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Fourier Method for Higher Order Quasi-Linear Parabolic Equation Subject with Periodic Boundary Conditions

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Abstract. In this paper, higher order inverse quasi-linear parabolic problem was investigated. It demonstrated the solution by the Fourier approximation. It proved the existence, uniqueness of the solution by Fourier and iteration method.

1. Introduction

In this study we present a high order scheme for determining unknown control parameter and unknown solution of two-dimensional parabolic inverse problem. Two- dimensional inverse parabolic problems are used especially in chemical diffusion applications, heat transfer processes have been used a lot such as population, medical area, electrochemistry, engineering, chemical area, plasma physics .This kind of problems with nonlocal boundary conditions are not easy to study. There are many papers on finding analytical and numerical solutions of inverse coefficient problems with nonlocal boundary conditions in one dimension [2, 5]. In these papers, Finite Difference Method, Boundary Element Method, Finite Element Method, etc. are examined to approximate numerical solutions.Finding of the unknown function in a nonlinear parabolic equation is used frequently by many engineers and scientists [1–5].

In this study, Fourier method is used for the for the solution of this problem.

Here $\Gamma := \{0 < x < \pi,\ 0 < y < \pi,\ 0 < t < T\}$, $\varphi(x, y)$, $f(x, y, t, u)$ are given functions.

$$\frac{\partial u}{\partial t} = b(t)\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y, t, u), (x, y, t) \in \Gamma \tag{1}$$

$$u(x, y, 0) = \varphi(x, y),\ x \in [0, \pi], y \in [0, \pi] \tag{2}$$

$$\begin{aligned} u(0, y, t) &= u(\pi, y, t),\ y \in [0, \pi], t \in [0, T] \\ u(x, 0, t) &= u(x, \pi, t),\ x \in [0, \pi], t \in [0, T] \end{aligned} \tag{3}$$

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$$\begin{aligned} u_x(0,y,t) &= u_x(\pi,y,t),\ y\in[0,\pi],t\in[0,T] \\ u_y(x,0,t) &= u_y(x,\pi,t),\ x\in[0,\pi],t\in[0,T] \end{aligned} \tag{4}$$

$$k(t)=\int_0^{\pi}\int_0^{\pi} xyu(x,y,t)dxdy,t\in[0,T] \tag{5}$$

where, in heat diffusion in a thin rod in which the law of variation $k(t)$ of the total quantity of heat in the bar is given. [6]

2. Solution of (1)-(4) Model

As known, in Fourier Method, the solution of problem (1)–(4) is considered in the following form :

$$\begin{aligned} u(x,y,t) &= \frac{u_0(t)}{4} \\ &+ \sum_{m,n=1}^{\infty}\left(u_{cmn}(t)\cos(2mx)\ \ \cos(2ny)+u_{csmn}(t)\cos(2mx)\ \ \sin(2ny)\right) \\ &+ \sum_{m,n=1}^{\infty}\left(u_{scmn}(t)\sin(2mx)\cos(2ny)+u_{smn}(t)\sin(2mx)\sin(2ny)\right). \end{aligned}$$

We have Fourier coefficients by applying the standart procedure of the Fourier method, as follows:

$$\begin{aligned} u_0(t) &= u_0(0)+\frac{4}{\pi^2}\int_0^t\int_0^{\pi}\int_0^{\pi} f(x,y,\tau,u)\,dxdyd\tau \\ u_{cmn}(t) &= u_{cmn}(0)e^{-\int_0^t\left[b(s)(2m)^2+(2n)^2\right]ds}+\frac{4}{\pi^2}\int_0^t\int_0^{\pi}\int_0^{\pi} e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds} f(x,y,\tau,u)\cos(2mx)\ \ \cos(2ny)\,dxdyd\tau \\ u_{csmn}(t) &= u_{csmn}(0)e^{-\int_0^t\left[b(s)(2m)^2+(2n)^2\right]ds}+\frac{4}{\pi^2}\int_0^t\int_0^{\pi}\int_0^{\pi} e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds} f(x,y,\tau,u)\cos(2mx)\ \ \sin(2ny)\,dxdyd\tau \\ u_{scmn}(t) &= u_{scmn}(0)e^{-\int_0^t\left[b(s)(2m)^2+(2n)^2\right]ds}+\frac{4}{\pi^2}\int_0^t\int_0^{\pi}\int_0^{\pi} e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds} f(x,y,\tau,u)\sin(2mx)\ \ \cos(2ny)\,dxdyd\tau \\ u_{smn}(t) &= u_{smn}(0)e^{-\int_0^t\left[b(s)(2m)^2+(2n)^2\right]ds}+\frac{4}{\pi^2}\int_0^t\int_0^{\pi}\int_0^{\pi} e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds} f(x,y,\tau,u)\sin(2mx)\ \ \sin(2ny)\,dxdyd\tau \end{aligned}$$

Then we obtain the solution:

$$
\begin{aligned}
u(x,y,t) &= \frac{1}{4}\left(\varphi_0+\frac{4}{\pi^2}\int_0^t f_0(\tau,u)\,d\tau\right)\\
&+\sum_{m,n=1}^{\infty}\left(\varphi_{cmn}+\frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds} f_{cmn}(\tau,u)\,d\tau\right)\cos(2mx)\cos(2ny)\\
&+\sum_{m,n=1}^{\infty}\left(\varphi_{csmn}+\frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds} f_{csmn}(\tau,u)d\tau\right)\cos(2mx)\ \sin(2ny) \qquad (6)\\
&+\sum_{m,n=1}^{\infty}\left(\varphi_{scmn}+\frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds} f_{scmn}(\tau,u)d\tau\right)\sin(2mx)\cos(2ny)\\
&+\sum_{m,n=1}^{\infty}\left(\varphi_{smn}+\frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds} f_{smn}(\tau,u)d\tau\right)\sin(2mx)\sin(2ny)
\end{aligned}
$$

where $\varphi_0 = u_0(0), \varphi_{cmn} = u_{cmn}(0)e^{-\int_0^t\left[b(s)(2m)^2+(2n)^2\right]ds}, \varphi_{csmn} = u_{csmn}(0)e^{-\int_0^t\left[b(s)(2m)^2+(2n)^2\right]ds}$,

$\varphi_{scmn} = u_{scmn}(0)e^{-\int_0^t\left[b(s)(2m)^2+(2n)^2\right]ds}, \varphi_{smn} = u_{smn}(0)e^{-\int_0^t\left[b(s)(2m)^2+(2n)^2\right]ds}$.

We have the following constraints for functions of the problem:

(C1) $k(t) \in C^1[0,T]$

(C2) $\varphi(x,y)\epsilon C^{1,1}([0,\pi]\times[0,\pi]), \varphi(0,y) = \varphi(\pi,y), \varphi_x(0,y) = \varphi_x(\pi,y), \varphi(x,0) = \varphi(x,\pi), \varphi_y(x,0) = \varphi_y(x,\pi)$ and

$\int_0^\pi\int_0^\pi xy\varphi(x,y)dxdy = k(0)$,

(C3) $f(x,y,t,u)$ is provided following conditions:

(1)$\left|\frac{\partial f(x,y,t,u)}{\partial x} - \frac{\partial f(x,y,t,\widetilde{u})}{\partial x}\right| \le l(x,y,t)\left|u-\widetilde{u}\right|$,

$\left|\frac{\partial f(x,y,t,u)}{\partial y} - \frac{\partial f(x,y,t,\widetilde{u})}{\partial y}\right| \le l(x,y,t)\left|u-\widetilde{u}\right|$,

$\left|\frac{\partial^2 f(x,y,t,u)}{\partial x\partial y} - \frac{\partial^2 f(x,y,t,\widetilde{u})}{\partial x\partial y}\right| \le l(x,y,t)\left|u-\widetilde{u}\right|$ where $l(x,y,t)\in L_2(\Gamma), l(x,y,t)\ge 0$,

(2) $f(x,y,t,u)\in C^{2,2,0}[0,\pi], t\in[0,T]$,

(3) $f(x,y,t,u)\big|_{x=0} = f(x,y,t,u)\big|_{x=\pi}, f_x(x,y,t,u)\big|_{x=0} = f_x(x,y,t,u)\big|_{x=\pi}, f_y(x,y,t,u)\big|_{y=0} = f_y(x,y,t,u)\big|_{y=\pi}$, $f_{xy}(x,y,t,u)\big|_{x=0} = f_{xy}(x,y,t,u)\big|_{x=\pi}, f_{xy}(x,y,t,u)\big|_{y=0} = f_{xy}(x,y,t,u)\big|_{y=\pi}$

(5) can be diffrentiated under the assumptions (C1)-(C3),

$$\int_0^\pi\int_0^\pi xyu_t(x,t)dxdy = k'(t),\ 0\le t\le T. \qquad (7)$$

then the unknown coefficient is obtained in this form

$$b(t) = \frac{k'(t) - \int_0^\pi\int_0^\pi xyf(x,y,t,u)dxdy - \frac{\pi^3}{2}u_y(\pi,t)}{\frac{\pi^3}{2}u_x(\pi,t)}. \qquad (8)$$

Definition 2.1. *Show the set* $\{u(t)\} = \{u_0(t), u_{cmn}(t), u_{csmn}(t), u_{scmn}(t), u_{smn}(t), m,n = 1,...\}$ *of continuous functions on* $[0,T]$ *which satisfy the condition*

$\max\limits_{0\le t\le T}\frac{|u_0(t)|}{4}+\sum\limits_{m,n=1}^{\infty}\left(\max\limits_{0\le t\le T}|u_{cmn}(t)|+\max\limits_{0\le t\le T}|u_{csmn}(t)|+\max\limits_{0\le t\le T}|u_{scmn}(t)|+\max\limits_{0\le t\le T}|u_{smn}(t)|\right)<\infty$.

$\|u(t)\|=\max\limits_{0\le t\le T}\frac{|u_0(t)|}{4}+\sum\limits_{m,n=1}^{\infty}\left(\max\limits_{0\le t\le T}|u_{cmn}(t)|+\max\limits_{0\le t\le T}|u_{csmn}(t)|+\max\limits_{0\le t\le T}|u_{scmn}(t)|+\max\limits_{0\le t\le T}|u_{smn}(t)|\right)$ is the norm in B. (B is the Banach spaces).

Theorem 2.2. *If the conditions (C1)-(C3) be implemented. Then it has a unique solution.*

Proof. If we apply an iteration to equation (6), the following functions are obtained:

$$
\begin{aligned}
u_0^{(N+1)}(t) &= \varphi_0+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi f\left(x,y,\tau,u^{(N)}\right)dxdyd\tau,\\
u_{cmn}^{(N+1)}(t) &= \varphi_{cmn}+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds}\cos(2mx)\cos(2ny)\ f\left(x,y,\tau,u^{(N)}\right)dxdyd\tau,\\
u_{csmn}^{(N+1)}(t) &= \varphi_{csmn}+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds}\cos(2mx)\sin(2ny)\,f\left(x,y,\tau,u^{(N)}\right)dxdyd\tau,\\
u_{scmn}^{(N+1)}(t) &= \varphi_{scmn}+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds}\sin(2mx)\cos(2ny)\ f\left(x,y,\tau,u^{(N)}\right)dxdyd\tau,\\
u_{smn}^{(N+1)}(t) &= \varphi_{smn}+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds}\sin(2mx)\sin(2ny)\ f\left(x,y,\tau,u^{(N)}\right)dxdyd\tau.
\end{aligned}
$$

According to the assumptions , we get $u^{(0)}(t)\in\mathbf{B}$, $t\in[0,T]$. Using Cauchy ,Hölder, Bessel inequalities and Lipschitzs condition, finally we get:

$$
\begin{aligned}
\left\|u^{(1)}(t)\right\|_{\mathbf{B}} &= \max_{0\le t\le T}\frac{\left|u_0^{(1)}(t)\right|}{4}+\sum_{m,n=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{cmn}^{(1)}(t)\right|+\max_{0\le t\le T}\left|u_{csmn}^{(1)}(t)\right|+\max_{0\le t\le T}\left|u_{scmn}^{(1)}(t)\right|+\max_{0\le t\le T}\left|u_{smn}^{(1)}(t)\right|\right)\\
&\le \frac{\left|\varphi_0\right|}{2}+\sum_{m,n=1}^{\infty}\left(\left|\varphi_{cmn}\right|+\left|\varphi_{csmn}\right|+\left|\varphi_{scmn}\right|+\left|\varphi_{smn}\right|\right)\\
&\quad+\sqrt{T}\left(\frac{3\sqrt{\pi}+16}{3\pi}\right)\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_B\\
&\quad+\sqrt{T}\left(\frac{3\sqrt{\pi}+16}{3\pi}\right)M.
\end{aligned}
$$

According to the assumptions of the theorem, we have $u^{(1)}(t)\in\mathbf{B}$. The same operations for the step N,

$$\begin{aligned}\left\|u^{(N+1)}(t)\right\|_{B} &= \max_{0\le t\le T}\frac{\left|u_0^{(N)}(t)\right|}{4} + \sum_{m,n=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{cmn}^{(N)}(t)\right| + \max_{0\le t\le T}\left|u_{csmn}^{(N)}(t)\right| + \max_{0\le t\le T}\left|u_{scmn}^{(N)}(t)\right| + \max_{0\le t\le T}\left|u_{smn}^{(N)}(t)\right|\right)\\ &\le \frac{\left|\varphi_0\right|}{2} + \sum_{m,n=1}^{\infty}\left(\left|\varphi_{cmn}\right| + \left|\varphi_{csmn}\right| + \left|\varphi_{scmn}\right| + \left|\varphi_{smn}\right|\right)\\ &+ \sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u^{(N)}(t)\right\|_{B}\\ &+ \sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})M.\end{aligned}$$

is obtained. We get $u^{(N+1)}(t)\in\mathbf{B}$ since $u^{(N)}(t)\in\mathbf{B}$,

$$\{u(t)\} = \{u_0(t), u_{cmn}(t), u_{csmn}(t), u_{scmn}(t), u_{smn}(t), m,n=1,...\}\in\mathbf{B}.$$

If we apply an iteration to equation (8), the following functions are obtained::

$$b^{(N+1)}(t) = \frac{k'(t) - \int_0^{\pi}\int_0^{\pi} xyf(x,y,t,u^{(N)})dxdy - \frac{\pi^3}{2}u_y^{(N)}(\pi,t)}{\frac{\pi^3}{2}u_x^{(N)}(\pi,t)}.$$

By using the same operations we obtain:

$$\left\|b^{(N+1)}(t)\right\|_{C[0,T]} \le \frac{\left|k'(t)\right| + \frac{\pi^4}{4}\left\|u^{(N)}(t)\right\|_B}{\frac{\pi^3}{2}\left\|u^{(N)}(t)\right\|_B}$$

$$\left\|b^{(N+1)}(t)\right\|_{C[0,T]} \le \frac{\pi}{2} + \frac{2\left|k'(t)\right|}{\left\|u^{(N)}(t)\right\|_B}$$

We get $b^{(N+1)}(t)\in C[0,T]$ since $u^{(N)}(t)\in B$.

Let us show that, $u^{(N+1)}(t)$, $b^{(N+1)}$ are converged for $N \to \infty$.

$$\begin{aligned}
u^{(1)}(t) - u^{(0)}(t) &= \frac{(u_0^{(1)}(t) - u_0^{(0)}(t))}{4} \\
&\quad + [(u_{cmn}^{(1)}(t) - u_{cmn}^{(0)}(t)) + (u_{csmn}^{(1)}(t) - u_{csmn}^{(0)}(t)) + (u_{scmn}^{(1)}(t) - u_{scmn}^{(0)}(t)) + (u_{smn}^{(1)}(t) - u_{smn}^{(0)}(t))] \\
&= \frac{1}{4}\left(\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi \left[f_{\alpha\beta}(x,y,\tau,u^{(0)}) - f_{\alpha\beta}(x,y,\tau,0)\right] dxdyd\tau\right) \\
&\quad + \sum_{m,n=1}^{\infty} \frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi \left[f_{xy}(x,y,\tau,u^{(0)}) - f_{xy}(x,y,\tau,0)\right] e^{-\int_\tau^t [b(s)(2m)^2+(2n)^2]ds} \cos(2mx)\cos(2ny)\, dxdyd\tau \\
&\quad + \sum_{m,n=1}^{\infty} \frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t [b(s)(2m)^2+(2n)^2]ds} \cos(2mx)\sin(2ny)\, dxdyd\tau \\
&\quad + \sum_{m,n=1}^{\infty} \frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi \left[f_{xy}(x,y,\tau,u^{(0)}) - f_{xy}(x,y,\tau,0)\right] e^{-\int_\tau^t [b(s)(2m)^2+(2n)^2]ds} \sin(2mx)\cos(2ny)\, dxdyd\tau \\
&\quad + \sum_{m,n=1}^{\infty} \frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi \left[f_{xy}(x,y,\tau,u^{(0)}) - f_{xy}(x,y,\tau,0)\right] e^{-\int_\tau^t [b(s)(2m)^2+(2n)^2]ds} \sin(2mx)\sin(2ny)\, dxdyd\tau \\
&\quad + \frac{1}{4}\left(\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi f_{xy}(x,y,\tau,0)\, dxdyd\tau\right) \\
&\quad + \sum_{m,n=1}^{\infty} \frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi f_{xy}(x,y,\tau,0) e^{-\int_\tau^t [b(s)(2m)^2+(2n)^2]ds} \cos(2mx)\cos(2ny)\; dxdyd\tau \\
&\quad + \sum_{m,n=1}^{\infty} \frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi f_{xy}(x,y,\tau,0) e^{-\int_\tau^t [b(s)(2m)^2+(2n)^2]ds} \cos(2mx)\sin(2ny)\; dxdyd\tau \\
&\quad + \sum_{m,n=1}^{\infty} \frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi f_{xy}(x,y,\tau,0) e^{-\int_\tau^t [b(s)(2m)^2+(2n)^2]ds} \sin(2mx)\cos(2ny)\; dxdyd\tau \\
&\quad + \sum_{m,n=1}^{\infty} \frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi f_{xy}(x,y,\tau,0) e^{-\int_\tau^t [b(s)(2m)^2+(2n)^2]ds} \sin(2mx)\sin(2ny)\, dxdyd\tau.
\end{aligned}$$

Let some inequalities(Bessel, Hölder, Lipschitzs) be implemented , the following estimations are obtained:

$$\left\|u^{(1)}(t) - u^{(0)}(t)\right\|_B \leq \sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\left(\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_B + M\right)$$

where

$$A = \sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\left(\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_B + M\right).$$

$$\left\|u^{(N+1)}(t) - u^{(N)}(t)\right\|_B \leq \frac{A\left\|l(x,y,t)\right\|_{L_2(\Gamma)}^N S^N}{\sqrt{N!}} \tag{9}$$

where

$$S = \sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\left(1 + \frac{\pi M}{2\left\|u^{(N)}(t)\right\|_B\left\|u^{(N+1)}(t)\right\|_B} + \frac{\pi\left\|l(x,y,t)\right\|_{L_2(\Gamma)}}{2\left\|u^{(N)}(t)\right\|_B}\right).$$

By using the same operations we obtain:

$$\left\|b^{(1)}(t)-b^{(0)}(t)\right\|_{C[0,T]}\leq C_1\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_B.$$

The same operations for the step N :

$$\left\|b^{(N+1)}(t)-b^{(N)}(t)\right\|_{C[0,T]}\leq C_N\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_B$$

where $C_1=\left(\frac{\pi M}{2\|u^{(0)}(t)\|_B\|u^{(1)}(t)\|_B}+\frac{\pi\|l(x,y,t)\|_{L_2(\Gamma)}}{2\|u^{(0)}(t)\|_B}\right),...,C_N=\left(\frac{\pi M}{2\|u^{(N)}(t)\|_B\|u^{(N+1)}(t)\|_B}+\frac{\pi\|l(x,y,t)\|_{L_2(\Gamma)}}{2\|u^{(N)}(t)\|_B}\right)$.The series which is consisting of the right hand side of (9) are convergent by ratio test. So, the series which is consisting of the left hand side of (9) are convergent by comparison test. Moreover, by the Weierstrass M test , the series $\sum_{N=0}^{\infty}\left|u^{(N+1)}(t)-u^{(N)}(t)\right|$ is uniformly convergent.

We obtain $u^{(N+1)}\to u^{(N)}, b^{(N+1)}\to b^{(N)}, N\to\infty$.

Therefore $u^{(N+1)}(t)$ and $b^{(N+1)}(t)$ are converged.

Now let's show that:

$$\lim_{N\to\infty}u^{(N+1)}(t)=u(t),\ \lim_{N\to\infty}b^{(N+1)}(t)=b(t).$$

By using Cauchy, Hölder, Bessel and Lipschitzs inequalities, we have

$$\begin{aligned}\left\|u(t)-u^{(N+1)}(t)\right\|_{\mathbf{B}} &\leq \sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u(t)-u^{(N+1)}(t)\right\|_B\\ &+\sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_B\\ &+\sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})M|T|\left\|u(t)-u^{(N+1)}(t)\right\|_B.\end{aligned}$$

By using the same operations we obtain:

$$\begin{aligned}\left\|b(t)-b^{(N+1)}(t)\right\|_{C[0,T]} &\leq C_N\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u(t)-u^{(N+1)}(t)\right\|_B\\ &+C_N\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_{B.}\end{aligned}$$

$$\begin{aligned}\left\|u(t)-u^{(N+1)}(t)\right\|_{\mathbf{B}} &\leq \sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u(t)-u^{(N+1)}(t)\right\|_B\\ &+\sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_B\\ &+\sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})M|T|\left\|b(t)-b^{(N+1)}(t)\right\|_B\end{aligned}$$

applying Gronwall's inequality to last inequality ,we have

$$\begin{aligned}\left\|u(t)-u^{(N+1)}(t)\right\|_{\mathbf{B}}^2 &\leq 2\frac{\left(A\sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\right)^2}{\sqrt{N!}}\left(\left\|l(x,y,t)\right\|_{L_2(\Gamma)}^{N+1}\right)^2\\ &\times\exp\left(\sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\right)^2\left\|l(x,y,t)\right\|_{L_2(\Gamma)}^2.\end{aligned}\qquad(10)$$

The series which is consisting of the right hand side of (10) are convergent by ratio test. So, the series which is consisting of the left hand side of (10) are convergent by comparison test. Moreover, by the Weierstrass M test , the series $\sum_{N=0}^{\infty}\left|u(t)-u^{(N+1)}(t)\right|$ is uniformly convergent.

We obtain $u^{(N+1)} \to u, b^{(N+1)} \to b, N \to \infty$.

To show the uniqueness, we get two solution pairs of the problem (1)–(5) as (c, u) and (b, v)

Applying Cauchy inequality, Hölder Inequality, Lipschitzs condition and Bessel inequality to the difference $|u(t) - v(t)|$, we obtain

$$\begin{aligned} \|u(t)-v(t)\|_{\mathbf{B}} \leq\ & \sqrt{T}\left(\frac{3\sqrt{\pi}+16}{3\pi}\right)\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\|u(t)-v(t)\|_B \\ & + \sqrt{T}\left(\frac{3\sqrt{\pi}+16}{3\pi}\right)M\,|T|\,\|b(t)-c(t)\|_B \end{aligned}$$

$$\|u(t)-v(t)\|_B \leq 0 \times \exp\left(\sqrt{T}\left(\frac{3\sqrt{\pi}+16}{3\pi}\right)\right)^2 \left\|l(x,y,t))\right\|^2_{L_2(\Gamma)}, \tag{11}$$

we get $u(t) = v(t)$ and $c(t) = b(t)$.

The proof is over. □

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