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Co-derivative operators on BL-algebras

Ahmet HAMAL*

Ege University, Faculty of Science, Department of Mathematics, Izmir, Turkey

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Abstract

In this study we establish and investigate multiplicative co-derivative operators on BL-algebras. We also indicate that multiplicative co-derivative operators are more general operators than multiplicative interior operators and modal operators on BL-algebras. Furthermore, we describe relations between multiplicative co-derivative operators on BL-algebras and on the algebras of their regular elements. Moreover, ɼ-filters (ɼ-derivative systems) will be introduced on BL-algebras depending on any multiplicative co-derivative operator ɼ on BL-algebras. We also show that some sets of BL-algebras are ɼ-filters (ɼ-deductive systems) on BL-algebras. Next, we will define quotient BL-algebra by means of any multiplicative co-derivative operator ɼ on BL-algebra and any ɼ-derivative systems on BL-algebra. Finally, we will define a new operator on the quotient BL-algebra with the aid of the operator ɼ and show that the new operator is a multiplicative co-derivative operator on the quotient BL-algebra.

Keywords: *BL-algebras, multiplicative interior operator, modal operator, multiplicative co-derivative operator.*

BL-cebirleri üzerindeki çarpımsal co-türev operatörleri

Öz

Bu çalışmada, BL-cebirleri üzerindeki çarpımsal co-türev operatörleri tanıtılacak ve incelenecektir. Aynı zamanda, çarpımsal co-türev operatörlerinin çarpımsal iç operatörler ve modal operatörlerden daha genel operatörler oldukları vurgulanacaktır. Ayrıca, BL-cebirleri ve bu cebirlerin regüler elemanlarının oluşturduğu cebirler üzerinde tanımlı çarpımsal co-türev operatörleri arasındaki ilişkiler betimlenecektir. Dahası, BL-cebirleri üzerindeki herhangi bir çarpımsal co-türev operatörü ɼ' e bağlı olarak BL-cebirleri üzerinde ɼ-süzgeçler (ɼ-türetim sistemleri) tanıtılacaktır. Ek olarak, BL-

*Ahmet HAMAL, ahmet.hamal@ege.edu.tr, https://orcid.org/ 0000-0002-4593-7054

cebirlerinin bazı kümelerinin ϝ-süzgeçler (ϝ-türetim sistemleri) oldukları gösterilecektir. Sonra, BL-cebiri üzerindeki herhangi bir çarpımsal türev operatörü ϝ ve BL-cebirinin herhangi bir ϝ-türetim sistemi yardımı ile bölüm BL-cebiri tanımlanacaktır. Son olarak, ϝ-operatörü yardımı ile bölüm BL cebiri üzerinde yeni bir operatör tanımlanacak ve bu yeni operatörün çarpımsal bir co-türev operatörü olduğu gösterilecektir.

Anahtar kelimeler: *BL-cebirleri, çarpımsal iç operatör, modal operatör, çarpımsal co-türev operatör.*

1. Introduction

BL-algebras as the algebric structures of basic fuzzy logic were introduced by Hájeck in [1]. Product algebras, MV-algebras and Gödel algebras are special cases of BL-algebras.

Topological Boolean algebras defined with topological closure operators and interior operators in [2] are generalizations of topological spaces. Closure and interior MV-algebras which were defined with the aid of multiplicative interior operators and additive closure operators in [3] were introduced qua generalizations of topological Boolean algebras. Since that time, this operators are considered on some algebras, such as, R*l*-monoids and commutative bounded residuated lattice. Derivative and coderivative MV-algebras defined with additive derivative and multiplicative coderivative operators in [4] as generalizations of closure and interior MV-algebras were introduced and investigated.

Modal operators were introduced and investigated on Heyting algebras which are the algebric counterpart of the intuitionistic propositional logic in [5]. Properties of modal operators are investigated on some algebras, for example, commutative residuated lattice, MV-algebras, commutative R*l*-monoids and so on.

In this study, we establish and study multiplicative co-derivative operators on BL-algebras. We show that multiplicative co-derivative operators are more general operators than multiplicative interior operators and modal operators. We describe relations between multiplicative co-derivative operators on BL-algebras and on the MV-algebras of their reguler elements. At the same time, we introduce ϝ-filters (ϝ-deductive systems) and study on them. Finally, we study multiplicative co-derivative operators on quotient BL-algebras.

2. Preliminaries

We remember that an algebra $(A,\odot,\wedge,\vee,\rightarrow,0,1)$ is named as a BL-algebra ([6-7]) if it satisfies the belows:

(B1) $(A,\odot,1)$ is a commutative monoid,
(B2) $(A,\wedge,\vee,0,1)$ is a bounded lattice,
(B3) $x\odot y\leq z$ iff $x\leq y\rightarrow z$,
(B4) $x\wedge y = x\odot(x\rightarrow y)$
(B5) $(x\rightarrow y)\vee(y\rightarrow x)=1$.

On any BL-algebra A we define a unary operator as $x^- = x \to 0$ and a binary operator $\oplus$ such that $x \oplus y = (x^- \odot y^-)^-$.

2.1. Proposition ([6-8]) Suppose that A be a BL-algebra. In that case the belows hold:

1) $x \odot (x \to y) \leq y$,
2) $x \odot y \leq x \wedge y \leq x \vee y$,
3) $1 \to x = x$, $x \to x = 1$, $x \to 1 = 1$,
4) $(x \oplus y)^- = x^- \odot y^-$, $(x \odot y)^- = x^- \oplus y^-$,
5) $(x \vee y)^- = x^- \wedge y^-$, $(x \wedge y)^- = x^- \vee y^-$,
6) $x \leq y$ *implies* $z \to x \leq z \to y$, $y \to z \leq x \to z$ *and* $x \odot z \leq y \odot z$,
7) $x \leq x^{--}, x^{---} = x^-$,
8) $1^- = 0$, $0^- = 1$, $1^{--} = 1$, $0^{--} = 0$,
9) $x \leq y$ *iff* $x \longrightarrow y = 1$,
10) $(x \to y)^{--} = x \to y^{--}$,
11) $x \odot x^- = 0$, $x \oplus x^- = 1$,
12) $(x \to y)^{--} = x^{--} \to y^{--}$,
13) $(x \oplus y)^{--} = x^{--} \oplus y^{--} = x^{--} \oplus y = x \oplus y$,

for any $x, y, z \in A$.

2.2. Definition ([8]) Suppose that A is a BL-algebra. If a nonempty subset F of A satisfies the below conditions

(F1) x, $y \in F$ *implies* $x \odot y \in F$,
(F2) $x \in F$, $y \in A$, $x \leq y$ *implies* $y \in F$,

Then it is named as a *filter* of A.

2.3. Definition ([7-8]) Suppose that A is a BL-algebra. If a subset D of A provides the following conditions

(D1) $1 \in D$,
(D2) x, $x \to y \in D \Rightarrow y \in D$,

Then it is named as a *deductive system* of A. We will write a *dsystem* of A shortly instead of a *deductive system* of A from now on.

It is known that a subset of A is a dsystem of A iff it is a filter of A.

2.4. Theorem ([7]) Assume that D is a dsystem of BL-algebra A. Define $x \sim_D y$ iff $(x \to y) \odot (y \to x) \in D$. Then $\sim_D$ is a congruence relation with respect to $\odot$, $\wedge$, $\vee$, $\to$, $^-$.

2.5. Theorem ([7]) Assume that D is a dsystem of a BL-algebra A. Define the operators on A/D, the set of equivalence classes, as follows:

$[x] \sqcap [y] = [x \wedge y], [x] \sqcup [y] = [x \vee y]$, $[x] \otimes [y] = [x \odot y]$, $[x] \rightharpoonup [y] = [x \to y]$ and $[x] \leq [y]$ iff $x \longrightarrow y \in D$ for all $[x], [y] \in A/D$.

Then $(A / D, \otimes, \sqcap, \sqcup, \rightharpoonup, [0], [1])$ is a BL-algebra.

3. BL-algebras with co-derivative operators

In this part of the study, we establish and study multiplicative co-derivative operators on BL-algebras.

3.1. Definition Suppose that A is a BL-algebra. We say that a mapping $\mathfrak{f}: A \to A$ is a multiplicative co-derivative operator (mcd-operator) if for all x, $y \in A$, it satisfies following conditions:

(t1) $\mathfrak{f}(x) \odot \mathfrak{f}(y) = \mathfrak{f}(x \odot y)$,
(t2) $x \odot \mathfrak{f}(x) \leq \mathfrak{f}\mathfrak{f}(x)$,
(t3) $\mathfrak{f}(1)=1$.

When $\mathfrak{f}$ has the following property

(t4) $\mathfrak{f}(x) \leq \mathfrak{f}\mathfrak{f}(x)$

call it is a stronger mcd-operator on A.

3.2. Theorem The conditions of an mcd-operator on BL-algebra A are independet.

Proof. We have to find a function on BL-algebra A where the property is false while the others are true.

(t1) Let $A = \{0, p, q, 1\}$ be a set, where $0 \leq p, q \leq 1$. Define $\odot$ and $\to$ as in Table 1:

Table 1. Cayley tables of the binary operations $\odot$ and $\to$

$\odot$	0	p	q	1
0	0	0	0	0
p	0	p	0	p
q	0	0	q	q
1	0	p	q	1

$\to$	0	p	q	1
0	1	1	1	1
p	q	1	q	1
q	p	p	1	1
1	0	p	q	1

Then A is a BL-algebra ([6]). We define $\mathfrak{f}: A \to A$ as in Table 2:

Table 2.

x	0	p	q	1
$\mathfrak{f}(x)$	0	q	q	1

$\mathfrak{f}$ satisfies (t2) and (t3) properties. We able to demonstrate that (t1) is not provided. Really, we get x=p and y=q.

$$\mathfrak{f}(x \odot y) = \mathfrak{f}(p \odot q) = \mathfrak{f}(0) = 0, \tag{1}$$

$$ƒ(x) \odot ƒ(y) = ƒ(p) \odot ƒ(q) = q \odot q = q, \tag{2}$$

Then,

$$ƒ(x) \odot ƒ(y) \neq ƒ(x \odot y) \tag{3}$$

(t2) Let $A = \{0, a, b, p, 1\}$ be a set, where $0 \leq p \leq a, b \leq 1$. Define $\odot$ and $\rightarrow$ as in Table 3 :

Table 3. Cayley tables of the binary operations $\odot$ and $\rightarrow$

$\odot$	0	p	a	b	1
0	0	0	0	0	0
p	0	p	p	p	p
a	0	p	a	p	a
b	0	p	p	b	b
1	0	p	a	b	1

$\rightarrow$	0	p	a	b	1
0	1	1	1	1	1
p	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	p	a	b	1

In that case, A is a BL-algebra ([9]). We set the function $ƒ: A \rightarrow A$ as in Table 4:

Table 4.

x	0	p	a	b	1
$ƒ(x)$	0	0	b	0	1

Then it is express that $ƒ$ satisfies (t1) and (t3) axioms. We take $x = a$. Then

$$a \odot ƒ(a) = a \odot b = p, \tag{4}$$

$$ƒƒ(a) = ƒ(ƒ(a)) = ƒ(b) = 0, \tag{5}$$

We have $p \nleq 0$. Thus, axiom (t2) is not satisfied.

(t3) Suppose that A is a BL-algebra. We describe the function $ƒ: A \rightarrow A$ by $ƒ(x) = 0$ for all $x \in A$. Then it is easy to see that $ƒ$ provides (t1) and (t3) axioms but axiom (t3) is not satisfied. □

3.3. Example Let $A = \{0, p, q, 1\}$ be a set, where $0 \leq p, q \leq 1$. Define $\odot$ and $\rightarrow$ as in Table 1. We know that A is a BL-algebra. We define $ƒ: A \rightarrow A$ as in Table 5:

Table 5.

x	0	p	q	1
$\mathfrak{f}(x)$	0	q	0	1

Then it is easy to see that $\mathfrak{f}$ provides (t1) - (t3) axioms. So, $\mathfrak{f}$ is an mcd-operator. However, $\mathfrak{f}$ does not satisfy (t4) axiom. Indeed, we take $x = p$. Then $\mathfrak{f}(p) = q \nleq \mathfrak{f}\big(\mathfrak{f}(p)\big) = \mathfrak{f}(q) = 0$. Thus, $\mathfrak{f}$ isn't a stronger mcd-operator.

3.4. Proposition If $\mathfrak{f}: A \to A$ is an mcd-operator on BL-algebra A and $x \leq y$ for x, y $\in A$, then $\mathfrak{f}(x) \leq \mathfrak{f}(y)$.

Proof. Let $x \leq y$ for x, y $\in A$. Then

$$\begin{aligned} \mathfrak{f}(x) &= \mathfrak{f}(x \wedge y) = \mathfrak{f}(y \wedge x) = \mathfrak{f}\big(y \odot (y \to x)\big) \text{ by (B4)} \\ &= \mathfrak{f}(y) \odot \mathfrak{f}(y \to x) \text{ by (t1)} \\ &\leq \mathfrak{f}(y) \wedge \mathfrak{f}(y \to x) \leq \mathfrak{f}(y) \text{ by 2.1. Proposition (2).} \end{aligned} \tag{6}$$

□

3.5. Definition ([3,8]) Suppose that A is a BL-algebra. If a mapping $f: A \to A$ satisfies the following conditions

(i1) $f(x) \odot f(y) = f(x \odot y)$,
(i2) $f(x) \leq x$,
(i3) $f(x) = ff(x)$,
(i4) f(1)=1,

for $x,\ y \in A$, then it is named as a multiplicative interior operator (mi-operator) on A.

3.6. Remark Let f be an mi-operator on BL-algebra A. It is clear that f satisfies (t1) - (t3) and (t4) axioms. Then f is a strong mcd-operator. But, an mcd-operator $\mathfrak{f}$ may not be an mi-operator.

3.7. Example Suppose that A is a BL-algebra. We take the operator $\mathfrak{f}: A \to A$ as $\mathfrak{f}(x) = 1$ for all $x \in A$. It is clear that $\mathfrak{f}$ is an mcd-operator. But $\mathfrak{f}$ isn't an mi-operator. Because $\mathfrak{f}$ does not satisfy (i2) axiom.

3.8. Definition ([10-11]) Suppose that A is a BL-algebra. We say that a unary mapping $f: A \to A$ is a modal operator on A if for any $x, y \in A$

(m1) $f(x) \odot f(y) = f(x \odot y)$,
(m2) $x \leq f(x)$,
(m3) $f(x) = ff(x)$,
for any $x, y \in A$.

3.9. Remark Let f be a modal operator on BL-algebra A. It is obvious that f satisfies (t1) - (t3) and (t4) axioms. Then f is a strong mcd-operator. But, an mcd-operator $\mathfrak{f}$ may not be an modal operator.

3.10. Example Suppose that A is a BL-algebra. We take the operator $\mathfrak{f}: A \to A$ as

$$\mathfrak{f}(x) = \begin{cases} 1, & x = 1 \\ 0, & otherwise \end{cases}, \tag{7}$$

for all $x \in A$. It is clear that $\mathfrak{f}$ is an mcd-operator. But $\mathfrak{f}$ isn't a modal operator. Because $\mathfrak{f}$ does not satisfy (m2) axiom.

3.11. Proposition Let $\mathfrak{f}$ be an mcd-operator on BL-algebra A. Then we get

(i) $\mathfrak{f}(x \to y) \leq \mathfrak{f}(x) \to \mathfrak{f}(y)$,
(ii) $\mathfrak{f}(x \wedge y) \leq \mathfrak{f}(x) \wedge \mathfrak{f}(y)$,
(iii) $x \leq \mathfrak{f}(x) \to \mathfrak{f}\mathfrak{f}(x)$ and $\mathfrak{f}(x) \leq x \to \mathfrak{f}\mathfrak{f}(x)$,

for any $x, y \in A$.

Proof. (i) Let $x,\ y \in A$. Then,

$\mathfrak{f}\big(x \odot (x \to y)\big) \leq f(y)$ by 2.1. Proposition (1) and 3.4. Proposition. (8)

$\mathfrak{f}\big(x \odot (x \to y)\big) = \mathfrak{f}(x) \odot \mathfrak{f}(x \to y) = \mathfrak{f}(x \to y) \odot \mathfrak{f}(x) \leq \mathfrak{f}(y)$.
(9)

We have,

$\mathfrak{f}(x \to y) \leq \mathfrak{f}(x) \to \mathfrak{f}(y)$ by (B3) .
(10)

(ii) $\mathfrak{f}(x \wedge y) = \mathfrak{f}(x \odot (x \to y)) = \mathfrak{f}(x) \odot \mathfrak{f}(x \to y)$ by (B4) and (t1)
$\leq \mathfrak{f}(x) \odot (\mathfrak{f}(x) \to \mathfrak{f}(y))$ by 2.1. Proposition (6)
$= \mathfrak{f}(x) \wedge \mathfrak{f}(y)$ by (B4).
(11)

(iii) We have,

$x \odot \mathfrak{f}(x) = \mathfrak{f}(x) \odot x \leq \mathfrak{f}\mathfrak{f}(x)$ by (t2) and (B1). (12)

It is clear that

$x \leq \mathfrak{f}(x) \to \mathfrak{f}\mathfrak{f}(x)$ and $\mathfrak{f}(x) \leq x \to \mathfrak{f}\mathfrak{f}(x)$ by (B3) (13)

□

3.12. Theorem Let $\mathfrak{f}_1$ and $\mathfrak{f}_2$ be two strong mcd-operator on BL-algebra A. If $\mathfrak{f}_1\mathfrak{f}_2 = \mathfrak{f}_2\mathfrak{f}_1$, then $\mathfrak{f}_1\mathfrak{f}_2$ is a strong mcd-operator on BL-algebra A.

Proof. (t1) $(\mathfrak{f}_1\mathfrak{f}_2)(x \odot y) = \mathfrak{f}_1\big(\mathfrak{f}_2(x \odot y)\big) = \mathfrak{f}_1(\mathfrak{f}_2(x) \odot \mathfrak{f}_2(y))$
$= \mathfrak{f}_1(\mathfrak{f}_2(x)) \odot \mathfrak{f}_1\big(\mathfrak{f}_2(y)\big) = (\mathfrak{f}_1\mathfrak{f}_2)(x) \odot (\mathfrak{f}_1\mathfrak{f}_2)(y)$. (14)

(t3) $(\mathfrak{f}_1\mathfrak{f}_2)(1) = \mathfrak{f}_1(\mathfrak{f}_2(1)) = \mathfrak{f}_1(1) = 1$ by (t3). (15)

(t4) We have,

$\mathfrak{f}_2(x) \leq \mathfrak{f}_2\mathfrak{f}_2(x)$ by (t4) (16)

Then,

$\mathfrak{f}_1(\mathfrak{f}_2(x)) \leq \mathfrak{f}_1(\mathfrak{f}_2\mathfrak{f}_2(x)) \leq \mathfrak{f}_1(\mathfrak{f}_1(\mathfrak{f}_2\mathfrak{f}_2(x)))$ by 3.4. Proposition and (t4) (17)

So,

$(\mathfrak{f}_1\mathfrak{f}_2)(x) \leq (\mathfrak{f}_1\mathfrak{f}_2)(\mathfrak{f}_1\mathfrak{f}_2)(x).$ (18)

Meanwhile, $\mathfrak{f}_1\mathfrak{f}_2$ satisfies (t2) axiom.

(t2) We know that $x \odot (\mathfrak{f}_1\mathfrak{f}_2)(x) \leq x \wedge (\mathfrak{f}_1\mathfrak{f}_2)(x) \leq (\mathfrak{f}_1\mathfrak{f}_2)(x)$ by 2.1. Proposition (2). Then $x \odot (\mathfrak{f}_1\mathfrak{f}_2)(x) \leq (\mathfrak{f}_1\mathfrak{f}_2)(x) \leq (\mathfrak{f}_1\mathfrak{f}_2)(\mathfrak{f}_1\mathfrak{f}_2)(x)$ by (t4). □

3.13. Proposition Suppose that A is a BL-algebra. In that case $^{--}: A \to A$ is a strong mcd-operator on BL-algebra A.

Proof. Let $x,\ y \in A$.

(t1) $(x \odot y)^{--} = x^{--} \odot y^{--}$ by 2.1. Proposition (4). (19)

(t3) $1^{--} = (1^{-})^{-} = 0^{-} = 1.$ (20)

(t4) We have $x^{----} = x^{--}$ by 2.1. Proposition (7). Then $x^{--} \leq x^{----}$. □

An element x of a BL-algebra A is called regular if $x^{--} = x$. We will denote by $Reg(A)$ to the set of all regular elements in A. We set $x \underline{\vee} y = (x \vee y)^{--}$, $x \underline{\wedge} y = (x \wedge y)^{--}$, $x \underline{\odot} y = (x \odot y)^{--}$, $x \rightsquigarrow y = (x \to y)^{--}$ for any $x, y \in Reg(A)$.

We known that If A is a BL-algebra, then $x \underline{\odot} y = (x \odot y)^{--} = x^{--} \odot y^{--}$ for all $x, y \in A$. Because of this, $x \underline{\odot} y = x \odot y$ for any $x, y \in Reg(A)$. If A is a BL-algebra, then $x \rightsquigarrow y = (x \to y)^{--} = x^{--} \to y^{--}$ by 2.1. Proposition (12). So, $x \rightsquigarrow y = x \to y$ for any $x,\ y \in Reg(A)$. It is obvious that $x \underline{\wedge} y = x \wedge y$.

3.14. Proposition ([12]) If A is a BL-algebra, then $(Reg(A), \underline{\odot}, \underline{\wedge}, \underline{\vee}, \rightsquigarrow, 0, 1)$ is the largest subalgebra of A that is an MV-algebra.

3.15. Theorem Suppose that A is a BL-algebra, $\mathfrak{f}: A \to A$ is an (a strong) mcd-operator on A and $\hat{\mathfrak{f}}: Reg(A) \to Reg(A)$ is the mapping such that $\hat{\mathfrak{f}}(x) = (\mathfrak{f}(x))^{--}$ for any $x \in Reg(A)$. Then $\hat{\mathfrak{f}}$ is an (a strong) mcd-operator on $Reg(A)$.

Proof. Let $x, y \in Reg(A)$.

(t1) $\hat{\mathfrak{f}}\left(x \underline{\odot} y\right) = (\mathfrak{f}(x \odot y))^{--} = (\mathfrak{f}(x) \odot \mathfrak{f}(y))^{--} = (\mathfrak{f}(x))^{--} \odot (\mathfrak{f}(y))^{--}$
$= \hat{\mathfrak{f}}(x) \odot \hat{\mathfrak{f}}(y) = \hat{\mathfrak{f}}(x) \underline{\odot} \hat{\mathfrak{f}}(y).$
(21)

(t2) $x \underline{\odot} \hat{\mathfrak{f}}(x) = x \odot \hat{\mathfrak{f}}(x) = x \odot (\mathfrak{f}(x))^{--} = x^{--} \odot (\mathfrak{f}(x))^{--} = (x \odot \mathfrak{f}(x))^{--}$
$\leq (\mathfrak{f}\mathfrak{f}(x))^{--} \leq (\mathfrak{f}(\mathfrak{f}(x))^{--}))^{--}$ by 2.1. Proposition (7) and 3.4. Proposition
$= (\mathfrak{f}(\hat{\mathfrak{f}}(x)))^{--} = \hat{\mathfrak{f}}(\hat{\mathfrak{f}}(x)) = \hat{\mathfrak{f}}\hat{\mathfrak{f}}(x).$ (22)

(t3) $\hat{\mathfrak{f}}(1) = (\mathfrak{f}(1))^{--} = (1)^{--} = 1.$ (23)

(t4) If $\mathfrak{f}$ has the stronger property, then

$$\hat{\mathfrak{f}}(x) = (\mathfrak{f}(x))^{--} \leq (\mathfrak{f}\mathfrak{f}(x))^{--} \leq (\mathfrak{f}(\mathfrak{f}(x))^{--}))^{--} = (\mathfrak{f}(\hat{\mathfrak{f}}(x)))^{--} = \hat{\mathfrak{f}}(\hat{\mathfrak{f}}(x)) = \hat{\mathfrak{f}}\hat{\mathfrak{f}}(x).$$
(24)

□

3.16. Theorem Suppose that A is a BL-algebra, $\mathfrak{f}^* : Reg(A) \to Reg(A)$ is an (a strong) mcd-operator on $Reg(A)$. Then the mapping $\mathfrak{f} : A \to A$ as $\mathfrak{f}(x) = \mathfrak{f}^*(x^{--})$ for any $x \in A$ is an (a strong) mcd-operator on A.

Proof. Let $x, y \in A$.

(t1) $\mathfrak{f}(x \odot y) = \mathfrak{f}^*((x \odot y)^{--}) = \mathfrak{f}^*(x^{--} \odot y^{--}) = \mathfrak{f}^*(x^{--} \underline{\odot} y^{--})$
$= \mathfrak{f}^*(x^{--}) \underline{\odot} \mathfrak{f}^*(y^{--}) = \mathfrak{f}^*(x^{--}) \odot \mathfrak{f}^*(y^{--}) = \mathfrak{f}(x) \odot \mathfrak{f}(y).$
(25)

(t2) $x \odot \mathfrak{f}(x) = x \odot \mathfrak{f}^*(x^{--}) \leq x^{--} \odot \mathfrak{f}^*(x^{--}) = x^{--} \underline{\odot} \mathfrak{f}^*(x^{--}) \leq \mathfrak{f}^*\mathfrak{f}^*(x^{--})$
$\leq \mathfrak{f}^*((\mathfrak{f}^*(x^{--}))^{--}) = \mathfrak{f}^*((\mathfrak{f}(x))^{--}) = \mathfrak{f}(\mathfrak{f}(x)) = \mathfrak{f}\mathfrak{f}(x).$ (26)

(t3) $\mathfrak{f}(1) = \mathfrak{f}^*(1^{--}) = \mathfrak{f}^*(1) = 1.$ (27)

(t4) If $\mathfrak{f}^*$ has the stronger property, then

$$\mathfrak{f}(x) = \mathfrak{f}^*(x^{--}) \leq \mathfrak{f}^*\mathfrak{f}^*(x^{--}) \leq \mathfrak{f}^*((\mathfrak{f}^*(x^{--}))^{--}) = \mathfrak{f}^*((\mathfrak{f}(x))^{--}) = \mathfrak{f}(\mathfrak{f}(x)) = \mathfrak{f}\mathfrak{f}(x).$$ (28)

□

3.17. Definition Let A be a BL-algebra with mcd-operator $\mathfrak{f}$ and F (D) be a filter (dsystem) of A. Then F (D) is called a $\mathfrak{f}$-*filter* ($\mathfrak{f}$-*dsystem*) if $\mathfrak{f}(x) \in F$ ($\mathfrak{f}(x) \in D$) for every $x \in F$ ($x \in D$).

3.18. Proposition Let A be a BL-algebra with mcd-operator $\mathfrak{f}$. Then the subset $D = \{x \in A : \mathfrak{f}(x) = 1\}$ of A is a $\mathfrak{f}$-dsystem of A.

Proof.

(D1) $1 \in D$ by (t3).

(D2) Let $x,\ x \to y \in D$. Then $\mathfrak{f}(x) = 1$, $\mathfrak{f}(x \to y)$ =1 We have $x \odot (x \to y) \le y$ from 2.1. Proposition (1). Since $\mathfrak{f}$ is a monotone mapping, we must have
$\mathfrak{f}\big(x \odot (x \to y)\big) = \mathfrak{f}(x) \odot \mathfrak{f}(x \to y) = 1 \ \odot\ 1 = 1 \le \mathfrak{f}(y)$. So, $\mathfrak{f}(y) = 1$ and $y \in D$.

Finally, Let $x \in D$. Then we get $\mathfrak{f}(x) = 1$. Hence, $\mathfrak{f}\mathfrak{f}(x) = \mathfrak{f}\big(\mathfrak{f}(x)\big) = \mathfrak{f}(1) = 1$. It means that $\mathfrak{f}(x) \in D$. □

3.19. Theorem Suppose that A is a BL-algebra, $\mathfrak{f}\colon A \to A$ be an (a strong) mcd-operator and D is a $\mathfrak{f}$-dsystem of A. Then the mapping $\mathfrak{f}^*\colon A/D \to A/D$ such that $\mathfrak{f}^*([x]) = [\mathfrak{f}(x)]$ is an (a strong) mcd-operator on the quotient BL-algebra A /D.

Proof. We know by 2.4. Theorem that $\sim_D$ is a congruence relation according to $\odot,\ \wedge,\ \vee,\ \to,\ ^{-}$. Suppose that $x \sim_D\ y$. Then $(x \to y)\ \odot\ (y \to x) \in D$. We have $(x \to y)\ \odot\ (y \to x) \le (x \to y)$ and $\mathfrak{f}\big((x \to y)\ \odot (y \to x)\big) \le \mathfrak{f}(x \to y) \le \mathfrak{f}(x) \to \mathfrak{f}(y)$ by 3.10. Proposition (i). Since D is a $\mathfrak{f}$-dsystem of A, $\mathfrak{f}(x) \to \mathfrak{f}(y) \in D$ and similarly, $\mathfrak{f}(y) \to \mathfrak{f}(x) \in D$. So, $(\mathfrak{f}(x) \to \mathfrak{f}(y)) \odot (\mathfrak{f}(y) \to \mathfrak{f}(x)) \in D$. That means $\mathfrak{f}(x) \sim_D\ \mathfrak{f}(y)$. Therefore, the relation $\sim_D$ is a congruence relation with respect to $\odot,\ \wedge,\ \vee,\ \to,\ ^{-}$ and $\mathfrak{f}$. Finally, we have to show that $\mathfrak{f}^*$ is an mcd operator on A/D.

(t1) $\mathfrak{f}^*([x] \otimes [y]) = \mathfrak{f}^*([x \odot y]) = [\mathfrak{f}(x \odot y)] = [\mathfrak{f}(x) \odot \mathfrak{f}(y)]$ by 2.5. Theorem
$= [\mathfrak{f}(x)] \otimes [\mathfrak{f}(y)] = \mathfrak{f}^*([x]) \otimes \mathfrak{f}^*([y])$ by 2.5. Theorem (29)

(t2) $[x] \otimes \mathfrak{f}^*([x]) = [x] \otimes [\mathfrak{f}(x)] = [x \odot \mathfrak{f}(x)] \le [\mathfrak{f}\mathfrak{f}(x)]$ by 2.5. Theorem
$= \big[\mathfrak{f}\big(\mathfrak{f}(x)\big)\big] = \mathfrak{f}^*[\mathfrak{f}(x)] = \mathfrak{f}^*\mathfrak{f}^*([x])$. (30)

(t3) $\mathfrak{f}^*([1]) = [\mathfrak{f}(1)] = [1]$. (31)

If $\mathfrak{f}$ has the stronger property, then

(t4) $\mathfrak{f}^*([x]) = [\mathfrak{f}(x)] \le [\mathfrak{f}\mathfrak{f}(x)] = \mathfrak{f}^*[\mathfrak{f}(x)] = \mathfrak{f}^*\mathfrak{f}^*([x])$. (32)
□

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