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Numerical Solution of Volterra Integral Equations Using Hosoya Polynomial

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Abstract. In this study, Volterra integral equation is solved by Hosoya Polynomials. The solutions obtained with Hosoya method were compared on the figure and table. And error analysis was done. Matlab package programming has been used to obtain results, tables and error analysis.

1. Introduction

Many mathematical models in disciplines such as engineering, physics and chemistry consist of integral equations [1]. Integral equations are equations in which the unknown function is under the integral sign [9]. Integral equations has been used in various applications such as geophysics, electricity and magnetism, kinetic theory of gases, regeneration theory, quantum mechanics, radiation, optimization, optimal control systems, mathematical economics, mathematical problems of radiative equilibrium, fluid mechanics, steady state heat [11]. One of most important integral equation is Volterra integral equation. Recently, Volterra integral equations have been increasingly used in engineering and applied mathematics studies. This equation has been studied in many fields of study such as Banach space, Haar functions problems, potential theory and Dirichlet problems, spectral methods, numerical computational problems and computer science problems [10]. In addition, the method studied in this paper was applied to the Volterra integral equation.

2. Volterra Integral Equations

The third kind of Volterra integral equations is of the form

$$u(x)h(x) = f(x) + \lambda \int_{\alpha}^{x} K(x,t)u(t)dt \tag{1}$$

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where the limits of integration are function of x and the unknown function $u(x)$ appears linearly under the integral sign. Second kind of Volterra integral equations is of the form

$$u(x) = f(x) + \lambda \int_{\alpha}^{x} K(x,t)u(t)dt \tag{2}$$

where $h(x) = 1$. First kind of Volterra integral equations is of the form

$$f(x) = \lambda \int_{\alpha}^{x} K(x,t)u(t)dt \tag{3}$$

3. Hosoya Polynomials

The Hosoya polynomial was initiated in 1988 by Haruo Hosoya [5, 8]. Hosoya polynomials count the distance between vertices of the path graph [12]. It is obtained from path graphs of certain pairs of graphs [3, 4]. Studies such as obtaining the physical and chemical properties of organic molecules with the Hosoya polynomial of the graph were carried out [12]. For a path graph with the Hosoya polynomial is described as,

$$H(P,\delta) = \sum_{l \geq 0} d(P,l)\delta^{l} \tag{4}$$

where $d(P,l)$ is the distance between vertex pairs in the path graph [6, 7]. Sum of the path graph vertices m with $1, 2, ..., m$ are multipled δ parameter. Then Hosoya values are calculated based on m vertex values [13]. For m integer values we represent path as ρ_m , then Hosoya polynomial of path compute as:

$$\begin{aligned}
&H(\rho_1,\delta) = \sum_{l \geq 0} d(\rho_1,l)\delta^{l} = 1\\
&H(\rho_2,\delta) = \sum_{l \geq 0} d(\rho_2,l)\delta^{l} = \delta + 2\\
&H(\rho_3,\delta) = \sum_{l \geq 0} d(\rho_3,l)\delta^{l} = \delta^2 + 2\delta + 3\\
&\vdots\\
&H(\rho_m,\delta) = m + (m-1)\delta + (m-2)\delta^2 + ...\\
&\ + (m-(m-2))\delta^{m-2} + (m-(m-1))\delta^{m-1}
\end{aligned}$$

A function $w(x) \in L_2[0;\ 1]$ is dilated as:

$$w(x) = \sum_{i=1}^{n} z_i H(\rho_i, x) = Z^T H_\rho(x), \tag{5}$$

where Z and $H_\rho(x)$ are $m \times 1$ matrices shown as:

$$Z = [z_1, z_2, z_3, ..., z_m]^T \tag{6}$$

and

$$H_\rho(x) = [H(\rho_1, x), H(\rho_2, x), ..., H(\rho_m, x)]^T. \tag{7}$$

4. Hosoya Polynomial Method

Consider The Volterra integral equation

$$y(x) = w(x) + \int_{1}^{x} L(x,t)y(t)dt, 0 \leq x, t \leq 1 \tag{8}$$

to solve equation (8), the method is as follows:

1. First we define $y(x)$ as defined in Equation (5). This equation is,

$$y(x) = Z^T H_\rho(x) \tag{9}$$

2. Then using place of (9) in (8), we get,

$$Z^T H_\rho(x) = w(x) + \int_{1}^{x} L(x,t)\left[Z^T H_\rho(t)\right]dt \tag{10}$$

3. Replacing the collocation point $x_j = \frac{j-0.5}{m}, j = 1, 2, \cdots, m$ in Equation (10). Then we get,

$$Z^T H_\rho(x_j) = w(x_j) + Z^T \left[\int_{1}^{x} L(x_j,t)H_\rho(t)dt\right] \tag{11}$$

$$Z^T(H_\rho(x_j) - Y) = w$$

where

$$Y = \int_{1}^{x} L(x_j,t)H_\rho(t)dt$$

4. In the last step, we get the conclutions of unknown Hosoya values,

$$Z^T L = w$$

where

$$L = H_\rho(x_j) - Y$$

solving this system of equations we get coefficients Z and then use in place of these coefficients in (9), we obtain the necessary result of (8) [2].

5. Numerical Example

5.1. Example

Consider Volterra integral equation,

$$u(x) = x + \int_{1}^{x} (t-x)u(t)dt \tag{12}$$

which has the exact solution $u(x) = \sin(x)$. First we substitute $u(x) = Z^T H_\rho(x)$ in equation (12). We get,

$$Z^T H_\rho(x) = x + \int_1^x (t-x)\left[Z^T H_\rho(t)\right] dt \tag{13}$$

Because of that reason for $m = 3$,

$$\begin{aligned} &Z_1[H_1(x) - \left(\int_1^x tH_1(t)dt - \int_1^x xH_1(t)dt\right)] \\ &+ Z_2[H_2(x) - \left(\int_1^x tH_2(t)dt - \int_1^x xH_2(t)dt\right)] \\ &+ Z_3[H_3(x) - \left(\int_1^x tH_3(t)dt - \int_1^x xH_3(t)dt\right)] = x \end{aligned} \tag{14}$$

Next, we achieve the Hosoya polynomials as

$$\begin{aligned} &Z_1[1 - \left(\int_1^x tdt - \int_1^x xdt\right)] \\ &+ Z_2[(x+2) - \left(\int_1^x t(t+2)dt - \int_1^x x(t+2)dt\right)] \\ &+ Z_3[(x^2+2x+3) - \left(\int_1^x t(t^2+2t+3)dt - \int_1^x x(t^2+2t+3)dt\right)] = x \end{aligned} \tag{15}$$

Next,

$$\begin{aligned} &Z_1[\frac{3}{2} + \frac{x^2}{2} - x] \\ &+ Z_2[\frac{x^3}{6} + x^2 - \frac{3x}{2} + \frac{10}{3}] \\ &+ Z_3[5 - \frac{x^2}{2} + \frac{53x}{12} - \frac{x^4}{4} - \frac{2x^3}{3}] = x \end{aligned} \tag{16}$$

If it is compute as $x_j = \frac{j-0.5}{m}$ and putting instead of the collocation points x_1, x_2, x_3 , we get the system of three equations with three unknowns as,

$$
\begin{aligned}
&Z_1[\frac{3}{2} + \frac{x_1{}^2}{2} - x_1] \\
&+ Z_2[\frac{x_1{}^3}{6} + x_1{}^2 - \frac{3x_1}{2} + \frac{10}{3}] \\
&+ Z_3[5 - \frac{x_1{}^2}{2} + \frac{53x_1}{12} - \frac{x_1{}^4}{4} - \frac{2x_1{}^3}{3}] = x_1 \\
\\
&Z_1[\frac{3}{2} + \frac{x_2{}^2}{2} - x_2] \\
&+ Z_2[\frac{x_2{}^3}{6} + x_2{}^2 - \frac{3x_2}{2} + \frac{10}{3}] \\
&+ Z_3[5 - \frac{x_2{}^2}{2} + \frac{53x_2}{12} - \frac{x_2{}^4}{4} - \frac{2x_2{}^3}{3}] = x_2 \\
\\
&Z_1[\frac{3}{2} + \frac{x_3{}^2}{2} - x_3] \\
&+ Z_2[\frac{x_3{}^3}{6} + x_3{}^2 - \frac{3x_3}{2} + \frac{10}{3}] \\
&+ Z_3[5 - \frac{x_3{}^2}{2} + \frac{53x_3}{12} - \frac{x_3{}^4}{4} - \frac{2x_3{}^3}{3}] = x_3
\end{aligned} \tag{17}
$$

resolving these systems we get the three unknown Hosoya values,

$$Z_1 = 0.5012, Z_2 = 0.8672, Z_3 = \text{-}0.4101$$

putting back with these coefficients in the approximation, we get

$$u(x) = Z_1[H_1(x)] + Z_2[H_2(x) + Z_3[H_3(x)]$$

If in (17) is written in place of the x_1, x_2, x_3 values, approximate solutions are achieved.

$$
\begin{aligned}
u_1(x) &= Z_1[H_1(x_1)] + Z_2[H_2(x_1) + Z_3[H_3(x_1)] \\
u_2(x) &= Z_1[H_1(x_2)] + Z_2[H_2(x_2) + Z_3[H_3(x_2)] \\
u_3(x) &= Z_1[H_1(x_3)] + Z_2[H_2(x_3) + Z_3[H_3(x_3)]
\end{aligned} \tag{18}
$$

We get the approximate values,

$$u_1 = \text{-}0.0682718, u_2 = 0.397826, u_3 = 0.820127$$

Maximum Error analyzed for $m = 3$ is,

$$
\begin{aligned}
E_{\max} = &\sqrt{\sum_{i=1}^{m} (u_e(x_i) - u_a(x_i))^2} = \\
&\sqrt{(x_1 - u_1)^2 + (x_2 - u_2)^2 + (x_3 - u_3)^2} = \ 0.2605
\end{aligned} \tag{19}
$$

and for $m = 3, 8, 10$ are shown in the Tables $1, 2, 3$ and Figures $1, 2, 3$.

Table 1: Conclution of Hosoya Polynomial Method, for $m = 3$

x	Hosoya Polynomial Method	Exact Solution
0.1667	0.1659	-0.0682718
0.5	0.4794	0.397826
0.8333	0.7402	0.820127

Figure 1: Example 5.1 for $m = 3$

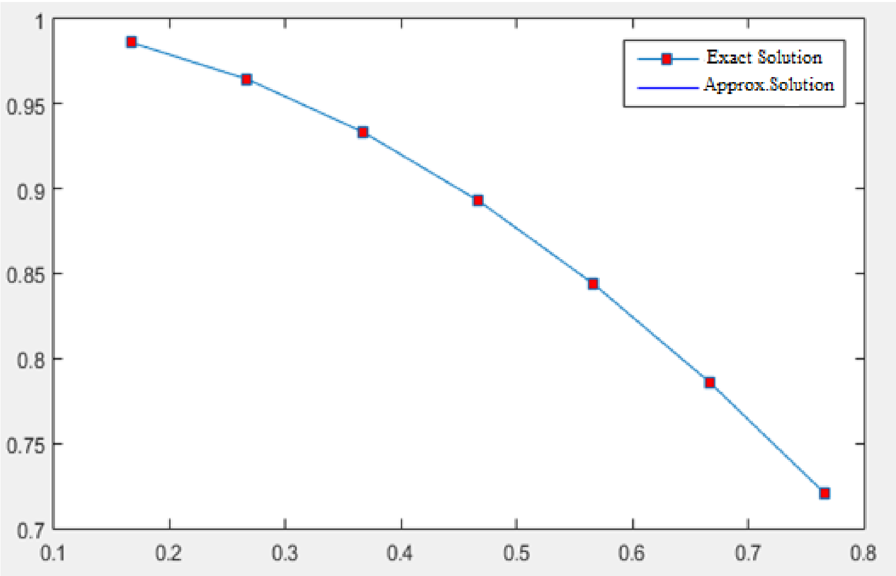


Table 2: Conclution of Hosoya Polynomial for $m = 8$

x	Hosoya Polynomial Method	Exact Solution
0.0625	0.5851	-0.214276
0.1875	0.1864	-0.0383231
0.3125	0.3125	0.138228
0.4375	0.3074	0.312622
0.5625	0.5333	0.482137
0.6875	0.6875	0.644129
0.8125	0.6346	0.79607
0.9375	0.9361	0.935588

Figure 2: Example 5.1 for $m = 8$

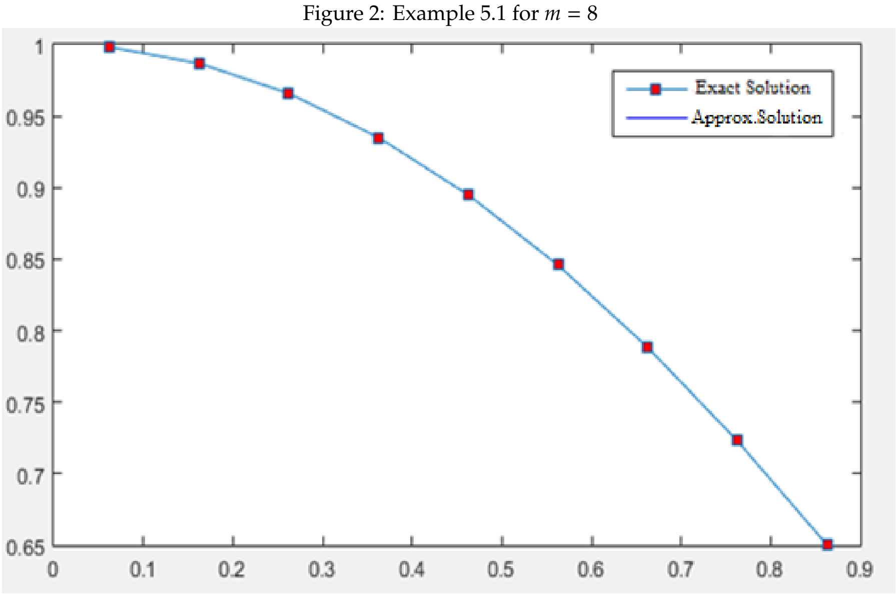
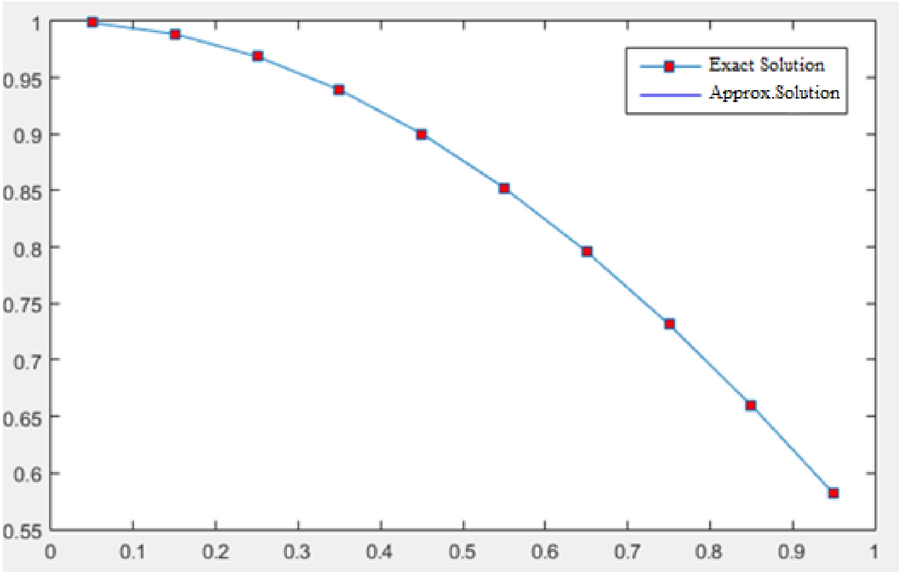


Table 3: . Conclution of Hosoya Polynomial Method for $m = 10$

x	**Hosoya Polynomial Method**	**Exact Solution**
0.050	0.050	-0.231732
0.150	0.1494	-0.0912973
0.250	0.2474	0.0500501
0.350	0.3429	0.190897
0.450	0.4350	0.329837
0.550	0.5227	0.465482
0.650	0.6052	0.596475
0.750	0.6816	0.721508
0.850	0.7513	0.839333
0.950	0.9134	0.948771

Figure 3: Example 5.1 for $m = 10$



6. Conclution

In this paper, the solution of Volterra integral equations with Hosoya method is examined. The method was applied to test problem in the matlab achieved with a certain algorithm. The method is solved for $m = 3$, $m = 8$, $m = 10$ values. The maximum error analysis was obtained according to the results exact and approximate solutions. The results exact and approximate solutions are shown with tables and figures. When the achieved conclutions are analyzed, it is seen that the Hosoya method is an useful method for solving the Volterra integral equation.

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