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Research Article

New aspects of weaving K-frames: the excess and duality

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Abstract

Weaving frames in separable Hilbert spaces have been recently introduced by Bemrose et al. to deal with some problems in distributed signal processing and wireless sensor networks. Likewise weaving K-frames have been proved to be useful during signal reconstructions from the range of a bounded linear operator K. In this paper, we study the notion of weaving and its connection to the duality of K-frames and construct several pairs of woven K-frames. Also, we find a unique biorthogonal sequence for every K-Riesz basis and obtain a K^*-frame which is woven by its canonical dual. Moreover, we describe the excess for K-frames and prove that any two woven K-frames in a separable Hilbert space have the same excess. Finally, we introduce the necessary and sufficient condition under which a K-frame and its image under an invertible operator have the same excess.

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1. Introduction and motivation

Frame theory has been converted as a useful tools in order to solve many problems from signal and image processing to differential equation and so on [9, 12, 16, 18].

The notion of K-frames has been introduced by *Găvrua* [28] to study the atomic system with respect to a bounded linear operator K in a separable Hilbert space $\mathcal{H}$. There exist many differences between frames and K-frames. Indeed, K-frames are more general than ordinary frames in the sense that the lower frame bound only holds for the elements in the range of K. Also, a K-frame is the image of an orthonormal basis under a bounded linear operator K, whereas a frame is the image of an orthonormal basis under a bounded linear surjection [28].

Traditionally, frame coefficients of a given frame have been used to represent every element of underlying Hilbert space as a linear combination of the frame elements. The concept of woven frames, which is motivated by some problems in signal processing [11], is used to write this linear combination by at least two frames. See [7, 17, 20, 23, 27, 29] for more results on K-frames and weaving. Study and analysis of woven K-frames is the main purpose of this article. Motivation of this work is study the dual of K-frames.

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2. Preliminaries and notations

2.1. Discrete frames

A sequence $\Phi = \{\varphi_i\}_{i\in I}$ in a separable Hilbert space $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist constants $0 < A_\Phi \leq B_\Phi < \infty$ such that

$$A_\Phi \|f\|^2 \leq \sum_{i\in I} |\langle f, \varphi_i\rangle|^2 \leq B_\Phi \|f\|^2, \qquad (f \in \mathcal{H}).$$

The constants A_Φ and B_Φ are called lower and upper frame bounds, respectively. If just the right inequality in the above holds, then Φ is called a *Bessel* sequence. A sequence $\Phi = \{\varphi_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ is called a *Riesz sequence* if there are constants $0 < A_\Phi \leq B_\Phi < \infty$ such that for every sequence $\{c_i\}_{i\in I} \in \ell^2$ we have

$$A_\Phi \sum_{i\in I} |c_i|^2 \leq \|\sum_{i\in I} c_i\varphi_i\|^2 \leq B_\Phi \sum_{i\in I} |c_i|^2.$$

The constants A_Φ and B_Φ are called lower and upper Riesz bounds, respectively. A subset A subset $\mathcal{H}$ is called *complete* whenever $\langle y, x\rangle = 0$ for all $x \in A$ implies that $y = 0$. In addition, if Φ is complete in $\mathcal{H}$, then it is called a *Riesz basis* for $\mathcal{H}$.

Given a Bessel sequence $\Phi = \{\varphi_i\}_{i\in I}$, the *synthesis operator* $T_\Phi : \ell^2 \to \mathcal{H}$ is defined by $T_\Phi\{c_i\} = \sum_{i\in I} c_i\varphi_i$. Its adjoint, $T_\Phi^* : \mathcal{H} \to \ell^2$, which is called the *analysis operator*, is given by $T_\Phi^* f = \{\langle f, \varphi_i\rangle\}_{i\in I}$. Moreover, $S_\Phi : \mathcal{H} \to \mathcal{H}$ the *frame operator* of Φ, is given by $S_\Phi f = T_\Phi T_\Phi^* f$. If Φ is a frame with frame bounds A_Φ and B_Φ, then S_Φ is invertible and $A_\Phi I_{\mathcal{H}} \leq S_\Phi \leq B_\Phi I_{\mathcal{H}}$, for more details see Subsection 5.1 of [15]. The sequence $\tilde{\Phi} = \{S_\Phi^{-1}\varphi_i\}_{i\in I}$, which is also a frame, is called the *canonical dual frame*. A frame $\{\psi_i\}_{i\in I}$ is called a *dual* of $\{\varphi_i\}_{i\in I}$ if

$$f = \sum_{i\in I} \langle f, \psi_i\rangle \varphi_i, \qquad (f \in \mathcal{H}).$$

Also if $\Phi = \{\varphi_i\}_{i\in I}$ is a frame, then every dual frame of Φ is of the form of $\Phi^d = \{S_\Phi^{-1}\varphi_i + u_i\}_{i\in I}$ [19] where $\{u_i\}_{i\in I}$ is a Bessel sequence such that

$$\sum_{i\in I} \langle f, \varphi_i\rangle u_i = 0, \qquad (f \in \mathcal{H}).$$

Throughout the paper, $\mathcal{H}$ is a separable Hilbert space, I a countable index set, $I_{\mathcal{H}}$ the identity operator on Hilbert space $\mathcal{H}$ and K is a closed range operator in $B(\mathcal{H})$, the set of all bounded operators on $\mathcal{H}$. Also, we denote the range of $K \in B(\mathcal{H})$ by $R(K)$, and the orthogonal projection of $\mathcal{H}$ onto a closed subspace $V \subseteq \mathcal{H}$ is denoted by π_V. Moreover we denote $\Phi = \{\varphi_i\}_{i\in I}$ for a frame with A_Φ and B_Φ as the lower and upper frame bounds. Also we use of $[m]$ to denote the set $\{1, 2, \ldots, m\}$.

2.2. K-frames

Now, we recall some definitions and primary results of K-frames, which are used in the present paper. For more information see [4, 21]. Let $K \in B(\mathcal{H})$, the set of all bounded operators on a Hilbert space $\mathcal{H}$. A sequence $\Phi := \{\varphi_i\}_{i=1}^\infty$ in $\mathcal{H}$ is called a *K-frame* for $\mathcal{H}$ if there exist constants $0 < A_\Phi \leq B_\Phi$ such that

$$A_\Phi \|K^* f\|^2 \leq \sum_{i\in I} |\langle f, \varphi_i\rangle|^2 \leq B_\Phi \|f\|^2, \qquad (f \in \mathcal{H}).$$

Every K-frame $\Phi = \{\varphi_i\}_{i\in I}$ is a Bessel sequence. Hence T_Φ, T_Φ^* and in particular S_Φ are well-defined. For a Bessel sequence Φ, it is proved that Φ is K-frame if and only if $R(K) \subseteq R(T_\Phi)$ [21] where $R(K)$ is the range of the operator K. Because of the higher generality of K-frames, the associated K-frame operator need not be invertible and if K has close range, then

$$B_\Phi^{-1}\|f\| \leq \|S_\Phi^{-1}f\| \leq A_\Phi^{-1}\|K^\dagger\|^2\|f\|, \qquad (f \in R(K)),$$

where $K^\dagger$ is the pseudo inverse of K, see [15] for more details. More precisely, $KK^\dagger$ is the orthogonal projection on $R(K)$, this easily follows

$$\|K^\dagger\|^{-1}\|Kf\| \leq \|K^*Kf\|, \qquad (f \in \mathcal{H}), \tag{2.1}$$

i.e., K^* is bounded below on $R(K)$. Thus, S_Φ is invertible on $R(K)$. However, $S_\Phi|_{R(K)} : R(K) \to S_\Phi(R(K))$ is not self-adjoint, in general. More precisely,

$$(S_\Phi|_{R(K)})^* = \pi_{R(K)}S_\Phi, \quad ((S_\Phi|_{R(K)})^{-1}\pi_{S_\Phi(R(K))})^* = ((S_\Phi|_{R(K)})^{-1})^*\pi_{R(K)}, \tag{2.2}$$

where $\pi_{R(K)}$ is the orthogonal projection of $\mathcal{H}$ onto a closed subspace $R(K)$. Indeed, for every $f, g \in \mathcal{H}$ we have

$$\pi_{R(K)}S_\Phi(S_\Phi^{-1})^*f = S_\Phi(S_\Phi^{-1})^*f = (S_\Phi^{-1}S_\Phi)f = f.$$

Hence,

$$(S_\Phi^{-1})^*\pi_{R(K)}S_\Phi g = (S_\Phi^{-1})^*S_\Phi^* g = ((S_\Phi S_\Phi^{-1})^* g = g.$$

This proves the first equality of (2.2) . Also, for every $f, g \in \mathcal{H}$ we have

$$\left\langle f, S_\Phi^{-1}\pi_{S_\Phi(R(K))}g \right\rangle = \left\langle f, \pi_{R(K)}S_\Phi^{-1}\pi_{S_\Phi(R(K))}g \right\rangle = \left\langle (S_\Phi^{-1})^*\pi_{R(K)}f, g \right\rangle.$$

So, we obtain the second equality in (2.2).

For simply, we denote $(S_\Phi|_{R(K)})^{-1}$ by S_Φ^{-1}. Let $\{\varphi_i\}_{i\in I}$ be a Bessel sequence. A Bessel sequence $\{\psi_i\}_{i\in I} \subset \mathcal{H}$ is called a *K-dual* of $\{\varphi_i\}_{i\in I}$ if

$$Kf = \sum_{i\in I}\langle f, \psi_i\rangle \pi_{R(K)}\varphi_i, \qquad (f \in \mathcal{H}). \tag{2.3}$$

In [4], it is shown that $\Phi := \{\varphi_i\}_{i=1}^\infty$ and $\Psi := \{\psi_i\}_{i=1}^\infty$ in (2.3) are interchangeable if and only if K is self adjoint [4]. In this case, Φ and Ψ are K-frame and K^*-frame with the lower bounds B_Ψ^{-1} and B_Φ^{-1} [4]. Let $K \in B(\mathcal{H})$ have close range and $\{\varphi_i\}_{i\in I}$ be a K-frame with bounds A_Φ and B_Φ. Then $\{K^*S_\Phi^{-1}\pi_{S_\Phi(R(K))}\varphi_i\}_{i\in I}$ is a K-dual of $\{\varphi_i\}_{i\in I}$ with the bounds B_Φ^{-1} and $B_\Phi A_\Phi^{-1}\|K\|^2\|K^\dagger\|^2$, respectively, [25]. It is called the canonical K-dual of $\Phi = \{\varphi_i\}_{i\in I}$ and is denoted by $\tilde{\Phi}$ for brevity.

The following theorem describes all K-duals of a K-frame with respect to its canonical dual.

Theorem 2.1 ([4, 25]). *Let K be a bounded linear operator on $\mathcal{H}$ with closed range. Suppose $\Phi = \{\varphi_i\}_{i\in I}$ is a K-frame. Then $\Psi = \{\psi_i\}_{i\in I}$ is K-dual of Φ if and only if*

$$\psi_i = \widetilde{\varphi_i} + u_i, \qquad (i \in I),$$

where $\{u_i\}_{i\in I}$ is a Bessel sequence such that

$$\sum_{i\in I}\langle f, \varphi_i\rangle \pi_{R(K)}u_i = 0, \qquad (f \in \mathcal{H}). \tag{2.4}$$

For more information about frames and K-frames and its application in pure mathematics and engineering such as image processing, signal processing and sampling see [2, 5, 6, 10, 12–14].

2.3. Woven frames

Recently a new notion in frame theory has been introduced by Bemrose et al. [11]. This fact help us to decompose elements of a Hilbert space by the partitions of frame coefficients of at least two frames.

A family of frames $\{\varphi_{ij}\}_{i\in I}$ for $j \in \{1, \dots, m\}$ for a Hilbert space $\mathcal{H}$ is said to be *woven* [11] if there are universal constants A and B such that for every partition $\{\sigma_j\}_{j=1}^m$ of I, the family $\{\varphi_{ij}\}_{i\in\sigma_j, j=1}^m$ is a frame for $\mathcal{H}$ with lower and upper frame bounds A and B, respectively [11]. Each family $\{\varphi_{ij}\}_{i\in\sigma_j, j=1}^m$ is called a *weaving*. Two frames $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ for Hilbert space $\mathcal{H}$ are *weakly woven* if for every subset $\sigma \subset I$, the family $\{\varphi_i\}_{i\in\sigma} \cup \{\psi_i\}_{i\in\sigma^c}$ is a frame for $\mathcal{H}$.

One of our aim is to find K-duals of a K-frame Φ which are woven with Φ, see Section 3. Moreover, in Section 4, we prove that the wovenness can be transferred from K-frames to their K-duals and vise versa. Finally in Section 5, we study the weaving property for K-Riesz bases. For example, a unique biorthogonal sequence for each K-Riesz basis is given. Roughly speaking we show that every K-Riesz basis is woven as K-frame with its canonical dual and two woven K-frames have the same excess.

3. Woven K-frames

In this section, the definition of woven K-frames is introduced. Then some results are presented in regards to weaving families of vectors. Throughout the rest of the paper for ease of notation, let $[m] = \{1, \dots, m\}$ and $[m]^c = N\backslash[m] = \{m+1, m+2, \dots\}$ for a given natural number m.

Definition 3.1. A family of K-frames $\left\{\{\varphi_{ij}\}_{j=1}^\infty : i \in [m]\right\}$ for $\mathcal{H}$ is said to be *woven* [17] if there exist universal positive constants A and B such that for any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{N}$, the family $\bigcup_{i\in[m]}\{\varphi_{ij}\}_{j\in\sigma_i}$ is a K-frame for $\mathcal{H}$ with the lower and upper K-frame bounds A and B, respectively. Each family $\bigcup_{i\in[m]}\{\varphi_{ij}\}_{j\in\sigma_i}$ is called a *weaving*. A family of K-frames $\{\{\varphi_{ij}\}_{j=1}^\infty : i \in [m]\}$ for $\mathcal{H}$ is said to be *weakly woven* if for any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{N}$, the family $\bigcup_{i\in[m]}\{\varphi_{ij}\}_{j\in\sigma_i}$ is a K-frame for $\mathcal{H}$. In fact, the frame bounds for weakly woven K-frames depend on the partition $\{\sigma_i\}_{i\in[m]}$.

To show two K-frames are woven, due to Proposition 3.1 of [11], we only need to prove the existence of a universal lower bound. In this section, we continue to study the concept of woven K-frames and try to find some conditions under which two K-frames are woven.

Theorem 3.2 ([17])**.** *Two K-frames are woven if and only if they are weakly woven.*

A strategy to find woven K-frames is that we consider K-frames small enough closed to each other. We begin with the following result whose proof is similar to Theorem 6.1 of [11].

Proposition 3.3. *Let $\Phi = \{\varphi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ be K-frames for $\mathcal{H}$, there exists $0 < \lambda < 1$ such that*

$$\lambda\left(\|T_\Phi\| + \|T_\Psi\|\right) \leq \frac{A_\Phi}{2}\|K^\dagger\|^{-2},$$

and for every $\{a_i\}_{i\in I} \in \ell^2$ we have

$$\left\|\sum_{i\in I} a_i(\varphi_i - \psi_i)\right\| \leq \lambda\|\{a_i\}\|. \tag{3.1}$$

Then Φ and Ψ are woven K-frames with the bounds $\frac{A_\Phi}{2}$ and $B_\Phi + B_\Psi$.

Proof. For each $\sigma \subset I$, denote

$$T_\Phi^\sigma\left(\{a_i\}_{i\in I}\right) = \sum_{i\in\sigma} a_i\varphi_i,$$
$$T_\Psi^\sigma(\{a_i\}_{i\in I}) = \sum_{i\in\sigma} a_i\psi_i.$$

By an argument similar to Theorem 6.1 of [11] we see that $\|T_\Phi^\sigma\| \leq \|T_\Phi\|$ and $\|T_\Psi^\sigma\| \leq \|T_\Psi\|$. Moreover by (3.1) we observe that $\|T_\Phi^\sigma - T_\Psi^\sigma\| \leq \|T_\Phi - T_\Psi\| < \lambda$. Hence, for every $f \in \mathcal{H}$ we have

$$\begin{aligned}
\|T_\Phi^\sigma(T_\Phi^\sigma)^* f - T_\Psi^\sigma(T_\Psi^\sigma)^* f\| &\leq \|T_\Phi^\sigma(T_\Phi^\sigma)^* f - T_\Phi^\sigma(T_\Psi^\sigma)^* f\| + \|T_\Phi^\sigma(T_\Psi^\sigma)^* f - T_\Psi^\sigma(T_\Psi^\sigma)^* f\| \\
&\leq \|T_\Phi^\sigma\|\,\|(T_\Phi^\sigma)^* - (T_\Psi^\sigma)^*\|\,\|f\| + \|T_\Phi^\sigma - T_\Psi^\sigma\|\,\|(T_\Psi^\sigma)^*\|\|f\| \\
&\leq \|T_\Phi\|\,\|T_\Phi - T_\Psi\|\,\|f\| + \|T_\Phi - T_\Psi\|\,\|T_\Psi\|\|f\| \\
&\leq \lambda\left(\|T_\Phi\| + \|T_\Psi\|\right)\|f\| \leq \frac{A_\Phi}{2}\|K^\dagger\|^{-2}.
\end{aligned}$$

For each $f \in \mathcal{H}$ by the last inequality we have

$$\begin{aligned}
&\left\|\sum_{i\in\sigma}|\langle f,\psi_i\rangle|^2 + \sum_{i\in\sigma^c}|\langle f,\varphi_i\rangle|^2\right\| \\
= &\left\|\sum_{i\in I}\|\langle f,\varphi_i\rangle|^2 + \left(\sum_{i\in\sigma}|\langle f,\psi_i\rangle|^2 - \sum_{i\in\sigma}|\langle f,\varphi_i\rangle|^2\right)\right\| \\
\geq &\left\|\sum_{i\in I}|\langle f,\varphi_i\rangle|^2\right\| - \left\|\sum_{i\in\sigma}|\langle f,\psi_i\rangle|^2 - \sum_{i\in\sigma}|\langle f,\varphi_i\rangle|^2\right\| \\
\geq\ & A_\Phi\|K^* f\|^2 - \langle\left(T_\Psi^\sigma(T_\Psi^\sigma)^* - T_\Phi^\sigma(T_\Phi^\sigma)^*\right) f, f\rangle \\
\geq\ & A_\Phi\|K^* f\|^2 - \frac{A_\Phi}{2}\|K^\dagger\|^{-2}\|f\|^2 \\
\geq\ & A_\Phi\|K^* f\|^2 - \frac{A_\Phi}{2}\|K^\dagger\|^{-2}\|K^* f\|^2\|K^\dagger\|^2 \\
=\ & \frac{A_\Phi}{2}\|K^* f\|^2.
\end{aligned}$$

So, the lower frame bound is $\frac{A_\Phi}{2}$. □

Let $\mathcal{H} = \mathbb{R}^3$ and $\{e_i\}_{i=1}^3$ be an orthonormal basis of $\mathcal{H}$. Also, let $Ke_1 = e_1 + e_2, Ke_2 = e_2, Ke_3 = 0$. Then $K^*e_1 = e_1, K^*e_2 = e_1 + e_2, K^*e_3 = 0$. So, for $\Phi = \{e_1, e_2, e_3\}$ and $\Psi = \{e_1, e_2, \frac{109}{108}e_3\}$ we have

$$\begin{aligned}
\|K^* f\|^2 &= \|K^* \sum c_i e_i\|^2 = \|c_1 e_1 + c_2(e_1 + e_2)\|^2 \\
&= |c_1 + c_2|^2 + |c_2|^2 \leq 2|c_1|^2 + 3|c_2|^2 \leq 3\sum_{i=1}^3 |c_i|^2.
\end{aligned}$$

Therefore,

$$\frac{1}{3}\|K^* f\|^2 \leq \sum_{i=1}^3 |\langle f,\varphi_i\rangle|^2 = \sum_{i=1}^3 |c_i|^2 = \|f\|^2.$$

Hence, Φ and Ψ are two K-frames with frame bounds $A_\Phi = A_\Psi = \frac{1}{3}, B_\Phi = B_\Psi = 1$. Also, $\|T_\Phi\| = \|T_\Psi\| = 1$. So if $\lambda \leq \frac{1}{108}$, we have

$$\lambda\left(\|T_\Phi\| + \|T_\Psi\|\right) < \frac{A_\Phi}{2}\|K^\dagger\|^{-1} = \frac{1}{54}.$$

Also,

$$\|\sum_{i=1}^{3} a_i\left(\varphi_i-\psi_i\right)\| = \frac{1}{108}|a_3| \leq \lambda\|\{a_i\}\|,$$

which satisfies to the condition of Propositions 3.3.

The following lemma is a generalization of Theorem 3.5 of [24].

Lemma 3.4. *Let Φ and Ψ be woven K-frames for Hilbert space $\mathcal{H}$ and $U \in B(\mathcal{H})$ a with closed range such that $UK = KU$. Then*

(1) *$U\Phi$ and $U\Psi$ are woven UK-frames for $\mathcal{H}$.*
(2) *If $R(K^*) \subseteq R(U)$ (for example U is onto), then $U\Phi$ and $U\Psi$ are woven K-frames for $\mathcal{H}$.*

Proof. Let Φ and Ψ be woven K-frames for $\mathcal{H}$ with a lower frame bound A. So for every $\sigma \subset I$ we have

$$\begin{aligned}\sum_{i\in\sigma}|\langle f, U\varphi_i\rangle|^2 + \sum_{i\in\sigma^c}|\langle f, U\psi_i\rangle|^2 &= \sum_{i\in\sigma}|\langle U^*f, \varphi_i\rangle|^2 + \sum_{i\in\sigma^c}|\langle U^*f, \psi_i\rangle|^2 \\ &\geq A\|K^*U^*f\|^2 \geq A\|(UK)^*f\|^2,\end{aligned}$$

and this proves (1). Since $U \in B(\mathcal{H})$ is closed range, so U^* is bounded below on $R(U)$ and by the assumption we have

$$\begin{aligned}\sum_{i\in\sigma}|\langle f, U\varphi_i\rangle|^2 + \sum_{i\in\sigma^c}|\langle f, U\psi_i\rangle|^2 &= \sum_{i\in\sigma}|\langle U^*f, \varphi_i\rangle|^2 + \sum_{i\in\sigma^c}|\langle U^*f, \psi_i\rangle|^2 \\ &\geq A\|K^*U^*f\|^2 \geq A\|U^*K^*f\|^2 \\ &\geq A\|(U^*)^{-1}\|^{-2}\|K^*f\|^2.\end{aligned}$$

This proves (2). □

Proposition 3.5. *Let $\Phi = \{\varphi_i\}_{i=1}^{\infty}$ be a K-frame for $\mathcal{H}$ and $U \in B(\mathcal{H})$ an onto operator such that $UK = KU$ and*

$$\|\left(I_{\mathcal{H}} - U^*\right)f\| < \alpha\|K^*f\|, \qquad (f \in \mathcal{H})$$

where $\alpha < \sqrt{\frac{A_\Phi}{B_\Phi}}$. Then $U\Phi$ is a K-frame woven by Φ with the universal lower bound $(\sqrt{A_\Phi} - \alpha\sqrt{B_\Phi})^2$.

Proof. By the above lemma we can conclude that the condition $UK = KU$ implies that $U\Phi = \{U\varphi_i\}_{i=1}^{\infty}$ is also a K-frame. Since $\Phi = \{\varphi_i\}_{i=1}^{\infty}$ is a K-frame, so for every $f \in \mathcal{H}$ and $\sigma \subset I$, a non trivial subset of I, we have

$$\begin{aligned}&\left(\sum_{i\in\sigma}|\langle f, \varphi_i\rangle|^2 + \sum_{i\in\sigma^c}|\langle f, U\varphi_i\rangle|^2\right)^{\frac{1}{2}} \\ =\ &\left(\sum_{i\in\sigma}|\langle f, \varphi_i\rangle|^2 + \sum_{i\in\sigma^c}|\langle f, \varphi_i\rangle - \langle\left(I-U^*\right)f, \varphi_i\rangle|^2\right)^{\frac{1}{2}} \\ \geq\ &\left(\sum_{i\in I}|\langle f, \varphi_i\rangle|^2\right)^{\frac{1}{2}} - \left(\sum_{i\in\sigma^c}|\langle\left(I-U^*\right)f, \varphi_i\rangle|^2\right)^{\frac{1}{2}} \\ \geq\ &\sqrt{A_\Phi}\|K^*f\| - \sqrt{B_\Phi}\|(I-U^*)f\| \\ \geq\ &\sqrt{A_\Phi}\|K^*f\| - \sqrt{B_\Phi}\alpha\|K^*f\| \\ \geq\ &\left(\sqrt{A_\Phi} - \alpha\sqrt{B_\Phi}\right)\|K^*f\|.\end{aligned}$$

□

Now we introduce an example that satisfies in condition of Proposition 3.5. Let $Ke_1 = e_2, Ke_2 = e_1, Ke_3 = 0$. Then, $K^*e_1 = e_2, K^*e_2 = e_1, K^*e_3 = 0$. Also, let $U \in B(\mathcal{H})$ as $Ue_1 = U^*e_1 = 1+\frac{\sqrt{2}}{2}e_1, Ue_2 = U^*e_2 = 1+\frac{\sqrt{2}}{2}e_2, Ue_3 = U^*e_3 = e_3$. Then $\Phi = \{2e_1, e_2, e_3\}$ is a K-frame with bounds $A_\Phi = 1, B_\Phi = 4$. Also, for every $f \in \mathcal{H}$ we have $UKf = KUf$, and

$$\|K^*f\|^2 = \|c_1e_2 + c_2e_1\|^2 = |c_1|^2 + |c_2|^2,$$

and

$$\|(I-U^*)f\|^2 = \frac{1}{2}\left(|c_1|^2 + |c_2|^2\right) < \alpha\|K^*f\| = \alpha\left(|c_1|^2 + |c_2|^2 + |c_3|^2\right),$$

which $\alpha < \sqrt{\frac{A_\Phi}{B_\Phi}} = \frac{1}{2}$. This gives the condition of the last proposition.

We end this section by discussing an example which shows that the woven property for K-frames is not transitive, in general.

Example 3.6. Let $\mathcal{H} = \ell^2(\mathbb{N})$ and K be the orthogonal projection of $\mathcal{H}$ onto $\{e_i\}_{i=2}^{\infty}$. Consider K-frames $\Phi = \{e_1, e_2, 0, e_3, e_4, \dots\}$, $\Psi = \{0, e_1, e_2, e_3, e_4, \dots\}$ and $\eta = \{e_1, 0, e_2, e_3, e_4, \dots\}$ on $\mathcal{H}$ where $\{e_1, e_2, \dots\}$ is the standard orthonormal basis of $\mathcal{H}$. Then Φ is woven with Ψ and Ψ is woven with η by the universal bounds $A_1 = A_2 = 1$ and $B_1 = B_2 = 2$. However, K-frames Φ and η are not woven. Indeed choose $\sigma = \mathbb{N}\backslash\{2\}$, then $\{\varphi_i\}_{i\in\sigma} \cup \{\eta_i\}_{i\in\sigma^c} = \{e_1, 0, 0, e_3, e_4, \dots\}$ which is not a K-frame.

In order to solve the above problem, we consider a condition on bounds. More precisely, suppose that $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ are woven K-frames by a universal lower bound A_1, and $\{\psi_i\}_{i\in I}$ is woven with a K-frame $\{\eta_i\}_{i\in I}$ by a universal lower bound A_2 such that $A_1 + A_2 - B_\Psi > 0$. Then for each $\sigma \subset I$ and $f \in \mathcal{H}$ we obtain

$$\begin{aligned}(A_1 + A_2 - B_\Psi) \parallel K^*f \parallel^2 &\leq \sum_{i\in\sigma} |\langle f, \varphi_i\rangle|^2 + \sum_{i\in\sigma^c} |\langle f, \psi_i\rangle|^2 + \sum_{i\in\sigma} |\langle f, \psi_i\rangle|^2 \\ &+ \sum_{i\in\sigma^c} |\langle f, \eta_i\rangle|^2 - \sum_{i\in I} |\langle f, \psi_i\rangle|^2 \\ &= \sum_{i\in\sigma} |\langle f, \varphi_i\rangle|^2 + \sum_{i\in\sigma^c} |\langle f, \eta_i\rangle|^2 \\ &\leq (B_\Phi + B_\eta) \parallel f \parallel^2 .\end{aligned}$$

Hence, $\{\varphi_i\}_{i\in I}$ is woven with $\{\eta_i\}_{i\in I}$.

4. Stability of dual Woven K-frames

In this section, we state some stability results for woven K-frames. Given two woven K-frames. The following questions naturally arise: how can we construct more woven K-frames and does the duality preserve the wovenness? The next proposition shows that under some condition, there are infinitely many K-dual of one of them which they are woven with the image of another one under a bounded operator.

Proposition 4.1. *Let K be a self-adjoint operator, also let $\Phi = \{\varphi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ be woven K-frames for $\mathcal{H}$ such that $S_\Phi(R(K)) \subseteq R(K)$. Then there are infinitely many K-dual frames of Φ which are woven with $K^*S_\Phi^{-1}\pi_{S_\Phi(R(K))}\Psi$.*

Proof. By the assumption $R(K)$ is invariant under S_Φ, hence

$$R((S_\Phi^{-1})^*) \subseteq S_\Phi(R(K)) \subseteq R(K).$$

Using the fact that K^* is bounded below on $R(K)$ (see (2.1)), we obtain

$$\|K^\dagger\|^{-1}\|S_\Phi\|^{-1}\|g\| \leq \left\|K^*(S_\Phi^{-1})^*g\right\|, \quad (g \in R(K)). \tag{4.1}$$

Assume that $U = \{u_i\}_{i\in I}$ is a Bessel sequence satisfying in (2.4). Then

$$\sum_{i\in I}\langle f, \varphi_i\rangle \pi_{R(K)} KK^{\dagger}\pi_{R(K)} u_i = KK^{\dagger}\sum_{i\in I}\langle f, \varphi_i\rangle \pi_{R(K)} u_i = 0, \qquad (f \in \mathcal{H}).$$

So, $\Phi^d_\epsilon = \{\widetilde{\varphi_i} + \epsilon KK^{\dagger}\pi_{R(K)}u_i\}_{i\in I}$ is also a K-dual frame of Φ by Theorem 2.1. Let $\epsilon > 0$ be small enough such that

$$A\|K^{\dagger}\|^{-1}\|S_\Phi\|^{-1} - \epsilon B_U\|K^{\dagger}\|^2 - 2\epsilon\sqrt{B_U B_\Phi \|S_\Phi^{-1}\|\|K^{\dagger}\|} > 0, \tag{4.2}$$

where A is a universal lower bound of weaving Φ and Ψ . To see $K^*S_\Phi^{-1}\pi_{S_\Phi(R(K))}\Psi$ and Φ^d_ϵ are woven, we only need to prove the existence of a universal lower bound. Suppose $\sigma \subset I$, applying (2.2) and (4.1) we have

$$\begin{aligned}
& \sum_{i\in\sigma}|\langle f, K^*S_\Phi^{-1}\pi_{S_\Phi(R(K))}\Psi\rangle|^2 + \sum_{i\in\sigma^c}|\langle f, \Phi^d_\epsilon\rangle|^2 \\
= \ & \sum_{i\in\sigma}|\langle (S_\Phi^{-1})^*Kf, \psi_i\rangle|^2 + \sum_{i\in\sigma^c}\left|\langle (S_\Phi^{-1})^*Kf, \varphi_i\rangle + \langle Kf, \epsilon K^{\dagger}\pi_{R(K)}u_i\rangle\right|^2 \\
\geq \ & \sum_{i\in\sigma}|\langle (S_\Phi^{-1})^*Kf, \psi_i\rangle|^2 + \sum_{i\in\sigma^c}\left|\left|\langle (S_\Phi^{-1})^*Kf, \varphi_i\rangle\right| - \left|\langle Kf, \epsilon K^{\dagger}\pi_{R(K)}u_i\rangle\right|\right|^2 \\
\geq \ & \sum_{i\in\sigma}|\langle (S_\Phi^{-1})^*Kf, \psi_i\rangle|^2 + \sum_{i\in\sigma^c}\left|\langle (S_\Phi^{-1})^*Kf, \varphi_i\rangle\right|^2 - \sum_{i\in\sigma^c}\left|\langle Kf, \epsilon K^{\dagger}\pi_{R(K)}u_i\rangle\right|^2 \\
- \ & 2\sum_{i\in\sigma^c}\left|\langle (S_\Phi^{-1})^*Kf, \varphi_i\rangle\right|\left|\langle Kf, \epsilon K^{\dagger}\pi_{R(K)}u_i\rangle\right| \\
\geq \ & A\left\|K^*(S_\Phi^{-1})^*Kf\right\|^2 - \epsilon B_U\|K^{\dagger}\|^2\|Kf\|^2 - 2\epsilon\sqrt{B_U B_\Phi\|S_\Phi^{-1}\|\|K^{\dagger}\|}\|Kf\|^2 \\
\geq \ & \left(A\|K^{\dagger}\|^{-1}\|S_\Phi\|^{-1} - \epsilon B_U\|K^{\dagger}\|^2 - 2\epsilon\sqrt{B_U B_\Phi\|S_\Phi^{-1}\|\|K^{\dagger}\|}\right)\|Kf\|^2,
\end{aligned}$$

So, by (4.2), we obtain infinitely many K-dual frames of Φ which satisfies the desired condition. This completes the proof. □

Now, we construct a family of woven K-duals from a pair of woven K-frames.

Theorem 4.2. *Let K be a self-adjoint operator,* $\Phi = \{\varphi_i\}_{i\in I}$ *and* $\Psi = \{\psi_i\}_{i\in I}$ *be woven K-frames with a universal lower bound A such that* $S_\Phi(R(K)) \subseteq R(K)$ *and*

$$\left\|\sum_{i\in I}c_i(\varphi_i - \psi_i)\right\| < \frac{\sqrt{A}\|K^{\dagger}\|^{-1}\|S_\Phi\|^{-1}}{\sqrt{B_\Psi}\|S_\Phi^{-1}\|\|S_\Psi^{-1}\|\left(\sqrt{B_\Psi} + \sqrt{B_\Phi}\right)}\sum_{i\in I}|c_i|^2, \tag{4.3}$$

for every sequence $\{c_i\}_{i\in I} \in \ell^2$. *Then there are infinitely many K-dual frames* Φ^d *of* Φ *and* Ψ^d *of* Ψ *which are woven* K^**-frame.*

Proof. By the assumption $S_\Phi : R(K) \to S_\Phi(R(K)) \subseteq R(K)$ is invertible and (4.1) holds. Choose arbitrary K-dual frames $\Phi^d = \{\widetilde{\varphi_i} + u_i\}_{i\in I}$ and $\Psi^d = \{\widetilde{\psi_i} + v_i\}_{i\in I}$ of Φ and Ψ, respectively such that $U = \{u_i\}_{i\in I}$ and $V = \{v_i\}_{i\in I}$ are Bessel sequences satisfy (2.4). By using (4.3) we have

$$\begin{aligned}
\|S_\Phi^{-1} - S_\Psi^{-1}\| &= \|S_\Psi^{-1}\left(S_\Phi - S_\Psi\right)S_\Phi^{-1}\| \\
&\leq \|S_\Psi^{-1}\|\|S_\Phi^{-1}\|\|S_\Phi - S_\Psi\| \\
&\leq \|S_\Psi^{-1}\|\|S_\Phi^{-1}\|\,\|T_\Phi T_\Phi^* - T_\Phi T_\Psi^* + T_\Phi T_\Psi^* - T_\Psi T_\Psi^*\| \\
&\leq \|S_\Psi^{-1}\|\|S_\Phi^{-1}\|\|T_\Phi - T_\Psi\|\left(\|T_\Phi\| + \|T_\Psi\|\right) \\
&< \|K^{\dagger}\|^{-1}\|S_\Phi\|^{-1}\sqrt{\frac{A}{B_\Psi}}.
\end{aligned}$$

Choose $0 < \alpha < 1$ such that

$$\|S_\Phi^{-1} - S_\Psi^{-1}\| + \alpha\|K^\dagger\| \left(\frac{\sqrt{B_U}+\sqrt{B_V}}{\sqrt{B_\Psi}}\right) < \|K^\dagger\|^{-1}\|S_\Phi\|^{-1}\sqrt{\frac{A}{B_\Psi}}. \tag{4.4}$$

So, $\Phi_\alpha^d = \{\widetilde{\varphi_i} + \alpha KK^\dagger\pi_{R(K)}u_i\}_{i\in I}$ and $\Psi_\alpha^d = \{\widetilde{\psi_i} + \alpha KK^\dagger\pi_{R(K)}v_i\}_{i\in I}$ are also K-dual frames of Φ and Ψ. Hence, by using (4.1) for every $\sigma \subset I$ we have

$$\begin{aligned}
&\sum_\sigma |\langle f, \Phi_\alpha^d\rangle|^2 + \sum_{\sigma^c} |\langle f, \Psi_\alpha^d\rangle|^2 \\
&= \sum_\sigma |\langle f, \widetilde{\varphi_i} + \alpha KK^\dagger\pi_{R(K)}u_i\rangle|^2 + \sum_{\sigma^c} |\langle f, \widetilde{\psi_i} + \alpha KK^\dagger\pi_{R(K)}v_i\rangle|^2 \\
&= \sum_\sigma \left|\langle (S_\Phi^{-1})^*Kf, \varphi_i\rangle + \langle \alpha(K^\dagger)^*K^*f, u_i\rangle\right|^2 \\
&\quad + \sum_{\sigma^c} \left|\langle (S_\Psi^{-1})^*Kf, \psi_i\rangle + \langle \alpha(K^\dagger)^*K^*f, v_i\rangle\right|^2 \\
&= \sum_\sigma \left|\langle (S_\Phi^{-1})^*Kf, \varphi_i\rangle + \langle \alpha(K^\dagger)^*K^*f, u_i\rangle\right|^2 \\
&\quad + \sum_{\sigma^c} \left|\langle (S_\Phi^{-1})^*Kf, \psi_i\rangle + \langle ((S_\Psi^{-1})^* - (S_\Phi^{-1})^*)Kf, \psi_i\rangle + \langle \alpha(K^\dagger)^*K^*f, v_i\rangle\right|^2.
\end{aligned}$$

Thus, using (2.1) follows that

$$\begin{aligned}
&\Big(\sum_\sigma |\langle f, \Phi_\alpha^d\rangle|^2 + \sum_{\sigma^c} |\langle f, \Psi_\alpha^d\rangle|^2\Big)^{\frac{1}{2}} \\
&\geq \left(\sum_\sigma \left|\langle (S_\Phi^{-1})^*Kf, \varphi_i\rangle\right|^2 + \sum_{\sigma^c} \left|\langle (S_\Phi^{-1})^*Kf, \psi_i\rangle\right|^2\right)^{\frac{1}{2}} \\
&\quad - \left(\sum_\sigma \left|\langle \alpha(K^\dagger)^*K^*f, u_i\rangle\right|^2\right)^{\frac{1}{2}} - \left(\sum_{\sigma^c} \left|\langle \alpha(K^\dagger)^*K^*f, v_i\rangle\right|^2\right)^{\frac{1}{2}} \\
&\quad - \left(\sum_{\sigma^c} \left|\langle \left((S_\Psi^{-1})^* - (S_\Phi^{-1})^*\right)Kf, \psi_i\rangle\right|^2\right)^{\frac{1}{2}} \\
&\geq \sqrt{A}\|K^*(S_\Phi^{-1})^*Kf\| - \left(\sqrt{B_U}+\sqrt{B_V}\right)\alpha\|K^\dagger\|\|K^*f\| - \sqrt{B_\Psi}\|S_\Psi^{-1} - S_\Phi^{-1}\|\|Kf\| \\
&\geq \left(\sqrt{A}\|K^\dagger\|^{-1}\|S_\Phi\|^{-1} - \alpha\|K^\dagger\|\left(\sqrt{B_U}+\sqrt{B_V}\right) - \sqrt{B_\Psi}\|S_\Psi^{-1} - S_\Phi^{-1}\|\right)\|Kf\| \\
&= \sqrt{B_\Psi}\left(\|K^\dagger\|^{-1}\|S_\Phi\|^{-1}\sqrt{\frac{A}{B_\Psi}} - \alpha\|K^\dagger\|\left(\frac{\sqrt{B_U}+\sqrt{B_V}}{\sqrt{B_\Psi}}\right) - \|S_\Psi^{-1} - S_\Phi^{-1}\|\right)\|Kf\|,
\end{aligned}$$

where in last inequality we have used the fact that K is self-adjoint. By attention to (4.4), K^*-frames Φ^d and Ψ^d are woven. □

We introduce an example that satisfies to the condition of the last theorem. Let

$$\varphi_i = \begin{cases} e_i & i = 2k \\ \frac{e_i}{i} & i = 2k+1 \end{cases}, \qquad \psi_i = \begin{cases} \varphi_i & i = 2k \\ 0 & i = 2k+1 \end{cases},$$

and $K = \pi_{span\{e_{2k}:k\in\mathbb{N}\}}$. Then $\Phi = \{\varphi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ are two K-frames with frame bounds $A_\Phi = B_\Phi = A_\Psi = B_\Psi = 1$. Then $R(K) = span\{e_{2k} : k \in \mathbb{N}\}$ and $S_\Phi|_{R(K)}$ with

$S_\Phi f = \sum_{i=1}^{\infty}\langle f, e_{2k}\rangle e_{2k}$ is the identity operator. Also,

$$\begin{aligned}\|\sum_{i\in I} c_i(\varphi_i - \psi_i)\| &= \|\sum_{i\in I} c_{2k+1}\frac{e_{2k+1}}{2k+1}\| = \sum_{i\in I}\frac{|c_{2k+1}|^2}{|2k+1|^2} < \frac{1}{2}\sum_{i\in I}|c_i|^2 \\ &= \frac{\sqrt{A}\|K^\dagger\|^{-1}\|S_\Phi\|^{-1}}{\sqrt{B_\Psi}\|S_\Phi^{-1}\|\|S_\Psi^{-1}\|\left(\sqrt{B_\Psi}+\sqrt{B_\Phi}\right)}\sum_{i\in I}|c_i|^2,\end{aligned}$$

which is satisfied in the condition of Theorem 4.2.

Corollary 4.3. *Let K be a self-adjoint operator and Φ, Ψ be woven K-frames with a universal lower bound A such that $S_\Phi(R(K)) \subseteq R(K)$ and*

$$\|S_\Phi^{-1} - S_\Psi^{-1}\| < \|K^\dagger\|^{-1}\|S_\Phi\|^{-1}\sqrt{\frac{A}{B_\Psi}}.$$

Then $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are woven K^*-frames.

In next theorem we check out under some condition the converse of the previous result holds.

Theorem 4.4. *Let K be a self-adjoint operator and $\Phi = \{\varphi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ be K-frames. If $\tilde{\Phi}$ and $\tilde{\Psi}$ are woven K^*-frames with a universal lower bound A such that $S_\Phi(R(K)) \subseteq R(K)$ and*

$$\|S_\Phi^{-1} - S_\Psi^{-1}\| \le \frac{\sqrt{A}}{\sqrt{B_\Psi}\|K^\dagger\|\|K\|\|S_\Phi^{-1}\|\|S_\Phi\|}. \tag{4.5}$$

Then Φ and Ψ are woven K-frames on $R(K)$.

Proof. Applying (2.1) easily shows that

$$\frac{\|K^*f\|}{\|S_\Phi^{-1}\|\|K\|} \le \frac{\|f\|}{\|S_\Phi^{-1}\|} \le \|S_\Phi^* f\| \le \|S_\Phi\|\|K^\dagger\|\|K^*f\|, \tag{4.6}$$

for all $f \in R(K)$. Now for every $\sigma \subset I$ we have

$$\begin{aligned}&\sum_\sigma |\langle f, \varphi_i\rangle|^2 + \sum_{\sigma^c}|\langle f, \psi_i\rangle|^2 \\ &= \sum_\sigma |\langle (S_\Phi^{-1})^* K K^\dagger S_\Phi^* f, \varphi_i\rangle|^2 + \sum_{\sigma^c}|\langle (S_\Phi^{-1})^* K K^\dagger S_\Phi^* f, \psi_i\rangle|^2 \\ &= \sum_\sigma |\langle K^\dagger S_\Phi^* f, \tilde{\varphi}_i\rangle|^2 + \sum_{\sigma^c}\left|\langle K^\dagger S_\Phi^* f, K^* S_\Phi^{-1}\pi_{R(K)}\psi_i\rangle\right|^2 \\ &= \sum_\sigma |\langle K^\dagger S_\Phi^* f, \tilde{\varphi}_i\rangle\rangle|^2 \\ &\quad + \sum_{\sigma^c}|\langle (S_\Psi^{-1})^* K K^\dagger S_\Phi^* f + \left((S_\Phi^{-1})^* - (S_\Psi^{-1})^*\right) K K^\dagger S_\Phi^* f, \psi_i\rangle|^2.\end{aligned}$$

By using (4.6) we obtain

$$
\begin{aligned}
&\sum_{\sigma}|\langle f,\varphi_i\rangle|^2+\sum_{\sigma^c}|\langle f,\psi_i\rangle|^2\\
&=\sum_{\sigma}\left|\langle K^{\dagger}S_{\Phi}^{*}f,\tilde{\varphi}_i\rangle\right|^2+\sum_{\sigma^c}\left|\langle K^{\dagger}S_{\Phi}^{*}f,\tilde{\psi}_i\rangle+\langle\left((S_{\Phi}^{-1})^{*}-(S_{\Psi}^{-1})^{*}\right)KK^{\dagger}S_{\Phi}^{*}f,\psi_i\rangle\right|^2\\
&\geq\Big[\Big(\sum_{\sigma}\left|\langle K^{\dagger}S_{\Phi}^{*}f,\tilde{\varphi}_i\rangle\right|^2+\sum_{\sigma^c}|\langle K^{\dagger}S_{\Phi}^{*}f,\tilde{\psi}_i\rangle|^2\Big)^{\frac{1}{2}}\\
&\qquad-\Big(\sum_{\sigma^c}|\langle\left((S_{\Phi}^{-1})^{*}-(S_{\Psi}^{-1})^{*}\right)S_{\Phi}^{*}f,\psi_i\rangle|^2\Big)^{\frac{1}{2}}\Big]^2\\
&\geq\left(\sqrt{A}\|KK^{\dagger}S_{\Phi}^{*}f\|-\sqrt{B_{\Psi}}\|S_{\Phi}^{-1}-S_{\Psi}^{-1}\|\|S_{\Phi}^{*}f\|\right)^2\\
&\geq\left(\sqrt{A}\|S_{\Phi}^{*}f\|-\sqrt{B_{\Psi}}\|S_{\Phi}^{-1}-S_{\Psi}^{-1}\|\|K^{\dagger}\|\|S_{\Phi}\|\|K^{*}f\|\right)^2\\
&\geq\left(\sqrt{A}\|S_{\Phi}^{-1}\|^{-1}\|K\|^{-1}-\sqrt{B_{\Psi}}\|K^{\dagger}\|\|S_{\Phi}\|\parallel S_{\Phi}^{-1}-S_{\Psi}^{-1}\parallel\right)^2\|K^{*}f\|^2.
\end{aligned}
$$

This completes the proof by using (4.5). □

Notice that the condition $S_{\Phi}(R(K))\subseteq R(K)$ in the above results can be reduced to the condition K^* is bounded below on $S_{\Phi}(R(K))$.

5. Weaving and excess

In this section we are focused on discussing the relation between weaving and the excess of K-frames. The following proposition plays a key role in this respect. First, we recall the definition of K-Riesz bases.

Definition 5.1. A family $\Phi=\{\varphi_i\}_{i\in I}$ is called a *K-Riesz sequence* for $\mathcal{H}$ if there exists an injective bounded operator $U:\mathcal{H}\to\mathcal{H}$ such that $\{\pi_{R(K)}\varphi_i\}_{i\in I}=\{Ue_i\}_{i\in I}$, where $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}$ [26]. In addition, if Φ is a K-frame, then Φ is called a *K-Riesz basis*. A family $\Phi=\{\varphi_i\}_{i\in I}$ is called *near K-Riesz basis* for $\mathcal{H}$ if there exists a finite set σ for which $\{\varphi_i\}_{i\notin\sigma}$ is a K-Riesz basis for $\mathcal{H}$.

Proposition 5.2 ([26]). *Let $\{\varphi_i\}_{i\in I}$ be a Bessel sequence in $\mathcal{H}$. The following are equivalent:*

(1) *$\{\varphi_i\}_{i\in I}$ is K-Riesz sequence for $\mathcal{H}$.*
(2) *$\{\pi_{R(K)}\varphi_i\}_{i\in I}$ is a Riesz sequence.*
(3) *$\{\pi_{R(K)}\varphi_i\}_{i\in I}$ is ω-independent.*

Moreover, let $\{\varphi_i\}_{i\in I}$ be a K-frame. Then $\{\varphi_i\}_{i\in I}$ is K-Riesz basis if and only if $\{\pi_{R(K)}\varphi_i\}_{i\in I}$ is ω-independent.

The following proposition shows that associated to each K-frame there exists an ordinary frame sequence. Applying this fact we obtain a pair of woven frames from woven K-frames.

Proposition 5.3. *Let $\Phi=\{\varphi_i\}_{i\in I}$ and $\Psi=\{\psi_i\}_{i\in I}$ be woven K-frames such that $span\{\varphi_i\}_{i\in I}=span\{\psi_i\}_{i\in I}$. Then $\{\pi_{R(K)}\varphi_i\}_{i\in I}$ and $\{\pi_{R(K)}\psi_i\}_{i\in I}$ are woven frames on $\pi_{R(K)}\overline{span}\{\varphi_i\}_{i\in I}$.*

Proof. Applying (2.1), for every $f\in R(K)$ we have

$$
\begin{aligned}
A_{\Phi}\|K^{\dagger}\|^{-1}\|f\|^2 &\leq A_{\Phi}\|K^{*}f\|^2\\
&\leq\sum_{i\in I}|\langle f,\varphi_i\rangle|^2\\
&=\sum_{i\in I}|\langle\pi_{R(K)}f,\varphi_i\rangle|^2=\sum_{i\in I}|\langle f,\pi_{R(K)}\varphi_i\rangle|^2.
\end{aligned}
$$

Hence, $\{\pi_{R(K)}\varphi_i\}_{i\in I}$ and $\{\pi_{R(K)}\psi_i\}_{i\in I}$ are frames on the Hilbert space

$$M := \pi_{R(K)}\overline{span}\{\varphi_i\}_{i\in I} \subset R(K).$$

Moreover, for every $\sigma \subset I$ and $f \in M$ we have

$$\begin{aligned} A\|K^\dagger\|^{-1}\|f\|^2 \le A\|K^* f\|^2 &\le \sum_{i\in\sigma} |\langle f, \varphi_i\rangle|^2 + \sum_{i\in\sigma^c} |\langle f, \psi_i\rangle|^2 \\ &= \sum_{i\in\sigma} |\langle \pi_{R(K)} f, \varphi_i\rangle|^2 + \sum_{i\in\sigma^c} |\langle \pi_{R(K)} f, \psi_i\rangle|^2 \\ &= \sum_{i\in\sigma} |\langle f, \pi_{R(K)}\varphi_i\rangle|^2 + \sum_{i\in\sigma^c} |\langle f, \pi_{R(K)}\psi_i\rangle|^2, \end{aligned}$$

where A is a universal lower bound for weaving Φ and Ψ. This completes the proof. □

It is well known that a Riesz basis has a unique biorthogonal sequence, see [15]. In the following we prove this fact for K-Riesz bases.

Theorem 5.4. *Let $\Phi = \{\varphi_i\}_{i\in I}$ be a K-Riesz basis. Then $\{K^\dagger\pi_{R(K)}\varphi_i\}_{i\in I}$ is the unique biorthogonal sequence of $\{\widetilde{\varphi_i}\}_{i\in I}$ in $R(K^\dagger)$.*

Proof. By using (2.3) for $f = K^\dagger\pi_{R(K)}\varphi_j$ we obtain

$$\pi_{R(K)}\varphi_j = KK^\dagger\pi_{R(K)}\varphi_j = \sum_{i\in I}\langle K^\dagger\pi_{R(K)}\varphi_j, \widetilde{\varphi_i}\rangle\pi_{R(K)}\varphi_i.$$

Applying Proposition 5.2 follows that $\{\pi_{R(K)}\varphi_i\}_{i\in I}$ is ω-independent. Hence,

$$\langle K^\dagger\pi_{R(K)}\varphi_j, \widetilde{\varphi_i}\rangle = \delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Now, suppose that $\Psi = \{\psi_i\}_{i\in I} \subseteq R(K^\dagger)$ is another biorthogonal sequence, then for every $f \in \mathcal{H}$ we have

$$\begin{aligned} \langle f, \pi_{R(K)}\varphi_j\rangle &= \langle f, KK^\dagger\pi_{R(K)}\varphi_j\rangle \\ &= \langle K^* f, K^\dagger\pi_{R(K)}\varphi_j\rangle \\ &= \left\langle \sum_{i\in I}\langle f, \pi_{R(K)}\varphi_i\rangle\widetilde{\varphi_i}, K^\dagger\pi_{R(K)}\varphi_j \right\rangle \\ &= \sum_{i\in I}\left\langle f, \pi_{R(K)}\varphi_i\right\rangle\left\langle \widetilde{\varphi_i}, K^\dagger\pi_{R(K)}\varphi_j\right\rangle \\ &= \sum_{i\in I}\left\langle f, \pi_{R(K)}\varphi_i\right\rangle\left\langle \psi_i, K^\dagger\pi_{R(K)}\varphi_j\right\rangle. \end{aligned}$$

Thus, by (2.3) we obtain

$$\begin{aligned} K^* f &= \sum_{j\in I}\left\langle f, \pi_{R(K)}\varphi_j\right\rangle\widetilde{\varphi_j} \\ &= \sum_{j\in I}\sum_{i\in I}\left\langle f, \pi_{R(K)}\varphi_i\right\rangle\left\langle \psi_i, K^\dagger\pi_{R(K)}\varphi_j\right\rangle\widetilde{\varphi_j} \\ &= \sum_{i\in I}\left\langle f, \pi_{R(K)}\varphi_i\right\rangle\sum_{j\in I}\left\langle (K^\dagger)^*\psi_i, \pi_{R(K)}\varphi_j\right\rangle\widetilde{\varphi_j} \\ &= \sum_{i\in I}\left\langle f, \pi_{R(K)}\varphi_i\right\rangle K^*(K^\dagger)^*\psi_i = \sum_{i\in I}\left\langle f, \pi_{R(K)}\varphi_i\right\rangle\psi_i. \end{aligned}$$

Therefore, Ψ is also a K-dual of Φ. On the other hand, every K-Riesz basis has a unique K-dual, see Proposition 2.5 of [26]. Hence, $\psi_i = K^\dagger\pi_{R(K)}\varphi_i$, for all $i \in I$. □

As a consequence, by attention to Theorem 3.5 of [3], we obtain the following result.

Corollary 5.5. *Let $\Phi=\{\varphi_i\}_{i\in I}$ be a K-Riesz basis for $\mathcal{H}$. Then*

(1) $\{\pi_{R(K)}\varphi_i\}_{i\in I}$ *and* $\{S_\Phi^{-1}\pi_{S_\Phi(R(K))}\varphi_i\}_{i\in I}$ *are woven frame on* $\pi_{R(K)}\overline{span}\{\varphi_i\}_{i\in I}$.
(2) $\{\varphi_i\}_{i\in I}$ *and* $\{S_\Phi^{-1}\pi_{S_\Phi(R(K))}\varphi_i\}_{i\in I}$ *are woven frames on* $R(K)$.
(3) $\{K^\dagger\pi_{R(K)}\varphi_i\}_{i\in I}$ *and* $\{\widetilde{\varphi_i}\}_{i\in I}$ *are woven* K^**-frames.*

The *excess* of a K-frame Φ, denoted by $E_K(\Phi)$, is the greatest integer n so that n elements can be deleted from the K-frame and still leave a K-frame, or $+\infty$ if there is no upper bound to the number of elements that can be removed. Every K-frame $\Phi=\{\varphi_i\}_{i\in I}$ with $E_K(\Phi)=n$ can be written as $\Phi=\{\varphi_i\}_{i\in I\setminus\{i_1,\dots i_n\}}\cup\{\varphi_{i_1},\dots\varphi_{i_n}\}$, where $\{\varphi_i\}_{i\in I\setminus\{i_1,\dots i_n\}}$ is a K-Riesz basis for $\mathcal{H}$ and $\{\varphi_{i_1},\dots\varphi_{i_n}\}$ are redundant elements of Φ. If $K=I_{\mathcal{H}}$, then Φ is a frame and $E_K(\Phi)$, denoted by $E(\Phi)$, coincides with the usual definition of excess of frames [8]. Dual frames and woven frames have the same excess [1,8].

The next lemma follows immediately from Theorem 5.4 of [11].

Lemma 5.6. *Let $\Phi=\{\varphi_i\}_{i\in I}$ be a Riesz sequence and $\Psi=\{\psi_i\}_{i\in I}$ be a frame sequence such that $span\{\varphi_i\}_{i\in I}=span\{\psi_i\}_{i\in I}$. If Φ and Ψ are woven frames, then Ψ is also a Riesz sequence.*

Proposition 5.7. *Let K be a closed range operator on a Hilbert space $\mathcal{H}$, $\Phi=\{\varphi_i\}_{i\in I}$ be a K-Riesz sequence and $\Psi=\{\psi_i\}_{i\in I}$ be a K-frame for $\mathcal{H}$. If Φ and Ψ are woven K-frames such that $span\{\varphi_i\}_{i\in I}=span\{\psi_i\}_{i\in I}$, then Ψ must actually be a K-Riesz sequence.*

Proof. Using Proposition 5.3 follows that $\{\pi_{R(K)}\varphi_i\}_{i\in I}$ and $\{\pi_{R(K)}\psi_i\}_{i\in I}$ are woven frames on $\pi_{R(K)}\overline{span}\{\varphi_i\}_{i\in I}$. On the other hand, $\{\pi_{R(K)}\varphi_i\}_{i\in I}$ is a Riesz-sequence by Proposition 5.2. Applying Lemma 5.6 gives that $\{\pi_{R(K)}\psi_i\}_{i\in I}$ is a Riesz sequence. Using again Proposition 5.2, follows that $\{\psi_i\}_{i\in I}$ is also a K-Riesz sequence. □

We are now ready to discuss the excess of K-frames.

Theorem 5.8. *Let $\Phi=\{\varphi_i\}_{i\in I}$ be a K-Riesz basis. Then $dim\left(ker(\pi_{R(K)}T_\Phi)\right)<\infty$. In fact, $ker\left(\pi_{R(K)}T_\Phi\right)$ is finite dimensional if and only if $\{\varphi_i\}_{i\in I}$ is near K-Riesz sequence.*

Proof. Assume that $\{\varphi_i\}_{i\in I}$ is a K-Riesz basis, then $\{\pi_{R(K)}\varphi_i\}_{i\in I}$ is a Riesz sequence by Proposition 5.2. In [15], it is introduced that if $\{\varphi_i\}_{i\in I}$ is a Riesz sequence such that $\sum_{i=1}^\infty a_i\varphi_i$ is convergent then $\{a_i\}_{i=1}^\infty\in\ell^2(\mathbb{N})$. So, if $\sum_{i=1}^\infty a_i\pi_{R(K)}\varphi_i$ is convergent then $\{a_i\}_{i=1}^\infty\in\ell^2(\mathbb{N})$. Hence, by Theorem 2.3 of [22], it follows that $ker\left(\pi_{R(K)}T_\Phi\right)=ker\left(T_{\pi_{R(K)}\Phi}\right)$ must be finite dimensional. For the second part, let $ker\left(T_{\pi_{R(K)}\Phi}\right)$ be finite dimensional. Then $\{\pi_{R(K)}\varphi_i\}_{i\in I}$ is near Riesz basis by Theorem 2.4 of [22] and so $\{\varphi_i\}_{i\in I}$ is near K-Riesz sequence by Proposition 5.2. Conversely, suppose that $\{\varphi_i\}_{i\in I}$ is near K-Riesz sequence for $\mathcal{H}$. So there is a finite set σ for which $\{\varphi_i\}_{i\notin\sigma}$ is a K-Riesz sequence for $\mathcal{H}$. Hence, by Proposition 5.2, $\{\pi_{R(K)}\varphi_i\}_{i\notin\sigma}$ is a Riesz sequence and so $dim\left(ker(\pi_{R(K)}T_\Phi)\right)<\infty$ by Theorem 2.4 of [22]. □

As a consequence, we obtain the following result.

Theorem 5.9. *Let $\Phi=\{\varphi_i\}_{i\in I}$ be a near K-Riesz basis. The following are equivalent:*

(1) $ker(\pi_{R(K)}T_\Phi)$ *is finite dimensional.*
(2) $E(\pi_{R(K)}\Phi)<\infty$.
(3) $E_K(\Phi)<\infty$.

Moreover, for a K-frame Φ we have

$$E_K(\Phi)=dim\left(ker\left(\pi_{R(K)}T_\Phi\right)\right)=E(\pi_{R(K)}\Phi). \tag{5.1}$$

Proof. (1) and (2) are equivalent by Theorem 5.8 and Proposition 5.2. Now suppose that $E_K(\Phi) < \infty$. Similar the given argument in Theorem 5.8 we obtain $E_K(\Phi) = E(\pi_{R(K)}\Phi)$. Now applying Theorem 3.1 of [22] follows that

$$E_K(\Phi) = E(\pi_{R(K)}\Phi) = dim(ker(T_{\pi_{R(K)}\Phi})) = dim\left(ker(\pi_{R(K)}T_\Phi)\right).$$

Moreover, if $E_K(\Phi) = \infty$ and $E(\pi_{R(K)}\Phi) < \infty$, then there exists a finite set $\sigma \subset I$ such that $\{\pi_{R(K)}\varphi_i\}_{i\notin\sigma}$ is a Riesz sequence. Using Proposition 5.2 follows that $\{\varphi_i\}_{i\notin\sigma}$ is a K-Riesz sequence. In particular, $E_K(\Phi) \leq card\sigma < \infty$ which is contradiction. So, $E(\pi_{R(K)}\Phi) = \infty$. In addition, by Lemma 4.1 of [8], we have

$$dim\left(ker\left(\pi_{R(K)}T_\Phi\right)\right) = dim\left(kerT_{\pi_{R(K)}\Phi}\right) \geq E\left(\pi_{R(K)}\Phi\right) = \infty.$$

The proof of other parts are similar. □

Corollary 5.10. *Let $\Phi = \{\varphi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ be woven K-frames such that $span\{\varphi_i\}_{i\in I} = span\{\psi_i\}_{i\in I}$. Then $E_K(\Phi) = E_K(\Psi)$.*

Proof. Since Φ and Ψ are woven K-frames and $span\{\varphi_i\}_{i\in I} = span\{\psi_i\}_{i\in I}$ then $\{\pi_{R(k)}\varphi_i\}_{i\in I}$ and $\{\pi_{R(k)}\psi_i\}_{i\in I}$ are wove frames on $\pi_{R(k)}\overline{span}\{\varphi_i\}_{i\in I}$ by Corollary 5.3. So, by using Theorem 3.1 of [1] and equation (5.1) we obtain

$$E_K(\Phi) = E\left(\pi_{R(K)}\Phi\right) = E\left(\pi_{R(K)}\Psi\right) = E_K(\Psi). \tag{5.2}$$

□

Let Φ be a frame on $\mathcal{H}$ and $U \in B(\mathcal{H})$ be an onto operator. Then $U\Phi$ is also a frame on $\mathcal{H}$. It is easy to see that $E(\Phi) = E(U\Phi)$ if and only if U is also injective. Combining our results with the fact that $U\pi_{R(K)} = \pi_{R(K)}U$ if and only if $R(K)$ and $(R(K))^\perp$ are invariant under U. We state this result for K-frames as following:

Corollary 5.11. *Let Φ be a K-frame and $U \in B(\mathcal{H})$ be an onto operator such that $UK = KU$. Then $U\Phi$ is also a K-frame. Moreover assume that $(R(K))^\perp$ is invariant under U. Then $E_K(\Phi) = E(U\pi_{R(K)}\Phi) = E_K(U\Phi)$ if and only if U is also an injective operator.*

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