ERCIYES JOURNAL OF EDUCATION (EJE)
2021, VOL 5, NO. 2, 208-220
http://dergipark.org.tr/eje

Matematik Öğretimi İçin Matematik Tarihini İncelemek: Logaritma Örneği

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ÖZ

Matematik tarihinin matematikteki kavramların anlaşılmasındaki rolü son yıllarda önem verilen konular arasındadır, ancak matematik tarihinin matematik öğretimine dâhil edilmesinde hala bazı kısıtlar mevcuttur. Logaritma, matematikteki en temel kavramlardan biridir, bilim ve teknoloji alanlarında sıklıkla kullanılmaktadır. Ancak, araştırmalar öğrencilerin logaritma kavramını anlamakta zorlandıklarını, kimi zaman logaritmik ifadelerle ilgili işlemlerde kimi zaman ise logaritmik ifadelerle ilgili işlemleri yapabildikleri halde logaritmanın anlamını kavramakta güçlük çektiklerini göstermektedir. Matematikteki diğer kavramlarda olduğu gibi, logaritma kavramının öğretiminde öğrencilere tanıtılma ve yapılandırılma şeklinin öğrencilerin onu nasıl anladığı ile yakından ilişkili olduğu bilinmektedir. Bir matematiksel kavramın ders kitabında nasıl yapılandırıldığı öğretmenlerin kavramı sınıfta nasıl öğrettiğini önemli oranda etkilemektedir. Dolayısıyla, ders kitaplarında matematiksel kavramların nasıl ele alındığı matematik öğretimindeki önemli parametrelerden biridir. Türkiye'de logaritma kavramı öğrencilere 12. sınıfta tanıtılmaktadır. 12. sınıf matematik ders kitabı incelendiğinde logaritma kavramının üstel fonksiyonun tersi olarak tanımlanarak öğretilmeye başlandığı görülmektedir. Logaritma öğretiminde yoğunluklu olarak cebirsel yaklaşımdan faydalanıldığı anlaşılmaktadır. Logaritma kavramının tarihi ile ilgili yapılan bu çalışmada, logaritma kavramının tarihsel gelişiminin kavramın ders kitaplarında ele alınma şeklinden farklı olduğu ortaya konulmaktadır. Bu çalışma, logaritmanın tarihsel gelişiminden faydalanarak logaritmayı tanımlamanın ve tanıtmanın alternatif bir yolunu önermeyi amaçlamaktadır. Çalışmanın bulgularının logaritmanın sayısal, cebirsel ve geometrik ifadeleri arasında ilişkinin anlaşılmasını kolaylaştırması, böylelikle logaritmanın anlamının kavratılmasına faydası olması beklenmektedir.

Anahtar Kelimeler: Logaritma, logaritma tarihi, matematik ders kitabı, matematik öğretimi, ortaokul matematiği.

Erciyes Üniversitesi, Eğitim Fakültesi, Kayseri/TÜRKİYE
Erciyes Journal of Education (EJE)
DOI: 10.32433/eje.1004600

SCREENED BY iThenticate Professional Plagiarism Prevention

Tür: Teorik Makale

Makale Geçmişi
Gönderim : 04.10.2021
Kabul : 25.10.2021
Yayınlanma : 31.10.2021

Önerilen Atıf

Ören Vural, D. (2021). Matematik öğretimi için matematik tarihini incelemek: Logaritma örneği. *Erciyes Journal of Education*, *5*(2), 208-220. https://doi.org/10.32433/eje.1004600

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ERCIYES JOURNAL OF EDUCATION (EJE)
2021, VOL 5, NO. 2, 208-220
http://dergipark.org.tr/eje

Investigating History of Mathematics for Teaching Mathematics: The Case of Logarithm

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ABSTRACT

The role of the history of mathematics in understanding the concepts in mathematics is among the topics that have been given importance in recent years, however incorporating history of mathematical concept in teaching is still limited. Logarithm is one of the most fundamental concepts in mathematics, it is frequently used in the fields of science and technology. However, studies show that students have difficulty in understanding the concept of logarithm, and they have difficulty in comprehending the meaning of logarithm even if they can perform operations on logarithmic expressions. As with other concepts in mathematics, it is known that the way in which the concept of logarithm is introduced and structured in teaching is closely related to how students understand it. How a mathematical concept is structured in the textbook significantly affects how teachers teach the concept in the classroom. Therefore, how mathematical concepts are structured in textbooks is one of the important parameters in mathematics teaching. In Turkey, the concept of logarithm is introduced to students in the 12th grade. When the 12th grade mathematics textbook is examined, it is seen that the concept of logarithm is defined as the inverse of the exponential function. It is also understood that the algebraic approach is heavily utilized in the teaching of logarithms. In this study on the history of the concept of logarithm, it is revealed that the historical development of the concept of logarithm is different from the way the concept is structured in textbooks. This study aims to propose an alternative way of defining and introducing the logarithm by making use of the historical development of logarithm. It is expected that the findings of the study will facilitate the understanding of the relationship between the numerical, algebraic, and geometric expressions of the logarithm, thus helping to comprehend the meaning of the logarithm.

Keywords: Logarithm, history of logarithm, mathematics textbooks, teaching mathematics, secondary school mathematics.

Erciyes University, Faculty of Education, Kayseri/TURKEY
Erciyes Journal of Education (EJE)
DOI: 10.32433/eje.1004600

SCREENED BY iThenticate Professional Plagiarism Prevention

Type: Review Article

Article History
Received : 04.10.2021
Accepted : 25.10.2021
Published : 31.10.2021

Suggested Citation

Ören Vural, D. (2021). Investigating history of mathematics for teaching mathematics: The case of logarithm. *Erciyes Journal of Education*, *5*(2), 208-220. https://doi.org/10.32433/eje.1004600

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EXTENDED ABSTRACT

Introduction

The concept of logarithm is considered as a fundamental concept used in science, technology, and mathematics (Williams, 2011). It is one of the core concepts for many mathematical courses as well as it is a useful tool for addressing problems in daily life (Weber, 2002; Smith & Confrey, 1994). In Turkey, students are introduced to the concept of logarithm at 12th grade (MEB, 2018). In mathematics textbooks, the concept logarithm was defined as a function which is the inverse of the exponential function. Whilst in the concept of exponents students are told that exponential operation represents "repeated multiplication" then the concept of logarithm is presented as "power/indices reducing" (MEB, 2018). However, researchers pointed out that this way of introducing the logarithm results in a limited conception of the concept (Confrey & Smith, 1995). It was argued because of this limited view, students fail to perform several expressions, or might perform operations in logarithmic expression successfully but not comprehend its meaning (Brezovski & Zazkis; 2006). Also, students have great difficulty to recognize non-standard problems based on logarithms without mentioning logarithmic function (Katzberg, 2002). The difficulties students are experiencing indicate that the ways how the logarithm was taught to students are very crucial. In this regard, how logarithm is structured in mathematics textbooks is important. In recent years importance of the history of mathematics for teaching and learning has been widely recognized (Katz, 1995; Radford & Puig, 2007). The idea that there is a close connection between historical development of mathematics and difficulties experienced by students had been largely advocated (Katz, 1995; Radford, 2001; Sfard, 1995). Investigating the history of an algebraic concept provides us both a lens through which we consider the concept itself and students' difficulties from a different perspective and a tool by which we develop a way of introducing and structuring algebra in the school. Based on this premise, investigating the historical development of the concept of logarithms can contribute to understanding students' difficulties and challenges and to designing effective instructional methods and strategies to overcome such difficulties and challenges.

Purpose

This study aims to propose an alternative way of defining and introducing logarithms by investigating the historical development of logarithm.

Method

This research is theoretical research. A critical review of related research on history of logarithm was conducted.

Findings

Most of works on the history of mathematics (Ahglin, 1994; Burton, 2008; Cajori, 1930; Eves, 1980; Hairer & Wanner, 1996) compromise on the idea that logarithms were invented by John Napier (1550-1617). To understand Napier's logarithms, one must understand the relationship between the arithmetic and geometric sequences. It was argued that the first emergence of arithmetic and geometric series date back to Babylonians. In an Old Babylonian tablet (c. 1800 B.C.) there was number patterns which answer the question "to what power must a certain number be raised in order to yield a given number". This number pattern can be considered earliest appearance of logarithms. However, Babylonians interested in logarithms as something to be used to solve certain types of problems (Boyer, 1968). Nonetheless, we cannot mention about comparison between arithmetic and geometric sequences in this tablet. It was argued that multiplication as an operation defined in Euclid's Book VII provided a ground for the comparison of two patterns operating parallel (Smith & Confrey, 1994). In Book VII, Euclid defined multiplication as "when that which is multiplied is added to itself as many times as there are units in the other" (Heath, 1956, cited in Smith & Confrey, 1994). With Euclid, the relation between multiplication and addition was begun to establish. Many historical documents argue that Archimedes (287-212 B.C.) was first who attempts to combine arithmetic and geometric series (Fauvel, 1995; Pierce, 1977; Simith & Confrey, 1994). Later, in 1484, Chuquet extended the Babylonian progression to the twentieth term, and explicitly note that addition in the arithmetic progression corresponded to multiplication in the geometric one (Fauvel, 1995). Chuquet was also one of the first mathematicians who recognize zero and negative and fractional numbers as exponents (Cajori, 1913). All these can be seen as taking a step towards the latter developments of the logarithm. By the end of sixteenth century, Napier had begun to investigate the most efficient way of multiplying two large numbers to facilitate the computations involved in astronomy (Fauvel, 1995). Napier's goal was to make a table wherein the multiplication of sines could be done by addition instead (Katz, 2004). He used certain properties of trigonometric identities to convert the multiplication problem into addition and subtraction. However, the calculations he carried out were very lengthy. After many years of slowly building up the concept, he finally developed logarithms (Smith, 2000). Napier set out to construct a table which consists of the logarithms of $\sin\theta$. It is important to highlight that the increments in the table were not arithmetic but rather "geometric". This idea proved to be an important benchmark in the history of mathematics. Napier's understanding of the relationship between the position (of a point) and the rate of change (of velocity) allowed him to construct the relationship between additive and multiplicative worlds. Although Napier's goal was to simplify the computations, his work had the greatest contribution to the development of the number e and the natural logarithms (Katz, 2004). After Napier, Henry Briggs (1561-1630) worked on the logarithms. Logarithm to base 10 came to be known as Briggsian logarithms.

Discussion & Conclusion

In the curriculum, the concept of logarithm follows the topic of exponents, and it is defined as the inverse of exponential function. However, the analysis of the historical and conceptual development of logarithms showed that what today seems like a simple base to exponent relationship really has a long history of work and improvements (Villarreal-Calderon, 2008). Based on the analysis of the historical and conceptual development of logarithm, followings are suggested an alternative way of defining and introducing logarithm.

- The historical development of the logarithm concept highlighted the importance of representations in the creation of logarithmic function. Explaining the historical development of the concept with examples can make it easier for students to understand logarithm.
- In the curriculum there is no reference to geometry or to table construction, the logarithm is defined in an algebraic way. The concept of logarithm should be taught to students in a way to establish a connection between geometric, numerical, and algebraic expressions.

- The relation between sine, cosine, and circle can be usefully reviewed to understand the logarithm. Reverting from ratios to sixteenth-century consideration of sines and cosines as lengths can help to demystify the subject (Fauvel,1995).
- The curriculum approach to multiplication as repeated addition and argued that such an approach results in an immature understanding of multiplication in exponential growth (Confrey and Smith, 1995). An example of splitting as a primitive model for multiplication and division can be used to facilitates students to see the relation between the arithmetic and geometric series.

INTRODUCTION

1. Importance of History of Mathematics

In recent years, the importance of the history of mathematics for teaching and learning mathematics has been widely recognized and promoted (Katz, 1995; Radford & Puig, 2007). As for algebra, it is also well documented that the history of algebra has a great contribution both to algebra as a branch of mathematics, and to teaching and learning algebra as a school subject. Regarding the contribution of the history of algebra to teaching and learning, Toumasis (1993) proposed that incorporating historical material into teaching helps both teachers and students to discover the beauty of the history of mathematics. However, the history of algebra not only allow us better to appreciate the beauty and complexity of algebraic concepts and the breakthroughs that occurred during its construction (Bernardz, Kieran & Lee, 1996), but it also leads to considering new items that have form part of competence, having new ways of understanding the performance of pupils and, developing teaching models (Filloy, Puig & Rojano, 2008, p-258).

Several researchers agreed that there is a close connection between the historical development of algebra and difficulties experienced by pupils (Katz, 1995; Radford, 2001; Sfard, 1995). Therefore, investigating the history of an algebraic concept provide us both a lens through which we consider the concept itself and students' difficulties from a different perspective and a tool by which we develop a way of introducing and structuring algebra in the school. As Toumasis (1993) said that "it was not always the way it is today and hence that it might have been otherwise as well as that mathematics might be taught in some different way" (p-428).

2. The concept of logarithm in the curriculum

The concept of logarithm is thought of as one of the crucial moments in mathematics (Eves, 1980). According to many resources on the history of mathematics, the logarithm was developed in order to simplify the operation of multiplication and division and save an immense amount of calculation, especially when large numbers are involved (Burton, 2007; Cajori, 1913; Eves, 1983; Katz, 1995). Today, the logarithm is considered as fundamental concept used in science, technology, and mathematics (Williams, 2011). It is a useful tool addressing problems in daily life such as modeling and understanding population growth, radioactive decay, and compound interest (Smith & Confrey, 1994; Weber, 2002). And it is also one of the core concepts for many mathematical courses such as calculus, differential equations, and complex analysis. Students start to learn the concept of logarithm when they are in secondary grades. In Turkey, students are introduced to the concept of logarithm at 12th grade (MEB, 2018).

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Although mathematics textbooks used in different countries follow a somehow different approaches for teaching logarithms (such as problem-solving approach or modeling approach), the focus in many of the mathematics textbooks is somehow similar. Toumais (1993) stated that:

> "In modern textbooks, logarithms are presented as a kind of dependence between two variables. First, the exponential function is presented and then the logarithmic function is defined as an inverse of the exponential function. In other words, if f(x)=a^x (a>0), then $f^{-1}(x)=\log_a x$ and the real number $\log_a x$ for each x>0, is called the logarithm of x to base a. In some textbooks logarithm is defined as exponent to base a (a>0) e.g., if $\log_a x = x$ then b=a^x. All the logarithmic properties are derived from these two definitions and from the fact that a^x and $\log_a x$ are inverses" (p-428).

Actually, not much has changed since Toumais (1993)'s statement. Today, many mathematics textbooks follow the same path portrayed by Toumasis. For example, in Turkey students were introduced to the concept of logarithm at 12th grade level (MEB, 2018). In mathematics textbooks, the concept logarithm was defined as a function which is the inverse function of the exponential function, that is y=a^x ⇔x=$\log_a x$ (see Figure 1 below).

$f:\mathbb{R}\to\mathbb{R}^+$, $a>0$ ve $a\neq 1$ olacak şekilde $f(x)=a^x$ üstel fonksiyonunun tersi olan $f^{-1}(x)$ fonksiyonuna, **a tabanına göre logaritma fonksiyonu** denir. Logaritma fonksiyonu, $f^{-1}:\mathbb{R}^+\to\mathbb{R}$, $\mathbf{f^{-1}(x)=\log_a x}$ şeklinde gösterilir.

$a\in\mathbb{R}^+-\{1\}$ için $\mathbf{y=a^x \Leftrightarrow x=\log_a y}$ olur.

Figure 1. *Description of Logarithmic function in 12th grade mathematics textbook* (MEB: 2019, p-22)

In the textbook, several modeling questions are also included. However, first, students are introduced to logarithms as an inverse of exponential functions, then they are presented to solve the modeling question as an application of logarithm. Moreover, at the end of the chapter separated for the logarithm, a piece of reading representing a historical anecdote describing John Napier as the person who found the logarithm was given.

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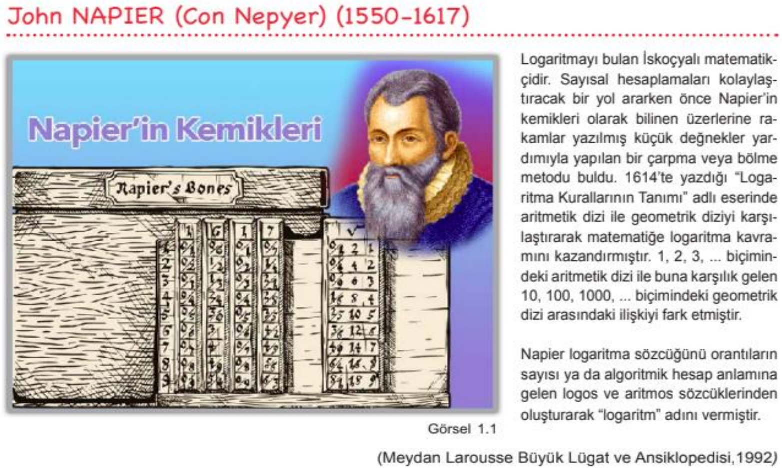

Figure 2. *Reading about John Napier, given in 12th grade mathematics textbook* (MEB: 2019, p-31)

3. Students' difficulties in logarithm

Whilst in the concept of exponents students are told that exponential operation represents "repeated multiplication" (e.g., $5^3 = 5.5.5$), then the concept of logarithm is presented as "power/indice reducing" (MEB, 2018). However, researchers pointed out that this conception is quite limited (Confrey & Smith, 1995), because, for students who see exponents as repeated multiplication, expressions such as 5^0, 5^{-1} and $5^{1/2}$ will be nonsense (Confrey, 1991). Similarly, though students often correctly answer what $\log_3 9$ is, they fail to answer what $\log_3 1/9$, $\log_3 1$ or $\log_{1/3} 1/9$ is (Confrey & Smith, 1995). Weber (2002) argued that students often forget many properties of exponents and logarithms shortly after they learn them and have difficulty in explaining why their properties are true. On the other hand, students might perform operations in logarithmic expression successfully even its meaning is not comprehended (Brezovski & Zazkis; 2006). Moreover, Confrey and Smith (1995) asserted that exponential and logarithmic functions are typically presented as formulas with which students learn to associate the rules for exponents/logarithms, a particular algebraic form, and routine algorithms. In this regard, Kastberg (2002) found that students characterize the logarithmic function as a symbol or a notation (such as $\log_{base} \#$) and understand logarithms as "problem to do", and they have great difficulty in recognizing non-standard problems based on logarithms without mentioning logarithmic function.

While students are experiencing such difficulties, the ways how we teach logarithms to students are very crucial. In this regard, investigating the historical development of the concept of logarithms can contribute to understanding students' difficulties and challenges and to designing effective instructional methods and strategies to overcome such difficulties and challenges.

4. Historical Development of the Concept "Logarithm"

There is much that has been said about the historical development of logarithms. Most of the works on the history of mathematics (Ahglin, 1994.; Burton, 2008; Cajori, 1930; Eves, 1980; Hairer & Wanner, 1996) compromise on the idea that logarithms were invented by John Napier (1550-1617). However, it is not improper to claim that none of the mathematical developments occurs suddenly. There are certain mathematical developments and the main works of mathematicians that laid the ground for Napier's invention of logarithms. It may seem unrelated for logarithms taught in today's classrooms, but a set of mathematical concepts and principles (such as arithmetic and geometric sequences, trigonometric functions, continuous ratios, etc.) formed a base for the emergence of logarithms. How did this happen?

To understand Napier's logarithms, one must understand the relationship between the arithmetic and geometric sequences. Since ancient times numbers were used to represent arithmetic sequences such as (1, 2, 3, 4 …). It was argued that the first emergence of arithmetic and geometric series dates back to Babylonians. Fauvel (1995) stated that "there is an Old Babylonian tablet (c. 1800 B.C.) which can provide a fruitful source for investigation of the pattern that in a sense underlies logarithms. It was found that numbers on one side of the tablet arranged like this:" (p-39).

15	2
30	4
45	8
1	16

It was emphasized that this table and its content is related with computation of interest. Fauvel (1995) underlined that number patterns seen in Babylonian tablet which answer the question "to what power must a certain number be raised in order to yield a given number" can be considered earliest appearance of logarithms, since this problem is identical with finding logarithm to the base a of a given number. However, Babylonians did not appear to be interested in logarithms as a computational aid but rather as something to be used to solve certain types of problems (Boyer, 1968).

Though Babylonian's tablet is an important step, we cannot mention a comparison between arithmetic and geometric sequences, yet. It was argued that multiplication as an operation defined in Euclid's Book VII provided a ground for the comparison of two patterns operating parallel (Smith & Confrey, 1994). In Book VII, Euclid defined multiplication as "when that which is multiplied is added to itself as many times as there are units in the other" (Heath, 1956, cited in Smith & Confrey, 1994). With Euclid, the relationship between multiplication and addition was begun to establish.

Although arithmetic and geometric sequences were considered distinctly, many historical documents argue that Archimedes (287-212 B.C.) was the first who attempts to juxtapose arithmetic and geometric series together (Fauvel, 1995; Pierce, 1977; Simith & Confrey, 1994;). The evidence of his use of exponents was revealed when he tried to estimate the number of grains of sand in the universe. He used the term "order" as the exponent in an exponential expression with a base of 100,000,000, and the addition of the "orders" corresponds to the product of the terms, which is known as the first law of the exponents (Boyer, 1968). Dover (1953, cited in Fauvel, 1995) further explained that Archimedes proved the basic logarithm property, essentially, for discrete series:

"If there be any number of terms of a series in continued proportion, say

$$A_1,\ A_2,\ A_3, \ldots.\ A_m, \ldots., A_n, \ldots.\ A_{m+n-1} \ldots.$$

of which A_1=1, A_2=10, and if any two terms as A_m , A_n will be taken and multiplied, the product A_m A_n will be a term in the same series and will be as many terms distant from A_n as A_m is distant from A_1: also will be distant from A_1 by a number of terms less by one than the sum of the numbers of terms by which A_m and A_n respectively are distant from A_1" (p-42).

Later, in 1484, Chuquet extended the Babylonian progression to the twentieth term, and explicitly note that addition in the arithmetic progression corresponded to multiplication in the geometric one (Fauvel, 1995). Using multiplication and the property of proportional numbers, he listed the first 20 powers of 2 and pointed out that when two numbers are multiplied, their indices are added (Cooke, 2005).

Numbers	Denomination	Numbers	Denomination	Numbers	Denomination
1	0	128	7	16384	14
2	1	256	8	32768	15
4	2	512	9	65536	16
6	3	1024	10	131072	17
16	4	2048	11	262144	18
32	5	4096	12	524288	19
64	6	8192	13	1048576	20

Chuquet was also one of the first mathematicians who recognize zero and negative and fractional numbers as exponents (Cajori, 1913). All these can be seen as taking a step towards the latter developments of logarithms (Flegg et al., 1985).

Fauvel (1995) highlighted Napier's logarithm as a confluence of the three conceptual streams: comparing arithmetic and geometric progressions, doing multiplication by means of addition, and using the geometry of motion. Before clarifying these aspects, it is better to note that logarithm Napier used as a concept is rather different from today's perception of a logarithms. There is strong evidence that by the end of the sixteenth century, Napier had begun to investigate the most efficient way of multiplying two large numbers in order to facilitate the computations involved in astronomy (Fauvel, 1995). As astronomers often deal with calculations that require trigonometric functions (particularly sines), Napier's goal was to make a table wherein the multiplication of sines could be done by addition instead (Katz, 2004). He used certain properties of trigonometric identities to convert the multiplication problem into addition and subtraction, such as the followings;

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$$
$$2 \sin A \sin B = \cos (A - B) - \cos (A + B)$$
$$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$$
$$2 \cos A \sin B = \sin (A - B) - \sin (A + B)$$

Ayoub (1993) denoted that in Napier's day, the sine of an angle was not viewed as a ratio, it was taken to be the leg of the right triangle as shown in Figure 3 below.

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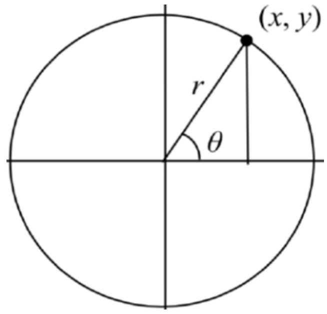

Figure 3. *Sine of an angle*

About using trigonometric identities, Villarreal-Calderon (2008) pointed out that "If one were to multiply 2994 by 3562, then sinα would be 0.2994 (the decimal is placed so that the value of α can be used later) and sinβ would be 0.3562—these would make $\alpha \approx 17.42$ and $\beta \approx 20.87$ (values obtainable in a table)" (p-338). After an intricate procedure, the result would yield an approximate value. Katz (2004) denoted that performing such calculation tricks allows astronomers to reduce errors and save time. However, the calculations Napier carried out were very lengthy. After many years of slowly building up the concept step by step, he finally developed logarithms (Smith, 2000). Napier (cited in Smith, 2000), explained why there was a need for logarithms as followings;

> "Seeing there is nothing…that is so troublesome to Mathematical practice, nor that doth more modest and hinder Calculators, than the Multiplications, Divisions, square and cubical Extractions of great numbers, which besides the tedious expense of time, are for the most part subject to many errors, I began therefore to consider in my mind by what certain and ready Art I might remove those hindrances".

Napier set out to construct a table that consists of the logarithms of sinθ. It is evident that in his table Napier chooses r=10^7 (r= radius of a unit circle) and his table consists of logarithm $10^7 \sin\theta$ (where $30^o \leq \theta \leq 90^o$) and increments of $1^{'}$ (one degree) and the table of logarithms was ranging from 10^7 to $\frac{10^7}{2}$. It is important to highlight that the increments in the table were not arithmetic but rather "geometric". This indicates that Napier's logarithm was based on the concept of "geometric progression" which will be described below.

The process of geometric progression is explained with two moving points, P and O, along two straight lines (see in Figure 4). P starts from A and moves geometrically along AB with decreasing velocity in proportion to its distance from B. Point O moves arithmetically along a second line CD at a constant velocity generating a number line. These two points start at the same time and begin moving at the same speed. However, point P slows down and takes infinitely longer to reach B, and at B its velocity would be zero. Point O continues to move at a constant speed (Cooke, 2005). Napier explained that suppose P is at the distance y from B at some instant in time t, while point O reaches the position x from C. If x is the Napierian logarithm of y, then x= $x = Nap \log \ y$. While point P moves to a new position, the division between two positions in the geometric model is mirrored by a subtraction in the arithmetic model in the corresponding position: thus, the diagram changes division into subtraction (Cooke, 2005).

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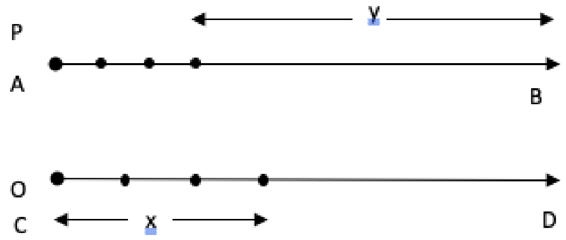

Figure 4. *A model of Napier's work*

As Napier's said (cited in Katz, 2004) "the logarithm of a given sine is that number which has increased arithmetically with the same velocity throughout as that with which radius began to decrease geometrically, and at the same time as radius has decreased to the given [number]" (p-132). This idea proved to be an important benchmark in the history of mathematics.

Napier did not think about a base for his logarithm. The tables he prepared were compiled through repeated multiplication, equivalent to powers of .9999999 (Cooke, 2005). His understanding of the relationship between the position (of a point) and the rate of change (of velocity) allowed him to construct the relationship between additive and multiplicative worlds. Although Napier's goal was to simplify the computations, his work had the greatest contribution to the development of the number e and the natural logarithms (Katz, 2004). After Napier, Henry Briggs (1561-1630) worked on the logarithms. "Logarithm of 1 is zero and the logarithm of 10 is 1" are the basic ideas added by Briggs, which we know common logarithms (Smith, 2000). Consequently, the logarithm to base 10 came to be known as Briggsian logarithms.

5. Contributions of using history as a tool in teaching a mathematical concept

In the mathematics curriculum, the concept of logarithm follows the topic of exponents, and it is defined as the inverse of the exponential function since the notion of the function has become the main concept of secondary school mathematics. However, the analysis of the historical and conceptual development of logarithms showed that what today seems like a simple base-to-exponent relationship really has a long history of work and improvements (Villarreal-Calderon, 2008). The analysis of the historical and conceptual development of the logarithm suggests an alternative way of defining and introducing logarithm. Some suggestion is listed as followings:

- Although it is not part of the curriculum, the historical development of the logarithm concept highlighted the importance of representations in the creation of logarithmic functions. Explaining the historical development of the concept with examples can make it easier for students to understand the logarithm.
- In the curriculum, the concept of logarithm is introduced in a formal algebraic way. There is no reference to geometry or to table construction. There is no connection between geometrical, numerical, and algebraic experience. The concept of logarithm should be taught to students in a way to establish a connection between geometric, numerical, and algebraic expressions.
- As Fauvel (1995) suggested that "to the extent that the original motivation for logarithms has disappeared from everyday experience, which on the face of it presents motivational difficulties in teaching them now" (p.42).

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$$

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B)$$

2 sin A cos B = sin (A + B) + sin (A – B)

2 cos A sin B = sin (A – B) – sin (A + B)

Fauvel (1995) proposed that exploring the process (presented above) offers several interesting opportunities for enlarging students' trigonometrical understanding. For example, the relationship between sine, cosine, and circle can be usefully reviewed. Reverting from ratios to sixteenth-century consideration of sines and cosines as lengths can help to demystify the subject.

- Confrey and Smith (1995) criticized the current curriculum approach to multiplication as repeated addition and argued that such an approach results in an immature understanding of multiplication in exponential growth. They proposed splitting as a primitive model for multiplication and division. In splitting, a particular quantity is split into equivalent multiple versions with respect to a growth rate. As an example of splitting, they consider folding a sheet of paper symmetrically, the paper is split into two similar rectangles, once (one indicates the level of splits), the folding process goes on, the paper is split twice so that there are 4 rectangles (two indicates the level of splits). Here, the connection between the levels of splitting and the geometrical view of the results (similar rectangles) distinguishes splitting from counting (Confrey, 1991). This also facilitates students to see the relationship between the arithmetic and geometric series.

In conclusion, using the historical approach in teaching logarithms offer important opportunities that as listed below (Toumasis, 1993).

1. Connects the new topic with the previous one.
2. Illustrates the practical examples for the origin of logarithms.
3. Helps students to apprehend the use of logarithms in numerical computation.
4. Justifies the basis of the term "logarithm".
5. Illustrate the meaning of the number e=lim(1+1/n)^n as a base of logarithmic systems.
6. Connects the initial aim of logarithms with their modern form.
7. Develops the awareness of mathematics as a developing subject.

Ethics Committee Report: Ethics Committee Report is not available since this research includes only a review of the literature on a mathematical concept.

Conflict of Interest Statement: The author has not declared any conflict of interest.

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