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B.-Y. Chen's Inequality for Kähler-like Statistical Submersions

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ABSTRACT

In this paper, we first define the notion of Lagrangian statistical submersion from a Kähler-like statistical manifold onto a statistical manifold. Then we prove that the horizontal distribution of a Lagrangian statistical submersion is integrable. Next, we establish Chen-Ricci inequality for Lagrangian statistical submersions from Kähler-like statistical manifolds onto statistical manifolds and discuss the equality case of the obtained inequality through a basic tensor introduced by O'Neill that plays the role of the second fundamental form of an isometric immersion. At the end, we give a nontrivial example of a Kähler-like statistical submersion.

Keywords: *Statistical submersions, Lagrangian statistical submersions, Kähler-like statistical submersions, statistical manifolds, Chen-Ricci Inequality.*
AMS Subject Classification (2020): *3B20; 53C43; 53C05; 53C40.*

1. Introduction

A statistical manifold (M, ∇, g_M) is a Riemannian manifold if ∇g_M is symmetric, where ∇ is a torsion-free affine connection. For (M, ∇, g_M), we define another torsion-free affine connection ∇^* satisfying [2]

$$Xg_M(Y,Z) = g_M(\nabla_X Y, Z) + g_M(Y, \nabla^*_X Z),$$

for any $X, Y, Z \in \Gamma(TM)$. The connections ∇ and ∇^* are called dual connections and satisfy $\left(\nabla^*\right)^* = \nabla$.

If (∇, g_M) is a statistical structure on M, then (∇^*, g_M) is also a statistical structure. Any torsion-free affine connection ∇ always has a dual connection given by [2]

$$2\nabla^0 = \nabla + \nabla^*,$$

where ∇^0 is the Levi-Civita connection on M.

Takano [18] defined a semi-Riemannian manifold (M, g_M) with almost complex structure J which has another tensor field J^* of type $(1,1)$ satisfying

$$g_M(JX, Y) + g_M(X, J^*Y) = 0,$$

for $X, Y \in \Gamma(TM)$. Then (M, g_M, J) is called an almost Hermite-like manifold. It is easy to verify the following relations [18]:

$$(J^*)^* = J, \quad (J^*)^2 = -Id, \quad g(JX, J^*Y) = g(X,Y).$$

Since, $J^2 = -Id$, the tensor field J is symmetric to g.

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Let (M, ∇, g_M, J) be an almost Hermite-like statistical manifold. Then (M, ∇, g_M, J) is called a Kähler-like statistical manifold [18] if J is parallel with respect to ∇. Also, we have

$$g_M((\nabla_X J)Y, Z) + g_M(Y, (\nabla_X J^*)Z) = 0.$$

On a Kähler-like statistical manifold (M, ∇, g_M, J), Takano [18] introduced the curvature tensor Rie with respect to ∇ such that

$$\begin{aligned} Rie(X,Y)Z = \frac{c}{4}\{&g_M(Y,Z)X - g_M(X,Z)Y - g_M(Y,JZ)JX + g_M(X,JZ)JY \\ &+ [g_M(X,JY) - g_M(Y,JX)]JZ\}, \end{aligned} \tag{1.1}$$

where c is a constant.

To find relationship between the extrinsic and intrinsic invariants of a submanifold has been very popular in the recent twenty five years (for example [5, 13, 14]). The first study in this direction was initiated by B.-Y. Chen in 1993. M.E. Aydin et al. [6] studied and proved inequalities for the scalar curvature and the Ricci curvature for statistical submanifolds in statistical manifolds of constant curvature associated with the dual connections. A.N. Siddiqui et al. [15, 16] worked with the statistical curvature tensor field, instead of the curvature tensor fields with respect to the dual connections and obtained geometric inequality by treating it as an optimization problem. On the other hand, statistical submersions between statistical manifolds are introduced and investigated by Abe and Hasegawa in [1]. In [17], Chen inequalities for statistical submersions between statistical manifolds are studied. Such inequalities derived by many authors (for instance, see, [4, 8, 9]) for Riemannian submersions.

Motivated by the affirmative studies, in the present paper, first we study some results on Lagrangian statistical submersions. Then we obtain the Chen-Ricci inequality for Lagrangian statistical submersions and characterize a basic tensor introduced by O'Neill and its dual introduced by Takano for which the equality case holds.

2. Statistical Submersion

The study of Riemannian submersions is a topic of great interest in differential geometry. Foundational works of O'Neill [10] and Gray [7] stated the fundamental tensors and equations relating the geometry of the total space, the base and the fibers of the submersion. Several generalizations of Riemannian submersions play a role in physics, particularly in Yang-Mills theory, String theory and Kaluza-Klein theory (cf. M. Falciteli, S. Ianus, A.M. Pastore, Riemannian submersions and related topics, World Scientific, 2004). Later, Riemannian submersions between manifolds endowed with various geometric structures were studied by many authors (see, [4, 8, 9]).

Let $\psi : (M, g_M) \to (B, g_B)$ be a Riemannian submersion between two Riemannian manifolds (M, g_M) and (B, g_B). We set $\dim(M) = m$, $\dim(B) = n$. For each point $q \in B$, $\psi^{-1}(q)$ is an n-dimensional Riemannian submanifold with the induced metric $\overline{g}$, called a *fiber* and denoted by $\overline{M}$. The dimension of each fiber is always $(m - n) = r$. A vector field on M is vertical if it is tangent to fibers and horizontal if orthogonal to fibers. For any $p \in M$, in the tangent space T_pM of M, the vertical and horizontal spaces are respectively denoted by $\mathcal{V}_p(M)$ and $\mathcal{H}_p(M)$. Then the tangent bundle TM is decomposed as

$$T(M) = \mathcal{V}(M) \oplus \mathcal{H}(M),$$

where $\mathcal{V}(M)$ and $\mathcal{H}(M)$ are the vertical and horizontal distributions. Moreover, let

$$\mathcal{V} : TM \to \mathcal{V}(M), \quad \mathcal{H} : TM \to \mathcal{H}(M)$$

be the projection mappings. We call a vector field X on M projectable if there exists a vector field X_* on B such that

$$\psi_*(X_p) = X_{*\psi(p)},$$

for each $p \in M$, in this situation X and X_* are ψ-related. A vector field X is said to be *basic* if it is projectable and horizontal [10, 11].

The geometry of Riemannian submersions is characterized by O'Neill's [10] tensors $\mathcal{T}$ and $\mathcal{A}$ of type $(1, 2)$, which are defined as follows.

$$\mathcal{T}_E F = \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F, \tag{2.1}$$

$$\mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F, \tag{2.2}$$

for any $E, F \in \Gamma(TM)$. It is easy to see that $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions. We summarize the properties of the tensor fields $\mathcal{T}$ and $\mathcal{A}$. Let U, V be vertical and X, Y be horizontal vector fields on M, then we have [18]

$$\mathcal{T}_U V = \mathcal{T}_V U, \quad \mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y]. \tag{2.3}$$

We recall that if $\mathcal{T}_U V = 0$, for all $U, V \in \mathcal{V}(M)$, then ψ is said to be a statistical submersion with isometric fibers [18].

Let (M, ∇, g_M) and $(B, \widetilde{\nabla}, g_B)$ be a statistical manifold and Riemannian manifold, respectively. Let $\psi : M \to B$ be a Riemannian submersion. Then the affine connections induced on fibers by the dual connections ∇ and ∇^* from M are respectively denoted by $\overline{\nabla}$ and $\overline{\nabla}^*$. Notice that both the affine connections $\overline{\nabla}$ and $\overline{\nabla}^*$ are defined by

$$\overline{\nabla}_U V = \mathcal{V}\nabla_U V, \quad \overline{\nabla}^*_U V = \mathcal{V}\nabla^*_U V.$$

Also, we observe that $\overline{\nabla}$ and $\overline{\nabla}^*$ are torsion free and conjugate to each other with respect to $\overline{g}$. We put $S = \nabla - \nabla^*$, then is a symmetric tensor.

Statistical manifolds with almost complex structure and its statistical submersions, statistical submersion of the space of the multivariate normal distribution, statistical manifolds with almost contact structures and its statistical submersions were studied by Takano in [18, 19, 20]. Recently, remarkable statistical submersions such as cosymplectic-like statistical [3], quaternionic Kähler-like statistical [23] and para-Kähler-like statistical submersions [22] have been investigated till now.

Definition 2.1. [18, 20] Let (M, ∇, g) and $(B, \widetilde{\nabla}, g_B)$ be two statistical manifolds. Then $\psi : (M, \nabla, g) \to (B, \widetilde{\nabla}, g_B)$ is a *statistical submersion* if ψ satisfies

$$\psi_*(\nabla_X Y)_p = (\widetilde{\nabla}_{\psi_* X}\psi_* Y)_{\psi(p)},$$

for basic vector fields X, Y on M and $p \in M$.

By changing ∇ for ∇^* in (2.1), (2.2), and (2.3), ones respectively define $\mathcal{T}^*$ and $\mathcal{A}^*$ (see [18, 20]). We remark that $\mathcal{A}$ and $\mathcal{A}^*$ are equal to zero if and only if $\mathcal{H}(M)$ is integrable with respect to ∇ and ∇^*, respectively. For $X, Y \in \mathcal{H}(M)$ and $V, W \in \mathcal{V}(M)$, we turn up [18]

$$g_M(\mathcal{T}_V W, X) = -g_M(W, \mathcal{T}^*_V X), \quad g_M(\mathcal{A}_X Y, V) = -g_M(Y, \mathcal{A}^*_X V). \tag{2.4}$$

In [18], Takano provided the following lemmas which are useful for this study. Therefore, for a statistical submersion $\psi : (M, \nabla, g) \to (B, \widetilde{\nabla}, g_B)$, we have

Lemma 2.1. *[18] If X and Y are horizontal vector fields, then*

$$\mathcal{A}_X Y = -\mathcal{A}^*_Y X.$$

Lemma 2.2. *[18] For $X, Y \in \mathcal{H}(M)$ and $U, V \in \mathcal{U}(M)$, we have*

$$\begin{aligned}
\nabla_U V &= \mathcal{T}_U V + \overline{\nabla}_U V, \quad \nabla^*_U V = \mathcal{T}^*_U V + \overline{\nabla}^*_U V, \\
\nabla_U X &= \mathcal{T}_U X + \mathcal{H}\nabla_U X, \quad \nabla^*_U X = \mathcal{T}^*_U X + \mathcal{H}\nabla^*{}_U X, \\
\nabla_X U &= \mathcal{A}_X U + \mathcal{U}\nabla_X U, \quad \nabla^*_X U = \mathcal{A}^*_X U + \mathcal{U}\nabla^*_X U, \\
\nabla_X Y &= \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \quad \nabla^*_X Y = \mathcal{H}\nabla^*_X Y + \mathcal{A}^*_X Y.
\end{aligned}$$

Furthermore, if X is basic, then $\mathcal{H}\nabla_U X = \mathcal{A}_X U$ and $\mathcal{H}\nabla^*_U X = \mathcal{A}^*_X U$.

Let Rie (resp. Rie^*) be the curvature tensor with respect to ∇ (resp. ∇^*) of M and $\overline{Rie}$ (resp. $\overline{Rie}^*$) be the curvature tensor with respect to the induced affine connection $\overline{\nabla}$ (resp. $\overline{\nabla}^*$) of each fiber. The Gauss equations are given by [18]

$$\begin{aligned} g_M(Rie(U,V)W,W') &= g_M(\overline{Rie}(U,V)W,W') + g_M(\mathcal{T}_U W, \mathcal{T}^*_V W') \\ &\quad - g_M(\mathcal{T}_V W, \mathcal{T}^*_U W'), \\ g(Rie^*(U,V)W,W') &= g_M(\overline{Rie}^*(U,V)W,W') + g_M(\mathcal{T}^*_U W, \mathcal{T}_V W') \\ &\quad - g_M(\mathcal{T}^*_V W, \mathcal{T}_U W'), \end{aligned} \tag{2.5}$$

The mean curvature vector fields of the fibre with respect to the affine connection ∇, its conjugate connection ∇^* and Levi-Civita connection ∇^0 are given by the horizontal vector fields $\mathcal{N} = \sum_{a=1}^r \mathcal{T}_{U_a} U_a$, $\mathcal{N}^* = \sum_{a=1}^r \mathcal{T}^*_{U_a} U_a$ and $\mathcal{N}^0 = \sum_{a=1}^r \mathcal{T}^0_{U_a} U_a$, respectively, [18]

$$\begin{aligned} H &= \frac{1}{r}\sum_{a=1}^r \mathcal{T}_{U_a} U_a = \frac{1}{r}\mathcal{N}, \quad H^* = \frac{1}{r}\sum_{a=1}^r \mathcal{T}^*_{U_a} U_a = \frac{1}{r}\mathcal{N}^*, \\ H^0 &= \frac{1}{r}\sum_{a=1}^r \mathcal{T}^0_{U_a} U_a = \frac{1}{r}\mathcal{N}^0. \end{aligned}$$

On the other hand, we have

$$\sum_{i=1}^n \sum_{a,b=1}^r \mathcal{T}^i_{ab} = \sum_{i=1}^n \sum_{a,b=1}^r g(\mathcal{T}_{U_a} U_b, X_i). \tag{2.6}$$

Let (M, ∇, g_M, J) and $(B, \widetilde{\nabla}, g_B, \widetilde{J})$ be two almost Hermite-like statistical manifolds. Then a semi-Riemannian submersion $\psi : M \to B$ is said to be an almost Hermite-like submersion [18] if $\psi_* J = \widetilde{J}\psi_*$. The horizontal and vertical distributions are J-invariant if and only if are $\widetilde{J}$-invariant. If X is basic on M which is ψ-related to X_* on B, then JX (resp. J^*X) is basic and ψ-related to $\widetilde{J}X_*$ (resp. $\widetilde{J}^*X_*$), where $\widetilde{J}$ and $\widetilde{J}^*$ are tensor fields of type $(1,1)$ such that

$$g_B(\widetilde{J}X_*, Y_*) + g_B(X_*, \widetilde{J}^*Y_*) = 0.$$

Takano [18] considered that a statistical submersion $\psi : (M, \nabla, g_M) \to (B, \widetilde{\nabla}, g_B)$ is a Kähler-like statistical submersion if (M, ∇, g_M, J) is a Kähler-like statistical manifold and each fibre is a J-invariant semi-Riemannian submanifold of M.

Theorem 2.1. *[18] If $\psi : (M, \nabla, g_M) \to (B, \widetilde{\nabla}, g_B)$ is a Kähler-like statistical submersion, then the base $(B, \widetilde{\nabla}, g_B, \widetilde{J})$ and each fibre $(\overline{M}, \overline{\nabla}, \overline{g}, \overline{J})$ are Kähler-like statistical manifolds.*

On combining (1.1) and (2.5), we obtain [18]

$$\begin{aligned} g_M(\overline{Rie}(U,V)W,W') &+ g_M(\mathcal{T}_U W, \mathcal{T}^*_V W') - g_M(\mathcal{T}_V W, \mathcal{T}^*_U W') \\ &= \frac{c}{4}\{g_M(Y,Z)X - g_M(X,Z)Y - g_M(Y,\overline{J}Z)\overline{J}X + g_M(X,\overline{J}Z)\overline{J}Y \\ &+ [g_M(X,\overline{J}Y) - g_M(Y,\overline{J}X)]\overline{J}Z\}. \end{aligned} \tag{2.7}$$

3. On Lagrangian Statistical Submersion

Similar to the classical definition of anti-invariant Riemannian and Lagrangian Riemannian submersions from a Kählerian manifold M onto a Riemannian manifold B (see [12], [21]), we give the following definitions.

Let (M, ∇, g_M, J) be an Hermite-like statistical manifold and $(B, \widetilde{\nabla}, g_B)$ be a statistical manifold. Suppose that there exists a statistical submersion $\psi : M \to B$ such that $ker(\psi_*)$ is anti-invariant with respect to J, then ψ is called an anti-invariant. On the other hand, an anti-invariant statistical submersion is called a Lagrangian statistical submersion, if $\dim(ker(\psi_*)) = \dim((ker(\psi_*))^\perp)$. In this case, J (respectively, J^*) of M reverses the vertical and the horizontal distributions.

In this section, we prove that the horizontal distribution of a Lagrangian statistical submersion from a Kähler-like statistical manifold M is integrable. First, we give the following lemmas:

Lemma 3.1. *Let $\psi : (M, \nabla, g_M) \to (B, \widetilde{\nabla}, g_B)$ be a Lagrangian statistical submersion from a Kähler-like statistical manifold M onto a statistical manifold B. Then we have $\mathcal{T}_U JE = J\mathcal{T}_U E$, for any $E \in \Gamma(TM)$ and $U \in \Gamma(\mathcal{V}(M))$.*

Lemma 3.2. *Let $\psi : (M, \nabla, g_M) \to (B, \widetilde{\nabla}, g_B)$ be a Lagrangian statistical submersion from a Kähler-like statistical manifold M onto a statistical manifold B. Then we have $\mathcal{A}_X JY = \overline{J}\mathcal{A}_X Y$, and $\mathcal{A}^*_X J^* Y = \overline{J}^* \mathcal{A}^*_X Y$, for any $X, Y \in \Gamma(\mathcal{H}(M))$.*

Proposition 3.1. *Let $\psi : (M, \nabla, g_M) \to (B, \widetilde{\nabla}, g_B)$ be a Lagrangian statistical submersion from a Kähler-like statistical manifold M onto a statistical manifold B. Then we have*

$$\mathcal{A}_X JY = -\mathcal{A}^*_Y JX, \tag{3.1}$$

for any $X, Y \in \Gamma(\mathcal{H}(M))$.

Proof. By Lemma 3.2 and Lemma 2.1, we have

$$\mathcal{A}_X JY = J\mathcal{A}_X Y = -J\mathcal{A}^*_Y X = -\mathcal{A}^*_Y JX,$$

for any $X, Y \in \Gamma(\mathcal{H}(M))$. Thus, we get our assertion. □

Similarly, the above result holds for $\mathcal{A}^*_X$.

Theorem 3.1. *Let $\psi : (M, \nabla, g_M) \to (B, \widetilde{\nabla}, g_B)$ be a Lagrangian statistical submersion from a Kähler-like statistical manifold M onto a statistical manifold B. Then $\mathcal{A}_X Y = \mathcal{A}^*_X Y = 0$, for any $X, Y \in \Gamma(\mathcal{H}(M))$, provided that $rank(\overline{J} + \overline{J}^*)$ coincides with the dimension of the fibers.*

Theorem 3.1 gives the following results.

Corollary 3.1. *If $\psi : (M, \nabla, g_M) \to (B, \widetilde{\nabla}, g_B)$ is a Lagrangian statistical submersion from a Kähler-like statistical manifold M onto a statistical manifold B such that $\overline{J} = \overline{J}^*$, then $\mathcal{A}_X Y = \mathcal{A}^*_X Y = 0$, for any $X, Y \in \Gamma(\mathcal{H}(M))$.*

The following result follows immediately from Corollary 3.1 and (2.3).

Corollary 3.2. *If $\psi : (M, \nabla, g_M) \to (B, \widetilde{\nabla}, g_B)$ is a Lagrangian statistical submersion from a Kähler-like statistical manifold M onto a statistical manifold B such that $\overline{J} = \overline{J}^*$, then the horizontal distribution $\mathcal{H}$ is completely integrable.*

4. Chen-Ricci Inequality

In this section, we derive Chen-Ricci inequality for Lagrangian statistical submersions from Kähler-like statistical manifolds onto statistical manifolds and discuss the equality case of the obtained inequality through O'Neill's tensor (and its dual introduced by Takano) that plays the role of the second fundamental form of an isometric immersion.

Now, we assume a Lagrangian statistical submersion $\psi : (M, \nabla, g_M) \to (B, \widetilde{\nabla}, g_B)$ from a Kähler-like statistical manifold $M(c)$ whose curvature tensor Rie is of the form (1.1) onto a statistical manifold B. Then, for each $p \in M$, the local orthonormal bases of horizontal $\mathcal{H}_p(M)$ and vertical $\mathcal{V}_p(M)$ subspaces are respectively given by $\{X_1, \ldots, X_n\}$ and $\{U_1 = U, \ldots, U_r\}$.

From (1.1), we have

$$\begin{aligned}\sum_{a,b=1}^{r} g_M(Rie(U_a, U_b)U_b, U_a) = \frac{c}{4}\sum_{a,b=1}^{r} \big\{ & g_M(U_a, U_a)g_M(U_b, U_b) - g_M(U_a, U_b)g_M(U_b, U_a) \\ & - g_M(U_b, \overline{J}U_b)g_M(U_a, \overline{J}U_a) + g_M(U_a, \overline{J}U_b)g_M(U_a, \overline{J}U_b) \\ & + g_M(U_a, \overline{J}U_b)g_M(U_a, \overline{J}U_b) - g_M(U_b, \overline{J}U_a)g_M(U_a, \overline{J}U_b)\big\}.\end{aligned} \tag{4.1}$$

By the relations (4.1), (2.7) and the symmetry of $\mathcal{T}$ and $\mathcal{T}^*$, we get

$$\frac{c}{4}r(r-1) = 2\overline{R} - r^2 g_M(H, H^*) + \sum_{a,b=1}^{r} g_M(\mathcal{T}_{U_a}U_b, \mathcal{T}^*_{U_a}U_b), \tag{4.2}$$

where $\overline{R}$ is the scalar curvature of each fibre with respect to the induced affine connection.

Since $2\mathcal{T}^0 = \mathcal{T} + \mathcal{T}^*$ and $2H^0 = H + H^*$, then equation (4.2) becomes

$$\begin{aligned}\frac{c}{4}r(r-1) &= 2\overline{R} - 2r^2 g_M(H^0, H^0) + \frac{r^2}{2}(||H||^2 + ||H^*||^2)\\ &+ \sum_{a,b=1}^{r} \left\{2g_M(\mathcal{T}^0_{U_a}U_b, \mathcal{T}^0_{U_a}U_b) - \frac{1}{2}\left[g_M(\mathcal{T}^0_{U_a}U_b, \mathcal{T}^0_{U_a}U_b) + g_M(\mathcal{T}^*_{U_a}U_b, \mathcal{T}^*_{U_a}U_b)\right]\right\}\\ &= 2\overline{R} - 2r^2||H^0||^2 + \frac{r^2}{2}(||H||^2 + ||H^*||^2) + \sum_{i=1}^{n}\sum_{a,b=1}^{r}\left\{2(\mathcal{T}^{0i}_{ab})^2\right.\\ &\left. - \frac{1}{2}\left[(\mathcal{T}^i_{ab})^2 + (\mathcal{T}^{*i}_{ab})^2\right]\right\}.\end{aligned} \tag{4.3}$$

For local orthonormal frame $\{X_i, U_a\}_{1\le i\le n, 1\le a\le r}$ on M, one can derive

$$\begin{aligned}\sum_{i=1}^{n}\sum_{a,b=1}^{r}(\mathcal{T}^i_{ab})^2 &= \frac{r^2}{2}||H||^2 + \frac{1}{2}\sum_{i=1}^{n}[\mathcal{T}^i_{11} - \cdots - \mathcal{T}^i_{rr}]^2\\ &+ 2\sum_{i=1}^{n}\sum_{b=2}^{r}(\mathcal{T}^i_{1b})^2 - 2\sum_{i=1}^{n}\sum_{2\le a\le b\le r}\left(\mathcal{T}^i_{aa}\mathcal{T}^i_{bb} - (\mathcal{T}^i_{ab})^2\right),\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n}\sum_{a,b=1}^{r}(\mathcal{T}^{*i}_{ab})^2 &= \frac{r^2}{2}||H^*||^2 + \frac{1}{2}\sum_{i=1}^{n}\left(\mathcal{T}^{*i}_{11} - \cdots - \mathcal{T}^{*i}_{rr}\right)^2\\ &+ 2\sum_{i=1}^{n}\sum_{b=2}^{r}(\mathcal{T}^{*i}_{1b})^2 - 2\sum_{i=1}^{n}\sum_{2\le a<b\le r}\left(\mathcal{T}^{*i}_{aa}\mathcal{T}^{*i}_{bb} - (\mathcal{T}^{*i}_{ab})^2\right).\end{aligned}$$

Combining the above two equalities, we find

$$\begin{aligned}\sum_{i=1}^{n}\sum_{a,b=1}^{r}[(\mathcal{T}^i_{ab})^2 + (\mathcal{T}^{*i}_{ab})^2] &\ge 2\sum_{i=1}^{n}\sum_{2\le a<b\le r}\mathcal{T}^i_{aa}\mathcal{T}^{*i}_{bb} + \sum_{i=1}^{n}\sum_{2\le a<b\le r}\left((\mathcal{T}^i_{ab})^2\right.\\ &\left. + (\mathcal{T}^{*i}_{ab})^2\right) - \sum_{i=1}^{n}\sum_{2\le a<b\le r}\left(\mathcal{T}^i_{aa} + \mathcal{T}^{*i}_{aa}\right)\left(\mathcal{T}^i_{bb} + \mathcal{T}^{*i}_{bb}\right)\\ &+ \frac{r^2}{2}\left(||H||^2 + ||H^*||^2\right).\end{aligned} \tag{4.4}$$

Substituting (4.4) into (4.3), it follows that

$$\begin{aligned}\frac{c}{4}r(r-1) &\le 2\overline{R} - 2r^2||H^0||^2 + \frac{r^2}{2}(||H||^2 + ||H^*||^2) + 2\sum_{i=1}^{n}\sum_{a,b=1}^{r}(\mathcal{T}^{0i}_{ab})^2\\ &- \sum_{i=1}^{n}\sum_{2\le a<b\le r}\mathcal{T}^i_{aa}\mathcal{T}^{*i}_{bb} - \frac{1}{2}\sum_{i=1}^{n}\sum_{2\le a<b\le r}[(\mathcal{T}^i_{ab})^2 + (\mathcal{T}^{*i}_{ab})^2]\\ &+ 2\sum_{i=1}^{n}\sum_{2\le a<b\le r}\mathcal{T}^{0i}_{aa}\mathcal{T}^{0i}_{bb} - \frac{r^2}{4}(||H||^2 + ||H^*||^2).\end{aligned} \tag{4.5}$$

Using (2.5), we derive

$$\begin{aligned}&\frac{c}{4}(r-1)(r-2)+\sum_{2\leq a<b\leq r}\big[-g_M(U_b,\overline{J}U_b)g_M(U_a,\overline{J}U_a)+g_M(U_a,\overline{J}U_b)g_M(U_a,\overline{J}U_b)\\&+g_M(U_a,\overline{J}U_b)g_M(U_a,\overline{J}U_b)-g_M(U_b,\overline{J}U_a)g_M(U_a,\overline{J}U_b)\big]\\&=\sum_{2\leq a<b\leq r}g_M(\overline{Rie}(U_a,U_b)U_b,U_a)-\sum_{i=1}^{n}\sum_{2\leq a<b\leq r}(\mathcal{T}^i_{aa}\mathcal{T}^{*i}_{bb}-\mathcal{T}^i_{ab}\mathcal{T}^{*i}_{ab}).\end{aligned}\tag{4.6}$$

With the help of (4.6), the equation (4.5) can be written as

$$\begin{aligned}\frac{c}{4}\left(r(r-1)\right)\leq{}&2\overline{R}-2r^2||H^0||^2+\frac{r^2}{4}(||H||^2+||H^*||^2)+2\sum_{i=1}^{n}\sum_{a,b=1}^{r}(\mathcal{T}^{0i}_{ab})^2\\&+2\sum_{i=1}^{n}\sum_{2\leq a<b\leq r}\mathcal{T}^{0i}_{aa}\mathcal{T}^{0i}_{bb}-\sum_{2\leq a<b\leq r}g_M(\overline{Rie}(U_a,U_b)U_b,U_a)\\&-\frac{1}{2}\sum_{i=1}^{n}\sum_{2\leq a<b\leq r}(\mathcal{T}^i_{ab}+\mathcal{T}^{*i}_{ab})^2+\frac{c}{4}(r-1)(r-2)\\&+\frac{c}{4}\sum_{2\leq a<b\leq r}\big[-g_M(U_b,\overline{J}U_b)g_M(U_a,\overline{J}U_a)+g_M(U_a,\overline{J}U_b)g_M(U_a,\overline{J}U_b)\\&+g_M(U_a,\overline{J}U_b)g_M(U_a,\overline{J}U_b)-g_M(U_b,\overline{J}U_a)g_M(U_a,\overline{J}U_b)\big].\end{aligned}\tag{4.7}$$

Thus, we have

$$\begin{aligned}\overline{Ric}(U)\geq{}&\frac{c}{4}\left((r-1)\right)+r^2||H^0||^2-\frac{r^2}{8}(||H||^2+||H^*||^2)-\sum_{i=1}^{n}\sum_{a,b=1}^{r}(\mathcal{T}^{0i}_{ab})^2\\&-\sum_{i=1}^{n}\sum_{2\leq a<b\leq r}[\mathcal{T}^{0i}_{aa}\mathcal{T}^{0i}_{bb}-(\mathcal{T}^{0i}_{ab})^2],\end{aligned}\tag{4.8}$$

where $\overline{Ric}$ is the Ricci curvature of each fibre with respect to the induced affine connection.

Again, by the Gauss equation with respect to the Levi-Civita connection, we arrive at

$$\sum_{1\leq a<b\leq r}g_M(Rie^0(U_a,U_b)U_b,U_a)=2\overline{R}^0-r^2||H^0||^2+\sum_{i=1}^{n}\sum_{a,b=1}^{r}(\mathcal{T}^{0i}_{ab})^2,\tag{4.9}$$

$$\begin{aligned}\sum_{2\leq a<b\leq r}g_M(Rie^0(U_a,U_b)U_b,U_a)={}&\sum_{2\leq a<b\leq r}g_M(\overline{Rie}^0(U_a,U_b)U_b,U_a)\\&-\sum_{i=1}^{n}\sum_{2\leq a<b\leq r}[\mathcal{T}^{0i}_{aa}\mathcal{T}^{0i}_{bb}-(\mathcal{T}^{0i}_{ab})^2],\end{aligned}\tag{4.10}$$

where $\overline{R}^0$ is the scalar curvature of each fibre with respect to the induced Levi-Civita connection.

On substituting (4.9) and (4.10) into (4.8), we obtain

$$\overline{Ric}(U)\geq 2\overline{Ric}^0(U)+\frac{c}{4}\left((r-1)\right)-\frac{r^2}{8}(||H||^2+||H^*||^2)-2\sum_{a=2}^{r}K^0(U\wedge U_a),$$

where $\overline{Ric}^0$ is the Ricci curvature of each fibre with respect to the induced Levi-Civita connection.

By summing up, we have the Chen-Ricci inequality for a Lagrangian statistical submersion as follows:

Theorem 4.1. *Let $\psi : (M, \nabla, g) \to (B, \widetilde{\nabla}, g_B)$ be a Lagrangian statistical submersion from Kähler-like statistical manifold whose curvature tensor Rie is of the form (1.1) onto a statistical manifold. Then, for each unit vector field $U \in \mathcal{V}_p(M)$, we have*

$$\overline{Ric}(U) \geq 2\overline{Ric}^0(U) + \frac{c}{4}(r-1) - \frac{r^2}{8}(||H||^2 + ||H^*||^2) - 2(r-1)\max K^0(U \wedge \cdot), \tag{4.11}$$

where $\max K^0(U \wedge \cdot)$ denotes the maximum of the sectional curvature function of M with respect to ∇ restricted to 2-plane sections of $\mathcal{V}_p(M)$, $p \in M$, which are orthogonal to U. The equality holds in the inequality (4.11) if and only if

(1) $2\mathcal{T}_U U = rH(p)$, $\quad \mathcal{T}_U V = 0$, $V \in \mathcal{V}_p(M)$ orthogonal to U.

(2) $2\mathcal{T}^*_U U = rH^*(p)$, $\quad \mathcal{T}^*_U V = 0$, $V \in \mathcal{V}_p(M)$ orthogonal to U.

An immediate consequence of Theorem 4.1 is the following:

Corollary 4.1. *Let $\psi : (M, \nabla, g) \to (B, \widetilde{\nabla}, g_B)$ be a Lagrangian statistical submersion from Kähler-like statistical manifold whose curvature tensor Rie is of the form (1.1) onto a statistical manifold. If*

$$\overline{Ric}(U) < 2\overline{Ric}^0(U) + \frac{c}{4}(r-1) - 2(r-1)\max K^0(U \wedge \cdot),$$

then neither $H \neq 0$ nor $H^* \neq 0$.

Remark 4.1. Similar inequalities can be stated for the Ricci curvature $\overline{Ric}^*$.

Now, we give the following non-trivial example of a Kähler-like statistical submersion:

Example 4.1. The Euclidean space $\mathbb{R}^4 = \left\{\{x_1, x_2, x_3, x_4\}, x_1, x_2 > 0\right\}$ is a Kähler-like statistical manifold with the following almost complex structures:

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J^* = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix},$$

the metric on $\mathbb{R}^4$ is defined as

$$g_{\mathbb{R}^4} = (x_1)^{-1}\left\{(dx_1)^2 + (dx_3)^2\right\} + (x_2)^{-1}\left\{(dx_2)^2 + (dx_4)^2\right\}$$

and the affine connection $\nabla^{\mathbb{R}^4}$ is given by

$$\begin{aligned} &\nabla^{\mathbb{R}^4}_{\partial x_1}\partial x_1 = \nabla^{\mathbb{R}^4}_{\partial x_2}\partial x_2 = 0, \quad \nabla^{\mathbb{R}^4}_{\partial x_3}\partial x_3 = \nabla^{\mathbb{R}^4}_{\partial x_4}\partial x_4 = 0, \\ &\nabla^{\mathbb{R}^4}_{\partial x_1}\partial x_3 = \nabla^{\mathbb{R}^4}_{\partial x_3}\partial x_1 = (x_1)^{-1}\partial x_3, \quad \nabla^{\mathbb{R}^4}_{\partial x_2}\partial x_4 = \nabla^{\mathbb{R}^4}_{\partial x_4}\partial x_2 = (x_2)^{-1}\partial x_4. \end{aligned}$$

Now, we define a Kähler-like statistical submersion

$$\psi : (\mathbb{R}^4, \nabla^{\mathbb{R}^4}, g_{\mathbb{R}^4}) \to (\mathbb{R}^2, \nabla^{\mathbb{R}^2}, g_{\mathbb{R}^2})$$

as the projection mapping

$$\psi(x_1, x_2, x_3, x_4) = (x_3, x_4).$$

Then ψ has isometric fibers.

5. Relevant problems

In section 3, we deal with Chen-Ricci inequality for a Lagrangian statistical submersion $\psi : (M, \nabla, g) \to (B, \widetilde{\nabla}, g_B)$, that is, the estimate of the Ricci curvature $\overline{Ric}$ of each fibre $(\overline{M}, \overline{\nabla}, \overline{g}, \overline{J})$ with respect to the induced affine connection. And we also consider equality case. As future projects, we can use these results to study the properties of the total space M and the base space B and also investigate other equality cases and their applications. We will also estimate $\overline{Ric}$ by changing M. The results stated here motivate further studies to obtain similar relationships for many kinds of invariants of similar nature for several statistical submersions.

We have the following question:

1. Is the converse of Theorem 4.1 true ?

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