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Research Article

The Third Order Variant Narayana Codes and Some Straight Lines Corresponding to These

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ABSTRACT

In this study, firstly, we examined the second order variant Narayana codes and we got some results from the tables were displayed by Das and Sinha[11]. Then, we studied on the third order variant Narayana code and we displayed these codes for some k positive integers and with tables. Also, we got some results from the tables. Then, we compared the results that we obtained from the tables for the third order variant Narayana universal code and the second order variant Narayana universal code in terms of cryptography. We found that third order variant Narayana universal codes are much more advantageous than the second order variant Narayana universal codes. Finally, we obtained some straight lines which yielding the some the third order Narayana codewords by considering (u, k) as a point in the (x, y) plane, from these tables.

Keywords: *Fibonacci sequence, Narayana universal code, cryptography*

Üçüncü Mertebeden Varyant Narayana Kodları ve Bunlara Karşılık Gelen Bazı Doğrular

ÖZ

Bu çalışmada, ilk olarak, ikinci mertebeden variant Narayana kodlarını inceledik ve bu kodlar ile ilgili Das ve Sinha[11] tarafından elde edilen tablolardan bazı sonuçlar elde ettik. Ardından üçüncü mertebeden variant Narayana kodları üzerine çalıştık ve bazı pozitif k tam sayıları için bu kodları tablolar ile belirledik. Ayrıca, bu tablolardan bazı sonuçlar elde ettik. Sonrasında, ikinci ve üçüncü mertebeden variant Narayana kodları için tablolardan elde ettiğimiz sonuçları kriptografik açıdan karşılaştırdık. Üçüncü mertebeden variant Narayana kodlarının çok daha avantajlı olduğunu elde ettik. Son olarak (u, k) yı (x, y) düzleminde bir nokta olarak kabul ederek tablolara göre bazı Narayana kodlarını veren bazı doğrular elde ettik.

Anahtar Kelimeler: Fibonacci dizisi, Narayana evrensel kodu, kriptografi

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I. INTRODUCTION

The Fibonacci sequence, $\{F_k\}_0^\infty$, is a sequence of numbers, beginning with the integer couple 0 and 1, in which the value of any element is computed by taking the summation of the two antecedent numbers. If so, for $k \geq 2$, $F_k = F_{k-1} + F_{k-2}$ [1]. This number sequence, which was previously found by Indian mathematicians in the sixth century. But the sequence was introduced by Fibonacci as a result of calculating the problem related to the reproduction of rabbits in the book called Liber Abaci in 1202. The first eight terms of this sequence are $1, 1, 2, 3, 5, 8, 13, 21$.

The Narayana sequence, $\{N_k\}_0^\infty$, is a sequence of numbers, beginning with the integers 1, 1 and 1, in which the value of any element is computed by taking the summation of the previous term and term two places before it. If so, for $k \geq 3$, $N_{k+1} = N_k + N_{k-2}$ [1]. This number sequence derived from the was introduced by Narayana as the result of calculating the problem related to the birth cows in the book called Gaṇita Kaumudi in 1356. The first eight terms of this sequence are $1, 1, 1, 2, 3, 4, 6, 9$.

By using these sequences, it can be obtained universal codes. A universal code transforms positive integers representing source messages into code words of different lengths. There are various universal codes such as the Elias codes, the Fibonacci universal code, Narayana code and non-universal codes such as Rice coding, Huffman coding and Golomb coding [2-4]. The best known of them is the Fibonacci code. The Fibonacci code is more useful in comparison with other universal codes. Because this code easily fixs data from damaged parts of codewords [5]. Fibonacci and Gopala-Hemachandra universal codes encode positive integers with binary representations and these code words are obtained based on Zeckendorf representation. Each positive integer has one and only one representation as the summation of non-sequential Fibonacci numbers according to Zeckendorf's theorem [6].

In this study, firstly, we examined the second order variant Narayana universal code which is described by Kirthi and Kak[5] and is extended by Das and Sinha[11]. And from the tables were obtained by Das and Sinha[11], we got some results. Then, we studied on the third order variant Narayana code and we displayed with tables these codes ($VN_u^{(3)}(k)$) of some k positive integers ($1 \leq k \leq 50$) for $-20 \leq u \leq -1$. Also, we got some results from the tables. Also, we compared the results that obtained from the tables for the third order variant Narayana universal code and the second order variant Narayana universal code cryptographically. Here, we obtained there is no Zeckendorf's representation for the first integer $k = 7$ in the third order variant Narayana sequence $VN_{-7}^{(3)}(k) = \{-7, 10, 8, 1, 11, 12, 23, 42\}$ and so $VN_{-1}^{(3)}, VN_{-2}^{(3)}, VN_{-3}^{(3)}, VN_{-4}^{(3)}, VN_{-5}^{(3)}, VN_{-6}^{(3)}$ can be used in cryptography. Since none of the second order variant Narayana codes can be used in cryptography, we found that third order variant Narayana universal code are much more advantageous than the second order variant Narayana universal code. Finally, we obtained some straight lines which yielding the some the third order Narayana codewords by considering (u, k) as a point in the (x, y) plane.

II. MATERIALS AND METHODS

Here, variant Fibonacci sequences and codes of different orders and variant Narayana sequences and codes of different orders are used as material.

Definition 2.1. The mth order Fibonacci numbers, that are represented by $F_k^{(m)}$, are described with iteration relation $F_k^{(m)} = F_{k-1}^{(m)} + F_{k-2}^{(m)} + \cdots + F_{k-m}^{(m)}$ for $k > 0$ and the boundary conditions $F_0^{(m)} = 1$ and $F_k^{(m)} = 0$ $(-m < k < 0)$ [7].

Fibonacci code can be obtained for each positive integer A with a binary string of length t, $l_1 l_2 \ldots l_{t-1} l_t$, such that $A = \sum_{i=1}^{t} l_i F_i^{(m)}$. The representation is unique only if one uses algorithm to find it as follows: When it is given the integer A, it is detected the largest Fibonacci number F_t equivalent or smaller to A; after that it is continued repeating with [7]. For instance $17 = 1 + 3 + 13$, hence its Fibonacci representation is 101001.

According to above algorithm, Fibonacci numbers aren't used consecutively in any of these summations, that is, in the binary representation, there are no contiguous 1 bits. When generalizing this procedure to higher orders, the same operations are realized as above. Additionally, it is appended $(m - 1)$ 1 bits to the mth order variant of Fibonacci representation of k to build the mth order variant of Fibonacci code of any positive integer k. But, unlike the Fibonacci representation, in the statement, codeword can't contain m contiguous 1 bits [7].
It was presented a second order variant Fibonacci code by using the Gopala-Hemachandra sequence for $m = 2$ and this sequence is defined as follows:

Definition 2.2. The second order variant Fibonacci sequence, $GH_u^{(2)}(k)$ is described with the sequence $\{u, v, u + v, u + 2v, 2u + 3v, 3u + 5v, ...\}$ where $v = 1 - u$ that is, $GH_u^{(2)}(1) = u$; $GH_u^{(2)}(2) = 1 - u$; and for $k \geq 3$, $GH_u^{(2)}(k) = GH_u^{(2)}(k - 1) + GH_u^{(2)}(k - 2)$ [8].

Afterwards, it was presented a third order variant Fibonacci code by using the Gopala-Hemachandra sequence for $m = 3$ and this sequence is defined as follows:

Definition 2.3. The third order variant of Fibonacci sequence, $GH_u^{(3)}(k)$ is described with the sequence $\{u, v, u + v, 2u + 2v, 3u + 4v, 6u + 7v, ...\}$ where $v = 1 - u$ that is, $GH_u^{(3)}(1) = u$; $GH_u^{(3)}(2) = 1 - u$; $GH_u^{(3)}(3) = 1$; and for $k \geq 4$, $GH_u^{(3)}(k) = GH_u^{(3)}(k - 1) + GH_u^{(3)}(k - 2) + GH_u^{(3)}(k - 3)$ [9].

For different values of u, it is obtained different sequences. We know that it was demonstrated that solely Fibonacci sequence forms a unique Fibonacci code for all positive integers by Daykin [10]. However, in variants of Fibonacci sequences, some integers have more than one Gopala-Hemachandra codes while some integers have Gopala-Hemachandra no code. For example, for the second order variant of Fibonacci sequence while $VF_{-5}^{(2)}(k) = \{-5, 6, 1, 7, 8, 15, 23, 38\}$ there is no Gopala-Hemachandra code for integer $k = 5{,}12$ [2], $VF_{-2}^{(2)}(k) = \{-2, 3, 1, 4, 5, 9, 14, 23\}$ there are two Gopala-Hemachandra codes for integer $k = 8$. These codes are 010011 and 1010011. Similarly, for the third order variant of Fibonacci sequence while $VF_{-11}^{(2)}(k) = \{-11, 12, 1, 13, 14, 27, 41, 68\}$ there is no Gopala-Hemachandra code for integer $k = 11$ [9], $VF_{-4}^{(3)}(k) = \{-4, 5, 1, 2, 8, 11, 21, 40\}$ there are two Gopala-Hemachandra codes for integer $k = 7$. These codes are 10000111 and 010111. In addition, the authors made cryptographic applications by using GH codes [9].

Narayana sequences are similar to the Fibonacci and Gopala-Hemachandra (GH) sequences in terms of their use in cryptographic applications and data coding. Kirthi and Kak described a variant Narayana coding scheme as follows:

Definition 2.4. A variant of Narayana coding scheme can be obtained by defining second order variant Narayana sequence $\{u, v, z, u + z, u + v + z, u + v + 2z, ...\}$, $u \in$ □ , $VN_u^{(2)}(k)$, such that $v = 3 - u$ and $z = 1 - u$. This yields $VN_u^{(2)}(1) = u$; $VN_u^{(2)}(2) = 3 - u$; $VN_u^{(2)}(2) = 3 - u$ and for $k \geq 3$, $VN_u^{(2)}(k) = VN_u^{(2)}(k - 1) + VN_u^{(2)}(k - 3)$ [5].
With the above definition, we obtain there is no Zeckendorf representation for integers 3 and 15 using the sequence $VN_{-1}^{(2)}(k) = \{-1, 4, 2, 1, 5, 7, 8, 13\}$, and integers 2, 13 and 19 can't be represented using sequence $VN_{-3}^{(2)}(k) = \{-3, 6, 4, 1, 7, 11, 12, 19\}$.

We know that variant Narayana universal coding is used in source coding as well as in cryptography. But some variant Narayana codes ($VN_u^{(3)}(k)$) which lack the ability to encode for certain values of k can't be used in cryptography. So, it is important to determine these codes. Here, we obtained there is no Zeckendorf's representation for the first integer $k = 7$ in the third order variant Narayana sequence $VN_{-7}^{(3)}(k) = \{-7, 10, 8, 1, 11, 12, 23, 42\}$. But we obtained there is no Zeckendorf's representation for the first integer $k = 3$ in the second order variant Narayana sequence $VN_{-1}^{(2)}(k) = \{-1, 4, 2, 1, 5, 7, 8, 13\}$. So, while $VN_{-1}^{(3)}, VN_{-2}^{(3)}, VN_{-3}^{(3)}, VN_{-4}^{(3)}, VN_{-5}^{(3)}, VN_{-6}^{(3)}$ can be used in cryptography for the third order variant Narayana codes, none of the second order variant Narayana codes can be used in cryptography.

III. RESULTS AND DISCUSSION

A. SOME RESULTS ABOUT THE SECOND ORDER VARIANT NARAYANA CODES

In this study, firstly, we examined the second order variant Narayana codes and from the tables were displayed by Das and Sinha[11]. We obtained

i. For the only positive integer $k = 1$, the second order variant Narayana code $VN_u^{(2)}(k)$ exactly exists for $u = -1, -2, \dots, -20$.

ii. For $1 \leq k \leq 50$, there is at most j consecutive undetectable values (NA) the second order variant of Narayana code in $VN_{-j}^{(2)}(k)$ column in which $1 \leq j \leq 20$.

iii. As long as j raises, the detectable of Narayana code is reduced in $VN_{-j}^{(2)}(k)$ column in which $1 \leq j \leq 20$.

B. THE THIRD ORDER VARIANT NARAYANA SEQUENCES AND CODES

We described the third order variant Narayana sequences as follows:

Definition 3.5. The third order variant Narayana sequences, $VN_u^{(3)}(k)$ is described with the sequence $\{u, v, z, u+z, u+v+z, 2u+v+2z, 3u+2v+3z, \dots\}$ where $v = 3 - u$ and $z = 1 - u$, that is, $VN_u^{(3)}(1) = u$; $VN_u^{(3)}(2) = 3 - u$; $VN_u^{(3)}(3) = 1 - u$; $VN_u^{(3)}(4) = 1$; $VN_u^{(3)}(5) = 4 - u$ and for $k \geq 6$, $VN_u^{(3)}(k) = VN_u^{(3)}(k-1) + VN_u^{(3)}(k-3) + VN_u^{(3)}(k-5)$.

The third order variant Narayana coding scheme can be obtained by defining third order variant Narayana sequence, $VN_u^{(3)}(k)$, such that $v = 3 - u$ and $z = 1 - u$. In variants Narayana sequences, some integers have more than one Narayana codes while others have no Narayana code. For example, for the second order variant Narayana sequence $VN_{-1}^{(2)}(k) = \{-1, 4, 2, 1, 5, 7, 8, 13\}$ while there is no Narayana code for integers $k = 2$ and $k = 11$ [11], there are two Narayana codes for integer $k = 4$. These codes are 011 and 100011. Similarly, for the third order variant Narayana sequence $VN_{-7}^{(3)}(k) = \{-7, 10, 8, 1, 11, 12, 23, 42\}$ while there is no Narayana code for integers $k = 7$ and $k = 40$, there are two Narayana codes for integer $k = 1$. These codes are 000111 and 10111. In this section, we obtained the third order variant of Narayana codes $VN_u^{(3)}(k)$ or undetectable values (NA) of the positive integer k for $1 \leq k \leq 50$ and for $u = -1, -2, \dots, -20$ with Tables 1-2. From the tables, we got the following results for the third order variant Narayana codes:

i. For the positive integers $k = 1, 2, 3, 4, 5, 6$, the third order variant of Narayana code $VN_u^{(3)}(k)$ exactly exists for $u = -1, -2, \dots, -20$.

ii. For $1 \leq k \leq 50$, there is at most j consecutive undetectable values (NA) the third order variant of Narayana code in $VN_{-(6+j)}^{(3)}(k)$ column in which $1 \leq j \leq 14$.

iii. For $1 \leq k \leq 50$, as long as j raises, the detectable of third order variant Narayana code is reduced in $VN^{(3)}_{-(6+j)}(k)$ column in which $1 \leq j \leq 14$.

Table 1. *The third order variant Narayana codes of* k *for* $(1 \leq k \leq 50)$ *and* $u = -1, -2, \ldots, -10$

k	$VN^{(3)}_{-1}(k)$	$VN^{(3)}_{-2}(k)$	$VN^{(3)}_{-3}(k)$	$VN^{(3)}_{-4}(k)$	$VN^{(3)}_{-5}(k)$	$VN^{(3)}_{-6}(k)$	$VN^{(3)}_{-7}(k)$	$VN^{(3)}_{-8}(k)$	$VN^{(3)}_{-9}(k)$	$VN^{(3)}_{-10}(k)$
1	**000111**	**000111**	**000111**	**000111**	**000111**	**000111**	**000111**	**000111**	**000111**	**000111**
2	**101111**	**101111**	**101111**	**101111**	**101111**	**101111**	**101111**	**101111**	**101111**	**101111**
3	**1111**	**1111**	**1111**	**1111**	**1111**	**1111**	**1111**	**1111**	**1111**	**1111**
4	**1000111**	**1000111**	**1000111**	**1000111**	**1000111**	**1000111**	**1000111**	**1000111**	**1000111**	**1000111**
5	**10000111**	**10000111**	**10000111**	**10000111**	**10000111**	**10000111**	**10000111**	**10000111**	**10000111**	**10000111**
6	**10010111**	**10010111**	**10010111**	**10010111**	**10010111**	**10010111**	**10010111**	**10010111**	**10010111**	**10010111**
7	**00010111**	**00000111**	**0000111**	**0111**	**001111**	**00111**	**NA**	**NA**	**NA**	**NA**
8	**00100111**	**00010111**	**00000111**	**0000111**	**0111**	**001111**	**00111**	**NA**	**NA**	**NA**
9	**0100111**	**0010111**	**00010111**	**00000111**	**0000111**	**0111**	**001111**	**00111**	**NA**	**NA**
10	**100000111**	**00100111**	**1100111**	**00010111**	**00000111**	**0000111**	**10111**	**001111**	**00111**	**NA**
11	**000000111**	**0100111**	**0010111**	**1100111**	**00010111**	**00000111**	**0000111**	**0111**	**001111**	**00111**
12	**000100111**	**01000111**	**00100111**	**11000111**	**1100111**	**00010111**	**00000111**	**0000111**	**0111**	**001111**
13	**001000111**	**000000111**	**0100111**	**0010111**	**11000111**	**1100111**	**00010111**	**00000111**	**0000111**	**0111**
14	**001100111**	**000100111**	**01000111**	**00100111**	**100000111**	**11000111**	**1100111**	**00010111**	**00000111**	**0000111**
15	**010000111**	**101100111**	**000000111**	**0100111**	**0010111**	**100000111**	**11000111**	**1100111**	**00010111**	**00000111**
16	**000010111**	**001000111**	**000100111**	**01000111**	**00100111**	**100100111**	**100000111**	**11000111**	**1100111**	**00010111**
17	**1000000111**	**001100111**	**010000111**	**000000111**	**0100111**	**0010111**	**100100111**	**100000111**	**11000111**	**1100111**
18	**0000000111**	**010000111**	**110000111**	**000100111**	**01000111**	**00100111**	**01111**	**100100111**	**100000111**	**11000111**
19	**0001000111**	**000010111**	**001000111**	**010000111**	**000000111**	**0100111**	**0010111**	**NA**	**100100111**	**100000111**
20	**0010000111**	**1000000111**	**001100111**	**110000111**	**000100111**	**01000111**	**00100111**	**01111**	**NA**	**100100111**
21	**0011000111**	**1001000111**	**010000111**	**100010111**	**010000111**	**000000111**	**0100111**	**0010111**	**NA**	**NA**
22	**0100000111**	**0000000111**	**000010111**	**001000111**	**110000111**	**000100111**	**01000111**	**00100111**	**01111**	**NA**
23	**1000100111**	**0001000111**	**1000000111**	**001100111**	**100010111**	**010000111**	**000000111**	**0100111**	**0010111**	**NA**
24	**0000010111**	**1011000111**	**1001000111**	**010000111**	**100001111**	**110000111**	**000100111**	**01000111**	**00100111**	**01111**
25	**00000000111**	**0010000111**	**011000111**	**000010111**	**001000111**	**100010111**	**010000111**	**000000111**	**0100111**	**0010111**
26	**00010000111**	**0011000111**	**0000000111**	**1000000111**	**001100111**	**100001111**	**110000111**	**000100111**	**01000111**	**00100111**
27	**10110000111**	**0100000111**	**0001000111**	**1001000111**	**010000111**	**100101111**	**100010111**	**010000111**	**000000111**	**0100111**
28	**11000000111**	**1000100111**	**1011000111**	**110010111**	**000010111**	**001000111**	**100001111**	**110000111**	**000100111**	**01000111**
29	**01000000111**	**0000010111**	**1100000111**	**011000111**	**1000000111**	**001100111**	**100101111**	**100010111**	**010000111**	**000000111**
30	**00001000111**	**00000000111**	**0010000111**	**0000000111**	**1001000111**	**010000111**	**01100111**	**100001111**	**110000111**	**000100111**
31	**00000100111**	**00010000111**	**0011000111**	**0001000111**	**110010111**	**000010111**	**001000111**	**100101111**	**100010111**	**010000111**
32	**00010100111**	**10110000111**	**0100000111**	**1011000111**	**110001111**	**1000000111**	**001100111**	**0110111**	**100001111**	**110000111**
33	**00100100111**	**11000000111**	**1000100111**	**1100000111**	**011000111**	**1001000111**	**010000111**	**01100111**	**100101111**	**100010111**
34	**00110100111**	**10001000111**	**0000010111**	**1000100111**	**0000000111**	**110010111**	**000010111**	**001000111**	**NA**	**100001111**
35	**10000010111**	**01000000111**	**00000000111**	**0010000111**	**0001000111**	**110001111**	**1000000111**	**001100111**	**0110111**	**100101111**
36	**00000010111**	**00001000111**	**00010000111**	**0011000111**	**1011000111**	**110101111**	**1001000111**	**010000111**	**01100111**	**NA**
37	**00010010111**	**00000100111**	**10110000111**	**0100000111**	**1100000111**	**011000111**	**110010111**	**000010111**	**001000111**	**NA**
38	**00100010111**	**00010100111**	**11000000111**	**1000100111**	**1000100111**	**0000000111**	**110001111**	**1000000111**	**001100111**	**0110111**
39	**00110010111**	**00101000111**	**10001000111**	**0000010111**	**1000010111**	**0001000111**	**110101111**	**1001000111**	**010000111**	**01100111**
40	**01000010111**	**00100100111**	**10000100111**	**00000000111**	**0010000111**	**1011000111**	**NA**	**110010111**	**000010111**	**001000111**
41	**000000000111**	**00110100111**	**01000000111**	**00010000111**	**0011000111**	**1100000111**	**011000111**	**110001111**	**1000000111**	**001100111**
42	**000100000111**	**01000100111**	**00001000111**	**10110000111**	**0100000111**	**1000100111**	**0000000111**	**110101111**	**1001000111**	**010000111**
43	**001000000111**	**00000010111**	**00000100111**	**11000000111**	**1000100111**	**1000010111**	**0001000111**	**NA**	**110010111**	**000010111**

44	001100000111	00010010111	00010100111	10001000111	0000010111	1001010111	1011000111	NA	110001111	1000000111
45	010000000111	10110010111	10110100111	10000100111	00000000111	0010000111	1100000111	011000111	110101111	1001000111
46	000010000111	00100010111	00101000111	0100010111	00010000111	0011000111	1000100111	0000000111	NA	110010111
47	000001000111	00110010111	00100100111	01000000111	10110000111	0100000111	1000010111	0001000111	NA	110001111
48	000101000111	01000010111	00110100111	00001000111	11000000111	1000100111	1001010111	1011000111	NA	110101111
49	001001000111	000000000111	01000100111	00000100111	10001000111	0000010111	10010000111	1100000111	011000111	NA
50	001101000111	000100000111	00000010111	00010100111	10000100111	00000000111	0010000111	1000100111	0000000111	NA

Table 2. *The third order variant Narayana codes of* k *for* $(1 \le k \le 50)$ *and* $u = -11, -12, \ldots, -20$

k	$VN_{-11}^{(3)}(k)$	$VN_{-12}^{(3)}(k)$	$VN_{-13}^{(3)}(k)$	$VN_{-14}^{(3)}(k)$	$VN_{-15}^{(3)}(k)$	$VN_{-16}^{(3)}(k)$	$VN_{-17}^{(3)}(k)$	$VN_{-18}^{(3)}(k)$	$VN_{-19}^{(3)}(k)$	$VN_{-20}^{(3)}(k)$
1	000111	000111	000111	000111	000111	000111	000111	000111	000111	000111
2	101111	101111	101111	101111	101111	101111	101111	101111	101111	101111
3	1111	1111	1111	1111	1111	1111	1111	1111	1111	1111
4	1000111	1000111	1000111	1000111	1000111	1000111	1000111	1000111	1000111	1000111
5	10000111	10000111	10000111	10000111	10000111	10000111	10000111	10000111	10000111	10000111
6	10010111	10010111	10010111	10010111	10010111	10010111	10010111	10010111	10010111	10010111
7	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
8	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
9	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
10	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
11	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
12	00111	NA	NA	NA	NA	NA	NA	NA	NA	NA
13	001111	00111	NA	NA	NA	NA	NA	NA	NA	NA
14	0111	001111	00111	NA	NA	NA	NA	NA	NA	NA
15	0000111	0111	001111	00111	NA	NA	NA	NA	NA	NA
16	00000111	0000111	0111	001111	00111	NA	NA	NA	NA	NA
17	00010111	00000111	0000111	0111	001111	00111	NA	NA	NA	NA
18	10110111	00010111	00000111	0000111	0111	001111	00111	NA	NA	NA
19	11000111	10110111	00010111	00000111	0000111	0111	001111	00111	NA	NA
20	100000111	11000111	10110111	00010111	00000111	0000111	0111	001111	00111	NA
21	100100111	100000111	11000111	10110111	00010111	00000111	0000111	0111	001111	00111
22	NA	100100111	100000111	11000111	10110111	00010111	00000111	0000111	0111	001111
23	NA	NA	100100111	100000111	11000111	10110111	00010111	00000111	0000111	0111
24	NA	NA	NA	100100111	100000111	11000111	10110111	00010111	00000111	0000111
25	NA	NA	NA	NA	100100111	100000111	11000111	10110111	00010111	00000111
26	01111	NA	NA	NA	NA	100100111	100000111	11000111	10110111	00010111
27	0010111	NA	NA	NA	NA	NA	100100111	100000111	11000111	10110111
28	00100111	01111	NA	NA	NA	NA	NA	100100111	100000111	11000111
29	0100111	0010111	NA	NA	NA	NA	NA	NA	100100111	100000111
30	01000111	00100111	01111	NA	NA	NA	NA	NA	NA	100100111
31	000000111	0100111	0010111	NA	NA	NA	NA	NA	NA	NA
32	000100111	01000111	00100111	01111	NA	NA	NA	NA	NA	NA
33	010000111	000000111	0100111	0010111	NA	NA	NA	NA	NA	NA
34	110000111	000100111	01000111	00100111	01111	NA	NA	NA	NA	NA
35	100010111	010000111	000000111	0100111	0010111	NA	NA	NA	NA	NA
36	100001111	110000111	000100111	01000111	00100111	01111	NA	NA	NA	NA
37	100101111	100010111	010000111	000000111	0100111	0010111	NA	NA	NA	NA
38	NA	100001111	110000111	000100111	01000111	00100111	01111	NA	NA	NA

39	**NA**	**100101111**	**100010111**	**010000111**	**000000111**	**0100111**	**0010111**	**NA**	**NA**	**NA**
40	**NA**	**NA**	**100001111**	**110000111**	**000100111**	**01000111**	**00100111**	**01111**	**NA**	**NA**
41	**0110111**	**NA**	**100101111**	**100010111**	**010000111**	**000000111**	**0100111**	**0010111**	**NA**	**NA**
42	**01100111**	**NA**	**NA**	**100001111**	**110000111**	**000100111**	**01000111**	**00100111**	**01111**	**NA**
43	**001000111**	**NA**	**NA**	**100101111**	**100010111**	**010000111**	**000000111**	**0100111**	**0010111**	**NA**
44	**001100111**	**0110111**	**NA**	**NA**	**100001111**	**110000111**	**000100111**	**01000111**	**00100111**	**01111**
45	**010000111**	**01100111**	**NA**	**NA**	**100101111**	**100010111**	**010000111**	**000000111**	**0100111**	**0010111**
46	**000010111**	**001000111**	**NA**	**NA**	**NA**	**100001111**	**110000111**	**000100111**	**01000111**	**00100111**
47	**1000000111**	**001100111**	**0110111**	**NA**	**NA**	**100101111**	**100010111**	**010000111**	**000000111**	**0100111**
48	**1001000111**	**010000111**	**01100111**	**NA**	**NA**	**NA**	**100001111**	**110000111**	**000100111**	**01000111**
49	**110010111**	**000010111**	**001000111**	**NA**	**NA**	**NA**	**100101111**	**100010111**	**010000111**	**000000111**
50	**110001111**	**1000000111**	**001100111**	**0110111**	**NA**	**NA**	**NA**	**100001111**	**110000111**	**000100111**

C. SOME STRAIGHT LINES CORRESPONDING TO THIRD ORDER SOME VARIANT NARAYANA CODES

The Narayana codewords for $k = 1, 2, 3, 4, 5, 6$ always exist and these are 000111, 101111, 1111, 1000111, 10000111, 10010111, respectively for any $u \leq -1$ since binary representations of $1, 2, 3, 4, 5, 6$ are independent of u. Here, we take u, k and i as integers. Now, we discuss the following properties of the Narayana code by considering (u, k) as a point in (x, y) plane and with the help of Tables 1 and 2.

Proposition 1. For $u \leq -1$ there are six straight lines $y + 0x = 0 + i$ for $i = 1, 2, 3, 4, 5, 6$ such that the six points $(u, 1 - 0u)$, $(u, 2 - 0u)$, $(u, 3 - 0u)$, $(u, 4 - 0u)$, $(u, 5 - 0u)$ and $(u, 6 - 0u)$ lie on these lines for $i = 1, 2, 3, 4, 5, 6$ respectively which give the respective Narayana codewords 000111, 101111, 1111, 1000111, 10000111 and 10010111.

Proposition 2. For $u \leq -1$ there are ten straight lines $y + x = 1 + i$ for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ such that the ten points $(u, 2 - u)$, $(u, 3 - u)$, $\ldots$ and $(u, 11 - u)$ lie on these lines for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ respectively which give the respective Narayana codewords 00111, 001111, 0111, 0000111,00000111, 00010111, 10110111, 11000111, 100000111 and 100100111.

Proposition 3. For $u \leq -1$, there are twelve straight lines $y + 2x = 5 + i$ for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ such that the twelve points $(u, 6 - 2u)$, $(u, 7 - 2u)$, …, $(u, 17 - 2u)$ lie on these lines for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ respectively which give the respective Narayana codewords 01111, 0010111, 00100111, 00110111, 0101111, 000000111, 000100111, 101100111, 110000111, 100010111, 100001111 and 100101111.

Proposition 4. For $u \leq -1$, there are eleven straight lines $y + 3x = 10 + i$ for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ such that the eleven points $(u, 11 - 3u)$, $(u, 12 - 3u)$, ..., $(u, 21 - 3u)$ lie on these lines for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ respectively which give the respective Narayana codewords 0110111, 01100111, 001000111, 001100111, 010000111, 000010111, 000001111, 1001000111, 110010111, 110001111 and 110101111.

Proposition 5. For $u \leq -1$, there are eight straight lines $y + 4x = 16 + i$ for $i = 1, 2, 3, 4, 5, 6, 7, 8$ such that the eight points $(u, 17 - 4u)$, $(u, 18 - 4u)$, ..., $(u, 24 - 4u)$ lie on these lines for $i = 1, 2, 3, 4, 5, 6, 7, 8$ respectively which give the respective Narayana codewords 011000111, 0000000111, 0001000111, 1011000111, 1100000111, 1000100111, 1000010111 and 1001010111.

Proposition 6. For $u \leq -1$, there are twelve straight lines $y + 5x = 19 + i$ for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ such that the twelve points $(u, 20 - 5u)$, $(u, 21 - 5u)$, …, $(u, 31 - 5u)$ lie on these lines for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ respectively which give the respective

Narayana codewords 0010000111, 0011000111, 0100000111, 0000100111, 0000010111, 00000000111, 00010000111,10110000111,11000000111, 10001000111, 10000100111 and 10010100111.

Proposition 7. For $u \leq -1$, there are thirteen straight lines $y + 6x = 23 + i$ for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$ such that the thirteen points $(u, 24 - 6u)$, $(u, 25 - 6u)$, ..., $(u, 36 - 6u)$ lie on these lines for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$ respectively which give the respective Narayana codewords 0110000111, 0010100111, 0010010111, 00100000111, 00110000111, 01000000111, 00001000111, 00000100111, 00010100111, 11001000111, 11000100111, 10000010111 and 10010010111.

Proposition 8. For $u \leq -1$, there are twelve straight lines $y + 7x = 30 + i$ for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ such that the twelve points $(u, 31 - 7u)$, $(u, 32 - 7u)$, …, $(u, 42 - 7u)$ lie on these lines for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ respectively which give the respective Narayana codewords 01100000111, 00101000111, 00100100111, 00110100111, 01000100111, 00000010111, 00010010111, 10110010111, 11000010111, 10001010111, 10000110111 and 10010110111.

Proposition 9. For $u \leq -1$, there are twelve straight lines $y + 8x = 35 + i$ for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ such that the twelve points $(u, 36 - 8u)$, $(u, 37 - 8u)$, ..., $(u, 47 - 8u)$ lie on these lines for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ respectively which give the respective Narayana codewords 01101000111, 01100100111, 00100010111, 00110010111, 01000010111, 000000000111, 000100000111, 101100000111, 110000000111, 100010000111, 100001000111 and 100101000111.

Also, we obtained the point (u, k) satisfying more than one Narayana code straight line, does not have unique Narayana codeword.

D. CRYPTOGRAPHIC COMPARISON OF THE THIRD AND THE SECOND ORDER VARIANT NARAYANA CODES

Cryptography is all of the methods used to make an understandable message incomprehensible to undesirable people [12]. The focus of cryptography is privacy. The main purpose of cryptography is to ensure the two in order to get into touch over an insecure canal such that a rival, Oscar, can't understand. The information which is sent is called as plaintext, and the information is selected arbitrary [13].

Every algorithm uses a key to audit encryption and decryption, and a message may only be decrypted when the key meets the encryption key used. So the most important issue in encryption is the key. According to key structure, cryptography is divided into two as symmetric cryptography and asymmetric cryptography. Examples to symmetric cryptography are Vernam, DES, AES, while examples to asymmetric cryptography are RSA, ElGamal. In addition, there are different cryptographic system built, too. One of them is a system created using source coding. Some examples of these applications are included in [9, 14].

We know that variant Narayana universal coding is used in source coding as well as in cryptography. But these codes ($VN_u^{(3)}(k)$) which lack the ability to encode for certain values of k can't be used in cryptography. So, it is important to determine these codes. Here, we obtained there is no Zeckendorf's representation for the first integer $k = 7$ in the third order variant Narayana sequence $VN_{-7}^{(3)}(k) = \{-7, 10, 8, 1, 11, 12, 23, 42\}$. But we obtained there is no Zeckendorf's representation for the first integer $k = 3$ in the second order variant Narayana sequence $VN_{-1}^{(2)}(k) = \{-1, 4, 2, 1, 5, 7, 8, 13\}$. So, while $VN_{-1}^{(3)}, VN_{-2}^{(3)}, VN_{-3}^{(3)}, VN_{-4}^{(3)}, VN_{-5}^{(3)}, VN_{-6}^{(3)}$ can be used in cryptography for the third order variant Narayana codes, none of the second order variant Narayana codes can be used in cryptography. At that rate, we obtained that the third order variant Narayana code are more useful than the second order variant Narayana code in terms of cryptography.

IV. CONCLUSION

In this study, firstly, we examined the second order variant Narayana codes and from the tables displayed by Das and Sinha [11]. And, we obtained for the only positive integer $k = 1$, the second order variant Narayana code $VN_u^{(2)}(k)$ exactly exists for $u = -1, -2, \ldots, -20$. For $1 \leq k \leq 50$, there is at most j consecutive undetectable values (NA) the second order variant Narayana code in $VN_{-j}^{(2)}(k)$ column in which $1 \leq j \leq 20$. As long as j raises, the detectable of Narayana code is reduced in $VN_{-j}^{(2)}(k)$ column in which $1 \leq j \leq 20$.

Then, we described the third order variant Narayana sequence. Then, we obtained the third order variant Narayana codes based on these sequences we described. Afterwards, we showed in tables $VN_u^{(3)}(k)$ we have defined for $1 \leq k \leq 50$ and $u = -1, -2, \ldots, -20$. From the tables, we got some important results. For $k = 1, 2, 3, 4, 5, 6$ the third order variant Narayana code $VN_u^{(3)}(k)$ exactly exists. There is at most j consecutive undetectable values (NA) the third order variant Narayana code in $VN_{-(6+j)}^{(3)}(k)$ column in which $1 \leq j \leq 14$. As long as j raises, the detectable of Narayana code is reduced in $VN_{-(6+j)}^{(3)}(k)$ column in which $1 \leq j \leq 14$.
Also, we compared the results that obtained from the tables for the third order variant Narayana universal code and the second order variant Narayana universal code cryptographically. While $VN_{-1}^{(3)}, VN_{-2}^{(3)}, VN_{-3}^{(3)}, VN_{-4}^{(3)}, VN_{-5}^{(3)}, VN_{-6}^{(3)}$ can be used in cryptography for the third order variant Narayana codes, none of the second order variant Narayana codes can be used in cryptography. And so, we found that third order variant Narayana universal codes are much more advantageous than the second order variant Narayana universal code. Finally, we obtained some straight lines which yielding some Narayana codewords by considering (u, k) as a point in the (x, y) plane, from these tables.

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