 5th order integration	$\frac{720}{T}\frac{z^3(z-1)}{251z^4+64z^3-264z^2+106z-19}$	$\frac{T}{720}\frac{251+64z^{-1}-264z^{-2}+106z^{-3}-19z^{-4}}{1-z^{-1}}$
Al-Alaoui – Alpha , $\alpha=0,1$ (Stabilized version of combined trapezoidal and Simpson rules)	$\frac{1,4301224}{T}\frac{z^2-1}{z^2+1,38245z+0,47779}$	$\frac{T}{1,4301224}\frac{1+1,38245z^{-1}+0,47779z^{-2}}{1-z^{-2}}$
Al-Alaoui – Schneider (Stabilized Adams-Moulton 3rd order integration rule)	$\frac{1,39818}{T}\frac{z^2-1}{z^2+0,46607z-0,06788}$	$\frac{T}{1,39818}\frac{1+0,46607z^{-1}-0,06788z^{-2}}{1-z^{-2}}$
Al-Alaoui – SKG (Stabilized Adams-Moulton 4th order integration rule)	$\frac{1,1272}{T}\frac{z^2(z-1)}{z^3+0,168z^2-0,0607z+0,0199}$	$\frac{T}{1,1272}\frac{1+0,168z^{-1}-0,0607z^{-2}+0,0199z^{-3}}{1-z^{-1}}$
Al-Alaoui – Two segment rule (combined Simpson 1/3 and trapezoidal integration rules) $H_2(z)=\alpha H_{S1/3}(z)+(1-\alpha)H_T(z)$	$\frac{15}{T}\frac{z^2-1}{7z^2+16z+7}$, $\alpha=0,2$	$\frac{T}{15}\frac{7+16z^{-1}+7z^{-2}}{1-z^{-2}}$, $\alpha=0,2$
Al-Alaoui – Three segment rule (combined Simpson 3/8 and trapezoidal integration rules) $H_3(z)=\alpha H_{S3/8}(z)+(1-\alpha)H_T(z)$	$\frac{80}{3T}\frac{z^3-1}{13z^3+27z^2+27z+13}$, $\alpha=0,1$	$\frac{3T}{80}\frac{13+27z^{-1}+27z^{-2}+13z^{-3}}{1-z^{-3}}$, $\alpha=0,1$
Al-Alaoui – Reduced four segment rule (combined backward rectangular and Two segment integration rules) $H_4(z)=\alpha H_B(z)+(1-\alpha)H_2(z)$	$\frac{945}{2T}\frac{z^2-1}{217z^2+512,8463z+217}$ $\alpha=1/20$	$\frac{2T}{945}\frac{217+512,8463z^{-1}+217z^{-2}}{1-z^{-2}}$ $\alpha=1/20$
Al-Alaoui – Stabilized two segment rule	$\frac{8,8438}{T}\frac{z^2-1}{7z^2+8,2543z+2,4333}$	$\frac{T}{8,8438}\frac{7+8,2543z^{-1}+2,4333z^{-2}}{1-z^{-2}}$
Al-Alaoui – Stabilized reduced four segment rule	$\frac{945}{2T}\frac{z^2-1}{393,0387z^2+434z+119,8075}$	$\frac{2T}{945}\frac{393,0387+434z^{-1}+119,8075z^{-2}}{1-z^{-2}}$
Hamming integrator-1	$\frac{24}{T}\frac{z(2z^2-z-1)}{17z^3+51z^2+3z+1}$	$\frac{T}{24}\frac{17+51z^{-1}+3z^{-2}+z^{-3}}{2-z^{-1}-z^{-2}}$
Hamming integrator-2	$\frac{24}{T}\frac{z(3z^2-2z-1)}{25z^3+91z^2+43z+9}$	$\frac{T}{24}\frac{25+91z^{-1}+43z^{-2}+9z^{-3}}{3-2z^{-1}-z^{-2}}$
Hamming integrator-3	$\frac{24}{T}\frac{3z^3-z^2-z-1}{26z^3+73z^2+30z+10}$	$\frac{T}{24}\frac{26+73z^{-1}+30z^{-2}+10z^{-3}}{3-z^{-1}-z^{-2}-z^{-3}}$
Graham-Lindquist integrator-1	$\frac{1}{T}\frac{5z^2-4z-1}{z(2z+4)}$	$T\frac{2+4z^{-1}}{5-4z^{-1}-z^{-2}}$
Graham-Lindquist integrator-2	$\frac{1}{T}\frac{17z^3-9z^2-9z+1}{z^2(6z+18)}$	$T\frac{6+18z^{-1}}{17-9z^{-1}-9z^{-2}+z^{-3}}$
Graham-Lindquist integrator-3	$\frac{1}{T}\frac{37z^4-8z^3-36z^2+8z-1}{z^3(12z+48)}$	$T\frac{12+48z^{-1}}{37-8z^{-1}-36z^{-2}+8z^{-3}-z^{-4}}$
NGO integrator	$\left(\frac{T}{2,7925}\right)\frac{(z+2,3658)(z-0,2167e^{j0,9427})(z-0,2167e^{-j0,9427})}{z^2(z-1)}$	
NGO differentiator	$\left(\frac{1}{T}\right)\left(\frac{2,7925}{2,3658}\right)\frac{z^2(z-1)}{(z+1/2,3658)(z-0,2167e^{j0,9427})(z-0,2167e^{-j0,9427})}$	

3. DESIGNED TOOL and APPLICATIONS

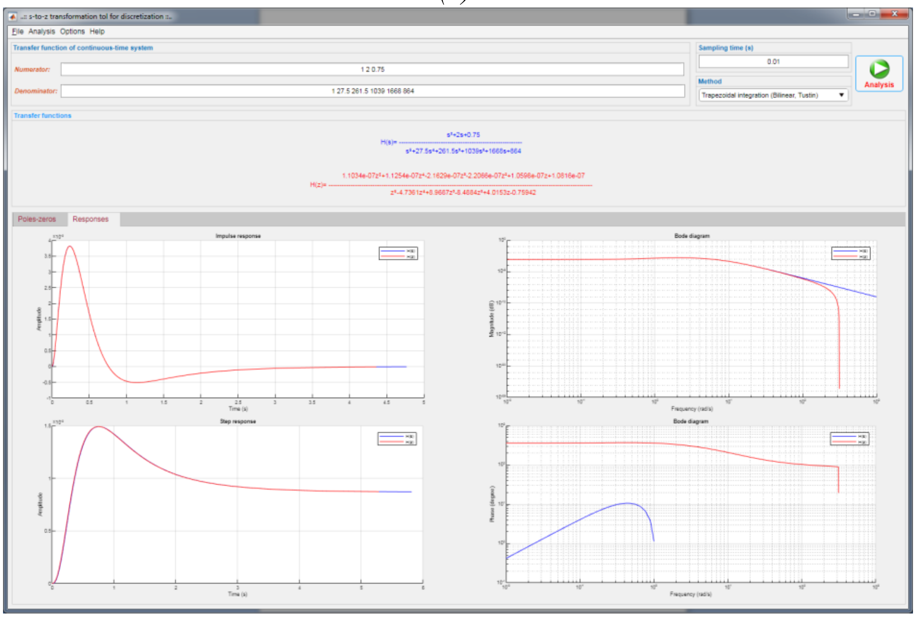
In this study, an application/simulator was designed using MATLAB App Designer [22] for $\mathrm{s-to-z}$ transformation/mapping, which is one of the most important application areas of signal processing. With this application, discretization operations can be performed with numerous different methods easily, quickly and effectively. In addition, the effectiveness of the methods can be clearly observed by performing single or comparative analyzes, because the application produces many numerical and graphical results (transfer functions of systems, zeros, poles, responses, etc.). Besides, it supports a better understanding of $\mathrm{s-to-z}$ transformation/mapping methods with its use in the field of education.

The first application is a fifth order filter with real poles, two real zeros and a dc gain of $1/1152$ (sampling time is $0.01\ s$) [55] which is given with following transfer function:

$$H(s) = \frac{s^2+2s+0.75}{s^5+27.5s^4+261.5s^3+1039s^3+1668s+864} \tag{5}$$



(a)



(b)

***Figure 2.** The screenshots for first application*

The results of discretization using bilinear (Tustin) method are given in Figure 2. As seen in Figure 2, in the designed application, the numerator and denominator coefficients of the transfer function of the continuous-time system are entered, the sampling time is determined, and the method to be used is selected. As a result of the analysis performed, the transfer functions of both continuous-time model and its discrete-time equivalent are shown and the pole-zero maps of these transfer functions are plotted, and also the pole-zero numbers are given (Figure 2a). In addition, the impulse and step responses and Bode diagrams of both continuous time model and its discrete-time equivalent are plotted (Figure 2b).

The second application is a sixth order Butterworth filter (sampling time is 0.01 s) [3] which is given with following transfer function:

$$H(s) = \frac{1}{(s^2+2Cos(5\pi/12)s+1)(s^2+2Cos(\pi/12)s+1)(s^2+\sqrt{2}s+1)} \quad (6)$$

The results of comparative discretization using Schneider transform, trapezoidal integration (bilinear, Tustin) method and Al-Alaoui-Schneider transform are given in Figure 3. In the comparison screen, the transfer function of the continuous time system is discretized with 3 different methods selected; the numerator and denominator coefficients of the discrete time transfer functions and their zeros and poles are listed. In addition, the unit impulse and step responses of discrete-time systems obtained by both continuous-time and comparative methods are plotted comparatively. Thus, the discretization efficiency of each method on the relevant system can be clearly observed.

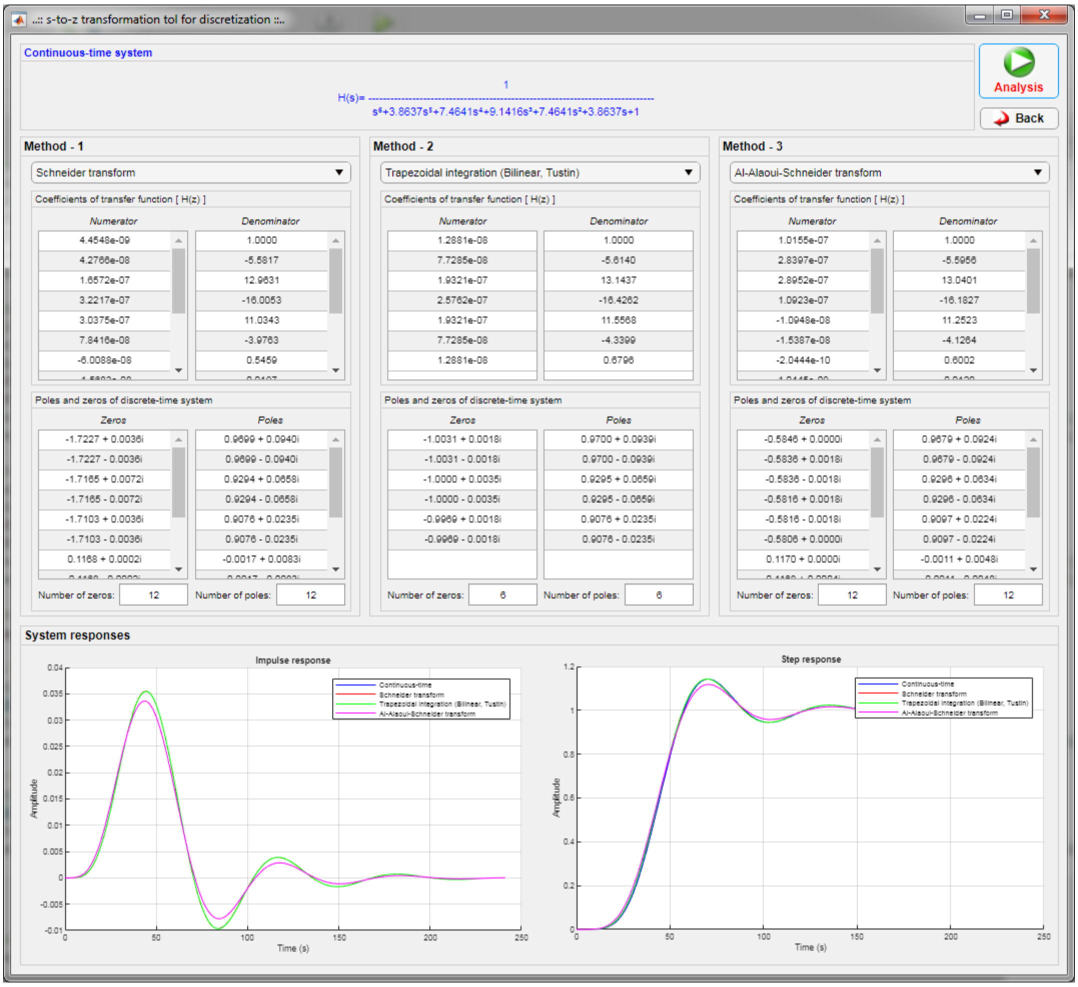


Figure 3. The screenshots for second application

4. CONCLUSIONS

In this study, an application/simulator was designed out for discretization, which is one of the most important application areas of signal processing. Continuous-time systems need to be discretized in order to be used with discrete time systems (microprocessor, computer, etc.). However, discrete-time equivalents of higher order continuous-time systems are extremely difficult to obtain manually. Complex mathematical operations are performed especially for obtaining of discrete models with high accuracy (equivalence). With the software tool designed in this study - regardless of the degree of the system - discretization can be done easily, quickly and effectively with many different methods. In addition, the most suitable models can be determined with comparative numerical and graphical results.

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