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Some new integral inequalities of the Simpson type for MT-convex functions

Siqintuya Jin[a], Aying Wan[b], Bai-Ni Guo[c,d]

[a] College of Mathematics and Physics, Inner Mongolia Minzu University, Tongliao 028043, Inner Mongolia, China.
[b] College of Mathematics and Statistics, Hulunbuir University, Hailaer 021008, Inner Mongolia, China.
[c] Institute of Mathematics, Henan Polytechnic University, Jiaozuo 454010, Henan, China.
[d] Corresponding author

Abstract

In the paper, with the aid of a known integral identity, the authors establish some new inequalities, similar to the celebrated Simpson's integral inequality, for differentiable MT-convex functions.

Keywords: integral inequality; MT-convex function; integral identity; Simpson's integral inequality.
2010 MSC: Primary 26D15; Secondary 26A51, 26E60, 41A55.

1. Introduction

In [13], Tunç and Yildirim introduced the concept of MT-convex functions.

Definition 1 ([13, Section 2])**.** Let $I \subseteq \mathbb{R}$ be an interval. A nonnegative function $f : I \to \mathbb{R}_0 = [0, \infty)$ is said to be MT-convex if the inequality

$$f(tx + (1-t)y) \le \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y)$$

hold for all $x, y \in I$ and $t \in (0, 1)$.

Email addresses: jinsiqintuya@126.com (Siqintuya Jin), wanaying1@aliyun.com (Aying Wan), bai.ni.guo@gmail.com (Bai-Ni Guo)

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In [4, 6, 7, 10, 11, 13, 14], integral inequalities of the Hermite–Hadamard type for MT-convex functions have been presented. In [9], integral inequalities of the Hermite–Hadamard type for MT-convex functions on differentiable coordinates were established. In [7, 8], integral inequalities of the Hermite–Hadamard type for MT-convex functions via classical integrals and fractional integrals were given.

Motivated by Definition 1, Bai, Wang, and Qi defined in [2] so-called HT-convex functions. Also motivated by Definition 1, Wang, Sun, and Guo defined in [15] so-called GT-convex functions. Meanwhile, some integral inequalities were set up in the papers [2, 15].

Due to wide applications of various convex functions, some mathematicians have dedicated to studying integral inequalities of the Hermite–Hadamard type for different classes of convex functions. For details, please refer to [1, 3, 5, 12, 16, 17, 18, 19, 20, 21, 22, 23] and closely related references therein.

In this paper, we will establish some new integral inequalities of the Simpson type for so-called MT-convex functions.

2. A lemma

In order to prove our main results, we need the following lemma.

Lemma 1 ([20, 22])**.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on the interior I° of an interval I and let $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then*

$$\begin{aligned}\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\\ =\frac{b-a}{4}\int_0^1\left[\left(\frac{2}{3}-t\right)f'\left(ta+(1-t)\frac{a+b}{2}\right)+\left(\frac{1}{3}-t\right)f'\left(t\frac{a+b}{2}+(1-t)b\right)\right]\mathrm{d}\,t.\end{aligned}$$

3. Inequalities of the Simpson type for MT-convex functions

Now we start out to establish some new integral inequalities of the Simpson type for MT-convex functions.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on the interior I° of an interval I and let $a, b \in I$ with $a < b$. If $f' \in L_1([a, b])$ and $|f'|^q$ for $q \ge 1$ is MT-convex on $[a, b]$, then*

$$\begin{aligned}\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\ \le\frac{b-a}{4}\left(\frac{8}{15}\right)^{1-1/q}\left\{\left[C_1|f'(a)|^q+C_2\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[C_2\left|f'\left(\frac{a+b}{2}\right)\right|^q+C_1|f'(b)|^q\right]^{1/q}\right\},\end{aligned}\tag{1}$$

where

$$C_1=\frac{1}{144}\left(20\sqrt{2}+3\pi-12\arcsin\sqrt{\frac{2}{3}}\right)$$

and

$$C_2=\frac{1}{144}\left(28\sqrt{2}-15\pi+60\arcsin\sqrt{\frac{2}{3}}\right).$$

Proof. Using Lemma 1 and noted Hölder's integral inequality, we have

$$\begin{aligned}\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\ \le\frac{b-a}{4}\int_0^1\left[\left|\frac{2}{3}-t\right|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|+\left|\frac{1}{3}-t\right|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\right]\mathrm{d}\,t\\ \le\frac{b-a}{4}\left\{\left(\int_0^1\left|\frac{2}{3}-t\right|\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1\left|\frac{2}{3}-t\right|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right.\\ \left.+\left(\int_0^1\left|\frac{1}{3}-t\right|\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1\left|\frac{1}{3}-t\right|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right\}.\end{aligned}\tag{2}$$

By the MT-convexity of $|f'|^q$ on $[a,b]$, direct calculation yields

$$\int_0^1\left|\frac{2}{3}-t\right|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q \mathrm{d}\,t \le \int_0^1\left|\frac{2}{3}-t\right|\left(\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(a)|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\mathrm{d}\,t \tag{3}$$

and

$$\int_0^1\left|\frac{1}{3}-t\right|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q \mathrm{d}\,t \le \int_0^1\left|\frac{1}{3}-t\right|\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right)\mathrm{d}\,t.$$

Further straightforward computation gives

$$\int_0^1\left|\frac{2}{3}-t\right|\frac{\sqrt{t}}{\sqrt{1-t}}\,\mathrm{d}\,t=\int_0^1\left|\frac{1}{3}-t\right|\frac{\sqrt{1-t}}{\sqrt{t}}\,\mathrm{d}\,t=\frac{20\sqrt{2}+3\pi-12\arcsin\sqrt{\frac{2}{3}}}{144},$$

$$\int_0^1\left|\frac{2}{3}-t\right|\frac{\sqrt{1-t}}{\sqrt{t}}\,\mathrm{d}\,t=\int_0^1\left|\frac{1}{3}-t\right|\frac{\sqrt{t}}{\sqrt{1-t}}\,\mathrm{d}\,t=\frac{28\sqrt{2}-15\pi+60\arcsin\sqrt{\frac{2}{3}}}{144},$$

and

$$\int_0^1\left|\frac{2}{3}-t\right|\mathrm{d}\,t=\int_0^1\left|\frac{1}{3}-t\right|\mathrm{d}\,t=\frac{5}{18}. \tag{4}$$

Substituting those inequalities and equalities between (3) and (4) into the inequality (2) results in the inequality (1). The proof of Theorem 1 is complete. □

Corollary 1. *Under conditions of Theorem 1, if $q=1$, then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left[C_1|f'(a)|^q+2C_2\left|f'\left(\frac{a+b}{2}\right)\right|^q+C_1|f'(b)|^q\right].$$

Theorem 2. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° and let $a,b\in I$ with $a<b$. If $f'\in L_1([a,b])$ and $|f'|^q$ for $q>1$ is MT-convex on $[a,b]$, then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{9}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\frac{3\pi}{8}\right)^{1/q}\left\{\left[|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right]^{1/q}\right\}.$$

Proof. By Lemma 1, noted Hölder's integral inequality, and the MT-convexity of $|f'|^q$ on $[a,b]$, we obtain

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\ &\le\frac{b-a}{4}\int_0^1\left[\left|\frac{2}{3}-t\right|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|+\left|\frac{1}{3}-t\right|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\right]\mathrm{d}\,t\\ &\le\frac{b-a}{4}\left\{\left(\int_0^1\left|\frac{2}{3}-t\right|^{q/(q-1)}\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right.\\ &\quad\left.+\left(\int_0^1\left|\frac{1}{3}-t\right|^{q/(q-1)}\mathrm{d}\,t\right)^{1-1/q}\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q\mathrm{d}\,t\bigg]^{1/q}\right\}\end{aligned}$$

$$
\begin{aligned}
&\le \frac{b-a}{4}\left[\frac{q-1}{2q-1}\left(\frac{2}{3}\right)^{(2q-1)/(q-1)}\right]^{1-1/q}\left\{\left[\int_0^1\left(\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(a)|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\mathrm{d}\,t\right]^{1/q}\right.\\
&\qquad\left.+\left[\int_0^1\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right)\mathrm{d}\,t\right]^{1/q}\right\}\\
&= \frac{b-a}{4}\left[\frac{q-1}{2q-1}\left(\frac{2}{3}\right)^{(2q-1)/(q-1)}\right]^{1-1/q}\left(\frac{\pi}{4}\right)^{1/q}\\
&\qquad\times\left\{\left[|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right]^{1/q}\right\}.
\end{aligned}
$$

The proof of Theorem 2 is complete. □

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Conflicts of Interest

The authors declare no conflict of interest.

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