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RESEARCH ARTICLE

ALMOST CONTACT STRUCTURES ON SOME LIE ALGEBRAS

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ABSTRACT

In this manuscript, we show that there are no almost contact structures with parallel characteristic vector field on certain 7 dimensional Lie algebras over the real field.

Keywords: 7-dimensional nilpotent Lie algebra, Almost contact metric structure, Parallel vector field

1. INTRODUCTION

Determining possible left invariant almost contact structures on Lie groups or on Lie algebras corresponding to these gropus is a recent research area. In 3-dimensions, almost contact metric structures which are homogeneous were studied by [1]. In [2], it was shown that the real Heisenberg group is the only odd dimensional nilpotent Lie group with a left-invariant Sasakian structure. Also, Sasakian Lie algebras were classified in 5 dimensions. K-contact structures on 5-dimensional Lie groups were given in [3]. In [4], existences of cosymplectic, nearly cosymplectic, $\alpha -$Sasakian, $\beta -$Kenmotsu, almost cosymplectic and semi-cosymplectic structures were studied on 5 dimensions. In [5], quasi-Sasakian structures on nilpotent Lie algebras were investigated in five dimensions. After the classification of 7-dimensional nilpotent Lie algebras by [6], there have been studies on almost contact metric structures on 7-dimensional nilpotent Lie algebras, see [7, 8, 9].

In this work we study almost contact metric structures having parallel characteristic vector field on indecomposable 7-dimensional nilpotent Lie algebras over the real field.

2. PRELIMINARIES

An almost contact structure (ϕ, ξ, η) on an odd dimensional manifold M consists of a vector field ξ, a 1-form η on M and an endomorphism ϕ such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1.$$

In addition, if M also has a Riemannian metric g satisfying

$$g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields X, Y, then M is said to be an almost contact metric manifold.

Almost contact metric manifolds were classified into 2^{12} classes due to symmetries of the covariant derivative of the fundamental 2-form, see [10] and [11]. The class of cosymplectic manifolds is the trivial class with the defining relation $\nabla\Phi = 0$ [12].

There are 12 basic classes C_i, i=1,…,12 and all classes are direct sums of these basic classes [10]. It can be seen from the defining relations of these classes that for an almost contact structure in C_1, C_2, C_3, C_4 and C_{11}, the characteristic vector field ξ of the almost contact metric structure is parallel. In this study

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we show that none of the 7-dimensional indecomposable nilpotent Lie algebras over the real field ℝ has a parallel vector field. Thus there are no almost contact metric structures of classes C_1, C_2, C_3, C_4 and C_{11}.

A left invariant almost contact metric structure (ϕ, ξ, η, g) on a connected Lie group L induces an almost contact metric structure (ϕ, ξ, η, g) on the corresponding Lie algebra ℓ of L, see [13]. We denote the structure on ℓ also by (ϕ, ξ, η, g).

3. ALMOST CONTACT STRUCTURES WITH PARALLEL VECTOR FIELD

Assume that $(\boldsymbol{\phi}, \boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{g})$ is an almost contact metric structure on a 7-dimensional indecomposable nilpotent Lie algebra over ℝ. For the list of these algebras, refer to [6].

Theorem There is no almost contact metric structure with parallel characteristic vector field on an indecomposable nilpotent Lie algebra over ℝ in dimension 7.

Proof We give the proof for Lie algebras with upper central series dimensions 37. Calculations for other series with upper central series dimensions 357, 27, 257, 247, 2457, 2357, 23457, 17, 157, 147, 1457, 137, 1357, 13457, 12457, 12357, 123457 are similar. We show that none of the algebras in [6] has a parallel vector field.

<u>The Lie algebra 37A</u>**:**

Consider the Lie algebra $\mathbf{37A}$ in the list of [6] with upper central series dimension 37. Assume that $(\boldsymbol{\phi}, \boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{g})$ is an almost contact metric structure on $\mathbf{37A}$. Choose the $\boldsymbol{g}$-orthonormal basis $\{\boldsymbol{b_1}, \dots, \boldsymbol{b_7}\}$ of this Lie algebra. The non-zero brackets are

$$[\boldsymbol{b_1}, \boldsymbol{b_2}] = \boldsymbol{b_5}, [\boldsymbol{b_2}, \boldsymbol{b_3}] = \boldsymbol{b_6}, [\boldsymbol{b_2}, \boldsymbol{b_4}] = \boldsymbol{b_7}.$$

We write the nonzero covariant derivatives by the Kozsul's formula.

$$\begin{gathered}
\nabla_{b_1} b_2 = \frac{1}{2} b_5, \quad \nabla_{b_1} b_5 = -\frac{1}{2} b_2, \quad \nabla_{b_2} b_1 = -\frac{1}{2} b_5, \quad \nabla_{b_2} b_3 = \frac{1}{2} b_6, \\
\nabla_{b_2} b_4 = \frac{1}{2} b_7, \quad \nabla_{b_2} b_5 = \frac{1}{2} b_1, \quad \nabla_{b_2} b_6 = -\frac{1}{2} b_3, \quad \nabla_{b_2} b_7 = -\frac{1}{2} b_4, \\
\nabla_{b_3} b_2 = -\frac{1}{2} b_6, \quad \nabla_{b_3} b_6 = \frac{1}{2} b_2, \quad \nabla_{b_4} b_2 = -\frac{1}{2} b_7, \quad \nabla_{b_4} b_7 = \frac{1}{2} b_2, \\
\nabla_{b_5} b_1 = -\frac{1}{2} b_2, \quad \nabla_{b_5} b_2 = \frac{1}{2} b_1, \quad \nabla_{b_6} b_2 = -\frac{1}{2} b_3, \quad \nabla_{b_6} b_3 = \frac{1}{2} b_2, \\
\nabla_{b_7} b_2 = -\frac{1}{2} b_4, \quad \nabla_{b_7} b_4 = \frac{1}{2} b_2.
\end{gathered}$$

Let $\boldsymbol{\xi} = \sum \boldsymbol{a_i}\, \boldsymbol{b_i}$ be a parallel vector field on $\mathbf{37A}$. Then for all basis elements, we have $\nabla_{b_i} \boldsymbol{\xi} = \mathbf{0}$.

$$\mathbf{0} = \nabla_{b_1} \xi = \nabla_{b_1}(a_1 b_1 + \dots + a_7 b_7) = \frac{a_2}{2} b_5 - \frac{a_5}{2} b_2$$

implies $\boldsymbol{a_2} = \boldsymbol{a_5} = \mathbf{0}$ from linear independence of vectors $\boldsymbol{b_5}$ and $\boldsymbol{b_2}$.

$$\mathbf{0} = \nabla_{b_2} \xi = \nabla_{b_2}(a_1 b_1 + \dots + a_7 b_7) = -\frac{a_1}{2} b_5 + \frac{a_3}{2} b_6 + \frac{a_4}{2} b_7 + \frac{a_5}{2} b_1 - \frac{a_6}{2} b_3 - \frac{a_7}{2} b_4$$

gives that remaining constants $\boldsymbol{a_i}$ are also zero.

The Lie algebra 37B:

Non-zero brackets of $\boldsymbol{g}$-orthonormal basis elements $\{\boldsymbol{b_1},\dots,\boldsymbol{b_7}\}$ are

$$[\boldsymbol{b_1},\boldsymbol{b_2}]=\boldsymbol{b_5},[\boldsymbol{b_2},\boldsymbol{b_3}]=\boldsymbol{b_6},[\boldsymbol{b_3},\boldsymbol{b_4}]=\boldsymbol{b_7}$$

and the nonzero covariant derivatives are:

$$\nabla_{b_1}b_2=\frac{1}{2}b_5,\quad \nabla_{b_1}b_5=-\frac{1}{2}b_2,\quad \nabla_{b_2}b_1=-\frac{1}{2}b_5,\quad \nabla_{b_2}b_3=\frac{1}{2}b_6,$$
$$\nabla_{b_2}b_5=\frac{1}{2}b_1,\quad \nabla_{b_2}b_6=-\frac{1}{2}b_3,\quad \nabla_{b_3}b_2=-\frac{1}{2}b_6,\quad \nabla_{b_3}b_4=\frac{1}{2}b_7,$$

$$\nabla_{b_3}b_6=\frac{1}{2}b_2,\quad \nabla_{b_3}b_7=-\frac{1}{2}b_4,\quad \nabla_{b_4}b_3=-\frac{1}{2}b_7,\quad \nabla_{b_4}b_7=\frac{1}{2}b_3,$$
$$\nabla_{b_5}b_1=-\frac{1}{2}b_2,\quad \nabla_{b_5}b_2=\frac{1}{2}b_1,\quad \nabla_{b_6}b_2=-\frac{1}{2}b_3,\quad \nabla_{b_6}b_3=\frac{1}{2}b_2,$$
$$\nabla_{b_7}b_3=-\frac{1}{2}b_4,\quad \nabla_{b_7}b_4=\frac{1}{2}b_3.$$

Let $\boldsymbol{\xi}=\sum \boldsymbol{a_i b_i}$ be a parallel vector field on $\boldsymbol{37B}$. Then from equations $\nabla_{b_i}\boldsymbol{\xi}=\boldsymbol{0}$, we obtain the followings.

$$0=\nabla_{b_1}\xi=\frac{a_2}{2}b_5-\frac{a_5}{2}b_2$$

implies $\boldsymbol{a_2=a_5=0}$.

$$0=\nabla_{b_2}\xi=-\frac{a_1}{2}b_5+\frac{a_3}{2}b_6+\frac{a_5}{2}b_1-\frac{a_6}{2}b_3$$

gives that $\boldsymbol{a_1=a_3=a_6=0}$.

From the equation

$$0=\nabla_{b_3}\xi=\frac{a_4}{2}b_7-\frac{a_7}{2}b_4,$$

we get $\boldsymbol{a_4=a_7=0}$. Thus $\boldsymbol{\xi=0}$.

The algebra 37C:

The non-zero brackets are

$$[\boldsymbol{b_1},\boldsymbol{b_2}]=\boldsymbol{b_5},[\boldsymbol{b_2},\boldsymbol{b_3}]=\boldsymbol{b_6},[\boldsymbol{b_2},\boldsymbol{b_4}]=\boldsymbol{b_7},[\boldsymbol{b_3},\boldsymbol{b_4}]=\boldsymbol{b_5}$$

and the nonzero covariant derivatives are:

$$\nabla_{b_1}b_2=\frac{1}{2}b_5,\quad \nabla_{b_1}b_5=-\frac{1}{2}b_2,\quad \nabla_{b_2}b_1=-\frac{1}{2}b_5,\quad \nabla_{b_2}b_3=\frac{1}{2}b_6,$$
$$\nabla_{b_2}b_4=\frac{1}{2}b_7,\quad \nabla_{b_2}b_5=\frac{1}{2}b_1,\quad \nabla_{b_2}b_6=-\frac{1}{2}b_3,\quad \nabla_{b_2}b_7=-\frac{1}{2}b_4,$$

$$\nabla_{b_3}b_2=-\frac{1}{2}b_6,\quad \nabla_{b_3}b_4=\frac{1}{2}b_5,\quad \nabla_{b_3}b_5=-\frac{1}{2}b_4,\quad \nabla_{b_3}b_6=\frac{1}{2}b_2,$$
$$\nabla_{b_4}b_2=-\frac{1}{2}b_7,\quad \nabla_{b_4}b_3=-\frac{1}{2}b_5,\quad \nabla_{b_4}b_5=\frac{1}{2}b_3,\quad \nabla_{b_4}b_7=\frac{1}{2}b_2,$$

$$\nabla_{b_5} b_1 = -\frac{1}{2} b_2, \quad \nabla_{b_5} b_2 = \frac{1}{2} b_1, \quad \nabla_{b_5} b_3 = -\frac{1}{2} b_4, \quad \nabla_{b_5} b_4 = \frac{1}{2} b_3,$$

$$\nabla_{b_6} b_2 = -\frac{1}{2} b_3, \quad \nabla_{b_6} b_3 = \frac{1}{2} b_2, \quad \nabla_{b_7} b_2 = -\frac{1}{2} b_4, \quad \nabla_{b_7} b_4 = \frac{1}{2} b_2.$$

Let $\xi = \sum a_i b_i$ be a parallel vector field on $37C$. Checking the condition $\nabla_{b_i} \xi = 0$ for all basis elements, we get that

$$0 = \nabla_{b_1} \xi = \frac{a_2}{2} b_5 - \frac{a_5}{2} b_2$$

which implies $a_2 = a_5 = 0$ and

$$0 = \nabla_{b_2} \xi = -\frac{a_1}{2} b_5 + \frac{a_3}{2} b_6 + \frac{a_4}{2} b_7 + \frac{a_5}{2} b_1 - \frac{a_6}{2} b_3 - \frac{a_7}{2} b_4$$

gives that remaining constants are also zero.

The algebra $37D$:

The non-zero brackets are

$$[b_1, b_2] = b_5, [b_1, b_3] = b_6, [b_2, b_4] = b_7, [b_3, b_4] = b_5$$

and the nonzero covariant derivatives are:

$$\nabla_{b_1} b_2 = \frac{1}{2} b_5, \quad \nabla_{b_1} b_3 = \frac{1}{2} b_6, \quad \nabla_{b_1} b_5 = -\frac{1}{2} b_2, \quad \nabla_{b_1} b_6 = -\frac{1}{2} b_3,$$
$$\nabla_{b_2} b_1 = -\frac{1}{2} b_5, \quad \nabla_{b_2} b_4 = \frac{1}{2} b_7, \quad \nabla_{b_2} b_5 = \frac{1}{2} b_1, \quad \nabla_{b_2} b_7 = -\frac{1}{2} b_4,$$

$$\nabla_{b_3} b_1 = -\frac{1}{2} b_6, \quad \nabla_{b_3} b_4 = \frac{1}{2} b_5, \quad \nabla_{b_3} b_5 = -\frac{1}{2} b_4, \quad \nabla_{b_3} b_6 = \frac{1}{2} b_1,$$
$$\nabla_{b_4} b_2 = -\frac{1}{2} b_7, \quad \nabla_{b_4} b_3 = -\frac{1}{2} b_5, \quad \nabla_{b_4} b_5 = \frac{1}{2} b_3, \quad \nabla_{b_4} b_7 = \frac{1}{2} b_2,$$
$$\nabla_{b_5} b_1 = -\frac{1}{2} b_2, \quad \nabla_{b_5} b_2 = \frac{1}{2} b_1, \quad \nabla_{b_5} b_3 = -\frac{1}{2} b_4, \quad \nabla_{b_5} b_4 = \frac{1}{2} b_3,$$
$$\nabla_{b_6} b_1 = -\frac{1}{2} b_3, \quad \nabla_{b_6} b_3 = \frac{1}{2} b_1, \quad \nabla_{b_7} b_2 = -\frac{1}{2} b_4, \quad \nabla_{b_7} b_4 = \frac{1}{2} b_2.$$

Let $\xi = \sum a_i b_i$ be a parallel vector field on $37D$. Then since $\nabla_{b_i} \xi = 0$ for all basis elements, we obtain

$$0 = \nabla_{b_1} \xi = \frac{a_2}{2} b_5 + \frac{a_3}{2} b_6 - \frac{a_5}{2} b_2 - \frac{a_6}{2} b_3$$

implying $a_2 = a_3 = a_5 = a_6 = 0$ and

$$0 = \nabla_{b_2} \xi = -\frac{a_1}{2} b_5 + \frac{a_4}{2} b_7 - \frac{a_7}{2} b_4$$

gives that remaining constants are also zero.

The algebra $37B_1$:

The non-zero brackets are
$[b_1, b_2] = b_5, [b_1, b_3] = b_6, [b_1, b_4] = b_7, [b_2, b_4] = b_6, [b_3, b_4] = -b_5$.

Some of the nonzero covariant derivatives are:

$$\nabla_{b_1} b_2 = \frac{1}{2} b_5, \quad \nabla_{b_1} b_3 = \frac{1}{2} b_6, \quad \nabla_{b_1} b_4 = \frac{1}{2} b_7, \quad \nabla_{b_1} b_5 = -\frac{1}{2} b_2,$$
$$\nabla_{b_1} b_6 = -\frac{1}{2} b_3, \quad \nabla_{b_1} b_7 = -\frac{1}{2} b_4, \quad \nabla_{b_2} b_1 = -\frac{1}{2} b_5, \quad \nabla_{b_2} b_4 = \frac{1}{2} b_6,$$

$$\nabla_{b_2} b_5 = \frac{1}{2} b_1, \ \nabla_{b_2} b_6 = -\frac{1}{2} b_4.$$

Let $\xi = \sum a_i b_i$ be a parallel vector field on $37B_1$. Then for all basis elements, we have $\nabla_{b_i} \xi = 0$.

$$0 = \nabla_{b_1} \xi = \frac{a_2}{2} b_5 + \frac{a_3}{2} b_6 + \frac{a_4}{2} b_7 - \frac{a_5}{2} b_2 - \frac{a_6}{2} b_3 - \frac{a_7}{2} b_4$$
implies $a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0$.
$$0 = \nabla_{b_2} \xi$$
gives $a_1 = 0$.

<u>The algebra $37D_1$</u>:

The non-zero brackets are

$[b_1, b_2] = b_5, [b_1, b_3] = b_6, [b_1, b_4] = b_7, \ [b_2, b_3] = -b_7, \ [b_2, b_4] = b_6, [b_3, b_4] = -b_5$.

Some of the nonzero covariant derivatives are:

$$\nabla_{b_1} b_2 = \frac{1}{2} b_5, \quad \nabla_{b_1} b_3 = \frac{1}{2} b_6, \quad \nabla_{b_1} b_4 = \frac{1}{2} b_7, \quad \nabla_{b_1} b_5 = -\frac{1}{2} b_2,$$
$$\nabla_{b_1} b_6 = -\frac{1}{2} b_3, \quad \nabla_{b_1} b_7 = -\frac{1}{2} b_4, \quad \nabla_{b_2} b_1 = -\frac{1}{2} b_5.$$

Let $\xi = \sum a_i b_i$ be a parallel vector field on $37D_1$. Then for all basis elements, we have $\nabla_{b_i} \xi = 0$.

$$0 = \nabla_{b_1} \xi = \frac{a_2}{2} b_5 + \frac{a_3}{2} b_6 + \frac{a_4}{2} b_7 - \frac{a_5}{2} b_2 - \frac{a_6}{2} b_3 - \frac{a_7}{2} b_4$$
implies $a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0$.
$$0 = \nabla_{b_2} \xi$$
gives $a_1 = 0$.

By similar calculations on each of the indecomposable nilpotent Lie algebras over ℝ in the list of [6], we see that there are no non-zero parallel vector fields on any of the algebras in this list in dimension 7. ■

As an example, we choose one of the algebras in the list of [6] and we investigate the existence of almost cosymplectic and almost α −Sasakian structures on this Lie algebra.

Example Consider the Lie algebra $37A$ in the list of [6] with upper central series dimension 37. Suppose that (ϕ, ξ, η, g) is an almost contact metric structure on $37A$. We use the basis $\{b_1, \dots, b_7\}$ of this Lie algebra such that basis elements are g-orthonormal. The non-zero brackets, covariant derivatives are given in the proof of the theorem.

Let us show that there are almost cosymplectic structures on $37A$. The defining relation of an almost cosymplectic structure is $d\Phi = 0$ and $d\eta = 0$. Since

$$\boldsymbol{0 = 2d\eta(X,Y) = (\nabla_X\eta)Y - (\nabla_Y\eta)X = g(\nabla_X\xi,Y) - g(\nabla_Y\xi,X)},$$

$\boldsymbol{d\eta = 0}$ if and only if the characteristic vector field $\boldsymbol{\xi}$ satisfies

$$\boldsymbol{g(\nabla_X\xi,Y) = g(\nabla_Y\xi,X)}$$

for all vector fields $\boldsymbol{X,Y}$, or equivalently, if and only if

$$\boldsymbol{g(\nabla_{b_i}\xi,b_j) = g(\nabla_{b_j}\xi,b_i)}$$

for all basis elements. Let $\boldsymbol{\xi = \sum a_i\, b_i}$ be the characteristic vector field of an almost cosymplectic structure. Then

$$\boldsymbol{g\big(\nabla_{b_1}\xi,b_2\big) = g\big(\nabla_{b_1}(a_1b_1 + \cdots + a_7b_7),b_2\big) = -\frac{a_5}{2}}$$

and

$$\boldsymbol{g\big(\nabla_{e_2}\xi,e_1\big) = \frac{a_5}{2}}$$

implies $\boldsymbol{a_5 = 0}$. Similarly checking the conditions

$$\boldsymbol{g(\nabla_{b_i}\xi,b_j) = g(\nabla_{b_j}\xi,b_i)}$$

for all basis elements gives $\boldsymbol{a_6 = a_7 = 0}$.

Let $\boldsymbol{\Phi = \sum a_{ij}b^{ij}}$, where $\boldsymbol{b^{ij}}$ denotes $\boldsymbol{b^i\wedge b^j}$, and $\boldsymbol{b^i}$ is the dual of the vector field $\boldsymbol{b_i}$ Then

$$\boldsymbol{d\Phi = (a_{16}+a_{35})b^{123} + (a_{17}+a_{45})b^{124} - a_{56}\big(b^{126}-b^{235}\big) - a_{57}\big(b^{127}-b^{245}\big) \\ +(a_{46}-a_{37})b^{234} - a_{67}(b^{237}-b^{246})}.$$

Thus $\boldsymbol{d\Phi = 0}$ if and only if $\boldsymbol{a_{16} = -a_{35}}$, $\boldsymbol{a_{17} = -a_{45}}$, $\boldsymbol{a_{46} = a_{37}}$ and $\boldsymbol{a_{56} = a_{57} = a_{67} = 0}$. For example, the structure $(\boldsymbol{\phi,\xi,\eta,g})$, where $\boldsymbol{\xi = e_1}$, $\boldsymbol{\eta = b^1}$, $\boldsymbol{\phi(b_1) = 0}$, $\boldsymbol{\phi(b_2) = -b_5}$, $\boldsymbol{\phi(b_3) = -b_6}$, $\boldsymbol{\phi(b_4) = -b_7}$, $\boldsymbol{\phi(b_5) = b_2}$, $\boldsymbol{\phi(b_6) = b_3}$, $\boldsymbol{\phi(b_7) = b_4}$ is such a structure.

Next we show that there is no almost $\boldsymbol{\alpha}$ −Sasakian structure on $\boldsymbol{37A}$, that is an almost contact metric structure such that $\boldsymbol{\alpha\Phi = d\eta}$, where $\boldsymbol{\alpha}$ is a differentiable function on $\boldsymbol{37A}$. Let $\boldsymbol{\eta = a_1b^1 + \cdots + a_7b^7}$. Then

$$\boldsymbol{d\eta = -a_5b^{12} - a_6b^{23} - a_7b^{24} = \alpha\Phi}.$$

In this case, we have $\boldsymbol{\Phi\wedge\Phi = 0}$, which can not be the case, since for an almost contact metric structure in 7-dimensions, we have $\boldsymbol{\eta\wedge\Phi^3 \neq 0}$, see [10]

CONFLICT OF INTEREST

The author stated that there are no conflicts of interest regarding the publication of this article.

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