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Numerical Solution of Bratu-Type Initial Value Problems by Aboodh Adomian Decomposition Method

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Bratu equation, Aboodh Transform, Adomian Decomposition Method.

Abstract

This study presents an efficient method for solving Bratu equations with an initial value problem. The procedure is based on the use of Aboodh transform and Adomian decomposition method. Adomian polynomials for the index n replace the nonlinear term. The dependent variable components are also replaced in the recurrence relation by their corresponding Aboodh transform components of the same index. Therefore, the nonlinear problem is solved directly without any linearization or discretization. Examples are presented to show the effectiveness and validity of this method. The derived results are compared with the existing exact solution.

1. Introduction

Bratu equation with initial value problem is of the form:

$$u'' + \lambda e^{u} = 0 \quad 0 < x < 1 \tag{1}$$

$$u(0) = u'(1) = 0 \tag{2}$$

The Bratu equation is one of the most examined mathematical problem [1,2]. It emerges in numerous physical and chemical problems such as chemical reactor theory, radiative heat transfer, nanotechnology, simplification of solid fuel ignition in thermal combustion theory, modeling the expansion of the universe, and the thermal reaction process [2-5]. The importance of the Bratu equation stems, in part, from its use in the combustion theory and, in part, from the fact that its exact solution is well-known. [1,2]. Therefore, it has been used to evaluate the effectiveness and precision of numerous approximate techniques of various levels of complexity like the perturbation techniques, Legendre wavelet method, Adomian decomposition method, the viral theorem, etc. [1,2]. Additionally, this solution has a bifurcation pattern that is unique to nonlinear differential equations. The reason to investigate the explicit and precise general solution to the Bratu equation is motivated by this significance. [1,2]. The common method for resolving a boundary or initial value problem is to compute the general solution to the differential equation and, secondly, by using the boundary or initial conditions to find the arbitrary parameters [1,2]. Diverse numerical techniques have been used to solve Bratu-type equations. These methods are Variational Iteration Technique (VIT) [6,7], Successive Differentiation Method (SDM) [1], Homotopy Permutation Method (HPM) [5,8], Chebyshev Wavelet Method (CWM), and also through the coupling of several methods like Laplace Adomian Decomposition Method [4], Aboodh Homotopy Permutation Method [9], Aboodh Differential Transform Method [10], Laplace Homotopy Permutation Method [5], etc.

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Khalid Aboodh introduced the Aboodh Transform in 2013 to make it easier to solve Ordinary and Partial differential equations in the time domain [11]. Aboodh Transform is unable to solve nonlinear differential equations like Bratu equations due to the challenges posed by the nonlinear variables [11], so we use a coupling of the Aboodh transform and Adomian decomposition method to solve the Bratu equations with initial value problems in this study. The Adomian Decomposition Method (ADM) decomposes the nonlinear terms so that the solution can be obtained as a rapidly converging infinite series.

2. Aboodh Transform Method

2.1. Aboodh Transform

Aboodh transform is an application of integral transforms obtained from classical Fourier integral. The function of exponential order refers to the new transform known as the Aboodh Transform. We take into account the functions in the set A denoted by

$$A = \{f(t) : \exists\, M, k_1, k_2 > 0, |f\,(t)| < Me^{-vt}\} > 0, \quad (3)$$

The constant M for a certain function in the set must be a finite value while k_1 and k_2 may be finite or infinite. The Aboodh transform is defined for a function $f\,(t)$ for all $t\ \geq 0$ as

$$A\,[\,f\,(t)] = K\,(v) = \frac{1}{v}\int_0^{\infty} f\,(t)\,e^{-vt}\,dt \qquad , k_1 \leq v\ \leq\ k_2 \quad (4)$$

where A is called the Aboodh transform operator.

Table 1. Aboodh Transform of some functions.

$f(t)$	$A[f(t)] = K(v)$
1	$\frac{1}{v^2}$
t	$\frac{1}{v^3}$
t^n	$\frac{n!}{v^{n+2}}$
e^{at}	$\frac{1}{v^2 - av}$
$\sin at$	$\frac{a}{v(v^2 + a^2)}$
$\cos at$	$\frac{1}{(v^2 + a^2)}$
sinh at	$\frac{a}{v(v^2 - a^2)}$
cosh at	$\frac{1}{(v^2 - a^2)}$

2.2. Aboodh Transform of the Derivatives of the Function *f(t)*

If K(v) is the Aboodh transform of $A(f(t))$, then

a. $A\,[f'(t)] = vk(v) - \frac{f(0)}{v}$ (5)

b. $A[f''(t)] = v^2k(v) - \frac{f'(0)}{v} - f(0)$ (6)

c. $A[f^n(t)] = v^nK(v) - \frac{f(0)}{v^{2-n}} - \frac{f'(0)}{v^{3-n}} - \cdots \ldots - \frac{f^{(n-1)}(0)}{v}$ (7)

3. Adomian Polynomial Decomposition Method

The decomposition method decomposes the solution $u(x)$and the non-linearity $N(u)$ into series

$$u(x) = \sum_{n=0}^{\infty} u_n, \qquad N(u) = \sum_{n=0}^{\infty} A_n$$

where A_n are Adomian polynomials and can be computed as

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\textstyle\sum_{n=0}^{\infty} \lambda^n u_n)]_{\lambda=0} \qquad n \geq 0 \tag{8}$$

To compute A_n, take $N_u = f(u)$ to be a nonlinear function in u, where $u = u(x)$ and consider the Taylor series expansion of $f(u)$ around u_0

$$f(u) = f(u_0) + f'(u_0)(u - u_0) + \frac{1}{2!}f''(u_0)(u - u_0)^2 + \frac{1}{3!}f'''(u_0)(u - u_0)^3 + \cdots ..$$

but $u = u_0 + u_1 + u_2 + \cdots$. (9)

Then,

$$f(u) = f(u_0) + f'(u_0)(u_1 + u_2 + u_3) + \frac{1}{2!}f''(u_0)(u_1 + u_2 + u_3)^2 + \frac{1}{3!}f'''(u_0)(u_1 + u_2 + u_3)^3 + \cdots .. \tag{10}$$

by expanding all terms, we get

$$f(u) = f(u_0) + f'(u_0)(u_1) + f'(u_0)(u_2) + f'(u_0)(u_3) + \cdots \ldots + \frac{1}{2!}f''(u_0)(u_{s1})^2 + \frac{2}{2!}f''(u_0)(u_1u_2) + \frac{1}{2!}f''(u_0)(u_1u_3) + \cdots \ldots + \frac{1}{3!}f'''(u_0)(u_1)^3 + \frac{3}{3!}f'''(u_0)(u_1{}^2u_2) + \frac{1}{3!}f'''(u_0)(u_1{}^2u_3) \ldots .. \tag{11}$$

now, let l_i be the order of u_l^i and $l(i) + m(j)$ be the order of $u_l^i u_m^j$. Then A_n consists of all terms of order n, so we have

$A_0 = f(u_0)$,

$A_1 = u_1f'(u_0)$,

$A_2 = u_2f'(u_0) + \frac{1}{2!}(u_1)^2 f''(u_0)$,

$A_3 = u_3f'(u_0) + \frac{2}{2!}u_1u_2f''(u_0) + \frac{1}{3!}(u_1)^3f'''(u_0)$,

$A_4 = u_4f'(u_0) + \left[\frac{1}{2!}u_2{}^2 + u_1u_3\right]f''(u_0) + \frac{1}{2!}u_1{}^2u_2f'''(u_0) + \frac{1}{4!}u_1{}^4f''''(u_0)$,

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Hence,

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{n=0}^{\infty}\lambda^n u_n)]_{\lambda=0} \qquad n \geq 0 \tag{12}$$

To find A_n's by Adomian general formula, these polynomials will be computed as follows :

$$A_0 = N(u_0),$$

$$A_1 = N'(u_0)u_1 = \frac{d}{d\lambda}N(u_0 + \lambda u_1)\Big|_{\lambda=0},$$

$$A_2 = N'(u_0)u_2 + \frac{1}{2!}N''(u_0)(u_1)^2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}N(u_0 + \lambda u_1 + \lambda^2 u_2)|_{\lambda=0}, \tag{13}$$

$$A_3 = N'(u_0)u_3 + \frac{2}{2!}N''(u_0)u_1u_2 + \frac{1}{3!}N'''(u_0)(u_1)^3 = \frac{1}{3!}\frac{d^3}{d\lambda^3}N(u_0 + \lambda u_1 + \lambda^2 u_2 + \lambda^3 u_3)|_{\lambda=0}$$

4. Aboodh Adomian Decomposition Method

The following is an example of Bratu equation:

$$u'' + \lambda e^u = 0 \tag{14}$$

Applying Aboodh transform to both sides,

$$A(u'') + A(\lambda e^u) = 0$$

using the properties of Aboodh transform,

$$v^2K(v) - \frac{f'(0)}{v} - f(0) + \lambda A(e^u) = 0 \tag{15}$$

Let $u'(0) = B$ and $u(0) = 0$

Applying the conditions, eqn (15) becomes

$$v^2K(v) = \frac{B}{v} - \lambda A(e^u)$$

$$K(v) = \frac{B}{v^3} - \frac{\lambda}{v^2}A(e^u) \tag{16}$$

The Aboodh transform defines the solution $y(x)$ as

$$y(x) = \sum_{n=0}^{\infty} y_{n(x)} \tag{17}$$

Using the Adomian decomposition method,

$$e^{y(x)} = \sum_{n=0}^{\infty} A_n \tag{18}$$

To obtain the values of A_n, we expand expand e^u about $u(x)$,

$$e^u = A_0 + A_1 + A_2 + \cdots ..$$

$$= e^{u_0} + \frac{e^{u_0}(u-u_0)}{1!} + \frac{e^{u_0}(u-u_0)^2}{2!} + \cdots$$

$$= e^{u_0} + \frac{e^{u_0}\sum_{n=1}^{\infty} u_n}{1!} + \frac{e^{u_0}\sum_{n=1}^{\infty} {u_n}^2}{2!} + \cdots \tag{19}$$

Thus, Adomian polynomials are provided by

$$A_0 = e^{u_0}$$

$$A_1 = u_1e^{u_0}$$

$$A_2 = \frac{{u_1}^2}{2!}e^{u_0} + u_2e^{u_0}$$

$$A_3 = \frac{{u_1}^3}{2!}e^{u_0} + \frac{2u_1u_2}{2!}e^{u_0} + u_3e^{u_0} \tag{20}$$

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As a result of the prior discussion, A_n yields

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{n=0}^{\infty}\lambda^n u_n)]_{\lambda=0} \qquad n \geq 0 \tag{21}$$

Substituting eqn (17) and (18) into eqn (16),

$$A[\sum_{n=0}^{\infty} u_n(x)] = \frac{B}{v^3} - \frac{\lambda}{v^2} A(\sum_{n=0}^{\infty} A_n) \quad (22)$$

Using the Aboodh transform's linearity property, eqn (22) becomes

$$[\sum_{n=0}^{\infty} A(u_n(x))] = \frac{B}{v^3} - \frac{\lambda}{v^2} (\sum_{n=0}^{\infty} A(A_n)) \quad (23)$$

Equating the terms,

$$A\big(u_0(x)\big) = \frac{B}{v^3}$$

$$A\big(u_1(x)\big) = \frac{-\lambda}{v^2} A(A_0)$$

$$A\big(u_2(x)\big) = \frac{-\lambda}{v^2} A(A_1)$$

$$A\big(u_3(x)\big) = \frac{-\lambda}{v^2} A(A_2)$$

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$$A\big(u_n(x)\big) = \frac{-\lambda}{v^2} A(A_{n-1}) \quad (24)$$

Taking the inverse of Aboodh transform to the system above, we get

$$u_0(x) = Bx \quad (25)$$

$$u_1(x) = \lambda\left(\frac{1}{B^2} + \frac{x}{B} - \frac{e^{Bx}}{B^2}\right) \quad (26)$$

$$u_2(x) = -\lambda^2\left(\frac{5}{4B^4} + \frac{x}{2B^3} - \frac{e^{Bx}}{B^4} + \frac{xe^{Bx}}{B^3} + \frac{e^{2Bx}}{4B^4}\right) \quad (27)$$

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From (25-27), B and λ are the unknown in the sequence $\{u_n\}_{n=0}^{\infty}$, so the solution of the Bratu equation is given by

$$u(x) = u_0(x) + u_2(x) + u_3(x) + \cdots.$$

$$u(x) = Bx + \lambda\left(\frac{1}{B^2} + \frac{x}{B} - \frac{e^{Bx}}{B^2}\right) - \lambda^2\left(\frac{5}{4B^4} + \frac{x}{2B^3} - \frac{e^{Bx}}{B^4} + \frac{xe^{Bx}}{B^3} + \frac{e^{2Bx}}{4B^4}\right) \quad (28)$$

5. Numerical Results

EXAMPLE 1

Examine the equation

$$y''(x) - 2e^{y(x)} = 0 \quad (29)$$

subject to $y(0) = 0\,, y'(0) = 0$ (30)

Exact solution $= -2In\cos(x)$

SOLUTION

$$y''(x) - 2e^{y(x)} = 0 \quad (31)$$

Take the Aboodh transform

$$A[y''(x)] - 2A\big[e^{y(x)}\big] = A[0]$$

$$v^2K(v) - \frac{f'(0)}{v} - f(0) - 2A\big[e^{y(x)}\big] = 0 \quad (32)$$

$y(0) = 0\,, y'(0) = 0 \quad v^2K(v) = 2A\big[e^{y(x)}\big]$

Take the inverse of the Aboodh transform to obtain:

$$y(x) = A^{-1}\left[\frac{2}{v^2}A\left(e^{y(x)}\right)\right] \tag{33}$$

The Aboodh transform defines the solution $y(x)$ as

$$y(x) = \sum_{n=0}^{\infty} y_{n(x)} \tag{34}$$

Using the Adomian decomposition method,

$$e^{y(x)} = \sum_{n=0}^{\infty} A_n \tag{35}$$

To obtain the values of A_n, we expand expand $e^{y(x)}$ about $y(x)$

$$e^u = A_0 + A_1 + A_2 + \cdots ..$$

$$= e^{y_0} + \frac{e^{y_0}(y-y_0)}{1!} + \frac{e^{y_0}(y-y_0)^2}{2!} + \cdots$$

$$= e^{y_0} + \frac{e^{y_0}\sum_{n=1}^{\infty} y_n}{1!} + \frac{e^{y_0}\sum_{n=1}^{\infty} {y_n}^2}{2!} + \cdots$$

Substitute (35) into (33)

$$\left[\sum_{n=0}^{\infty} y_n(x)\right] = A^{-1}\left[\frac{2}{v^2}A\left[\sum_{n=0}^{\infty} A_n\right]\right]$$

$$A\left[\sum_{n=0}^{\infty} y_n(x)\right] = \frac{2}{v^2}A\left[\sum_{n=0}^{\infty} A_n\right]$$

where n=0, y_0(x) = 0, A_0=1

n=1, $y_1(x) = x^2$, $A_1 = x^2$

n=2, $y_2(x) = \frac{x^4}{6}$, $A_2 = \frac{2x^4}{3}$

n=3, $y_3(x) = \frac{2x^6}{45}$, $A_3 = \frac{17x^6}{45}$

n=4, $y_4(x) = \frac{17x^8}{1260}$, $A_4 = \frac{62x^8}{315}$

n=5, $y_5(x) = \frac{62x^{10}}{14175}$

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \cdots ...$$

$$= x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \frac{17x^6}{45} + \frac{62x^8}{315} + \frac{62x^{10}}{14175} + \cdots. \tag{36}$$

Table 4.1. Numerical results

x	Exact Error	Approximate Error (AADM)	Absolute Error (AADM)
0	0	0	0
0.1	0.01001671125	0.01001671111	1.4 $E-10$
0.2	0.0402695461	0.04026951111	3.499 $E-8$
0.3	0.09138331185	0.0913824	9.1185 $E-7$
0.4	0.1644580382	0.1644487111	9.3271 $E-6$
0.5	0.2611684809	0.2611111111	5.73698 $E-5$
0.6	0.3839303388	0.3836736	2.567388 $E-4$
0.7	0.5361715151	0.535245511	9.260041 $E-4$
0.8	0.7227814936	0.7199175111	2.8639825 $E-3$
0.9	0.9508848872	0.9429696	7.9152872 $E-3$
1.0	1.231252941	1.211111111	0.02014183

EXAMPLE 2

Solve $y''(x) - e^{2y(x)} = 0$ (37)

$y(0) = 0$, $y'(0) = 0$ (38)

Exact solution is $y(x) = In(sec(x))$

SOLUTION

$y''(x) - e^{2y(x)} = 0$ (39)

Take the Aboodh transform,

$A[y''(x)] - A[e^{2y(x)}] = A[0]$

$v^2K(v) - \frac{f'(0)}{v} - f(0) - A[e^{2y(x)}] = 0$ (40)

Inserting the initial conditions,

$v^2K(v) = A[e^{2y(x)}]$

Take the inverse of the Aboodh transform to obtain:

$y(x) = A^{-1}\left[\frac{1}{v^2}A(e^{2y(x)})\right]$ (41)

The Aboodh transform defines the solution $y(x)$ as

$y(x) = \sum_{n=0}^{\infty} y_n(x)$ (42)

Using the Adomian decomposition method,

$e^{2y(x)} = \sum_{n=0}^{\infty} A_n$ (43)

To obtain the values of A_n, we expand expand $e^{2y(x)}$ about $y(x)$,

$e^{2y(x)} = A_0 + A_1 + A_2 + \cdots.$

$$= e^{2y_0} + \frac{e^{2y_0}(2y-2y_0)}{1!} + \frac{e^{2y_0}(2y-2y_0)^2}{2!} + \cdots$$

$$= e^{2y_0} + \frac{e^{2y_0}\sum_{n=1}^{\infty} 2y_n}{1!} + \frac{e^{2y_0}\sum_{n=1}^{\infty} 2{y_n}^2}{2!} + \cdots$$

Substitute eqn. (43) into (42)

$$[\sum_{n=0}^{\infty} y_n(x)] = A^{-1}\left[\frac{1}{v^2}A[\sum_{n=0}^{\infty} A_n]\right]$$

where n=0, y_0(x) = 0, A_0=1

n=1, $y_1(x) = \frac{x^2}{2}$, $A_1 = x^2$

n=2, $y_2(x) = \frac{x^4}{12}$, $A_2 = \frac{2x^4}{3}$

n=3, $y_3(x) = \frac{x^6}{45}$

$y(x) = y_0(x) + y_1(x) + y_2(x) + \cdots$ ...

$= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots.$ (44)

Table 4.2. Numerical results

x	Exact Error	Approximate Error	Absolute Error
0	0	0	0
0.1	$5.008355623\ E\ 10^{-3}$	$5.008355556\ E\ 10^{-3}$	$6.7444445\ E-11$
0.2	0.02013477305	0.02013475556	$1.749240765\ E-8$
0.3	0.04569165593	0.0456912	$4.559260565\ E-7$
0.4	0.08222901908	0.08222435556	$4.663515054\ E-6$
0.5	0.1305842404	0.1305555556	$2.868484372\ E-5$
0.6	0.1919651694	0.1918368	$1.283694194\ E-4$
0.7	0.2680857576	0.2676227556	$4.630019679\ E-4$
0.8	0.3613907468	0.3599587556	$1.431991211\ E-3$
0.9	0.4754424436	0.4714848	$3.957643586\ E-3$
1.0	0.6156264704	0.6055555556	0.0100709148

EXAMPLE 3

$$y''(x) - \pi^2 e^{y(x)} = 0 \quad (45)$$

$$y(0) = 0 \quad y'(0) = \pi \quad (46)$$

Exact solution is $y(x) = -In(1 - \sin(\pi x))$

SOLUTION

$$y''(x) - \pi^2 e^{y(x)} = 0 \quad (47)$$

Take the Aboodh transform

$$A[y''(x)] - A\left[\pi^2 e^{y(x)}\right] = A[0]$$

$$v^2 K(v) - \frac{f'(0)}{v} - f(0) - A\left[\pi^2 e^{y(x)}\right] = 0 \quad (48)$$

Applying the initial conditions,

$$v^2 K(v) - \frac{\pi}{v} - A\left[\pi^2 e^{y(x)}\right] = 0$$

$$v^2 K(v) = \frac{\pi}{v} + A\left[\pi^2 e^{y(x)}\right]$$

Take the inverse of the Aboodh transform to obtain:

$$y(x) = A^{-1}\left[\frac{\pi}{v^3} + \frac{\pi^2}{v^2} A\left(e^{y(x)}\right)\right] \quad (49)$$

The Aboodh transform defines the solution $y(x)$ as

$$y(x) = \sum_{n=0}^{\infty} y_{n(x)} \quad (34)$$

Using the Adomian decomposition method,

$$e^{y(x)} = \sum_{n=0}^{\infty} A_n \quad (50)$$

To obtain the values of A_n, we expand expand $e^{y(x)}$ about y_0,

$$e^{y(x)} = A_0 + A_1 + A_2 + \cdots ..$$

$$= e^{y_0} + \frac{e^{y_0}(y - y_0)}{1!} + \frac{e^{y_0}(y - y_0)^2}{2!} + \cdots$$

$$= e^{y_0} + \frac{e^{y_0}\sum_{n=1}^{\infty} y_n}{1!} + \frac{e^{y}\sum_{n=1}^{\infty} {y_n}^2}{2!} + \cdots$$

$$A[\sum_{n=0}^{\infty} y_n(x)] = \frac{\pi}{v^3} + \frac{\pi^2}{v^2} A[\sum_{n=0}^{\infty} A_n]$$

using the linearity property of the Aboodh transform,

$$A[\sum_{n=0}^{\infty} y_n(x)] = \frac{\pi}{v^3} + \frac{\pi^2}{v^2} [\sum_{n=0}^{\infty} A(A_n)]$$

where n=0, $y_0(x) = \pi x$, $A^0 = e^{\pi x}$

n=1, $y_1(x) = -1 - \pi x + e^{\pi x}$, $A^1 = -e^{\pi x} - \pi x e^{\pi x} + e^{2\pi x}$

n=2, $y_2(x) = -\frac{5}{4} - \frac{\pi x}{2} + e^{\pi x} - \pi x e^{\pi x} + \frac{e^{2\pi x}}{4}$

$$y(x) = y_1(x) + y_2(x) + y_3(x) + \cdots.$$

$$y(x) = (\pi x) + (-1 - \pi x + e^{\pi x}) + (-\frac{5}{4} - \frac{\pi x}{2} + e^{\pi x} - \pi x e^{\pi x} + \frac{e^{2\pi x}}{4})$$

$$y(x) = -\frac{9}{4} - \frac{\pi x}{2} + 2e^{\pi x} - \pi x e^{\pi x} + \frac{e^{2\pi x}}{4} \quad (51)$$

Table 4.3. Numerical results

x	Exact Error	Approximate Error	Absolute Error
-0.5	-0.6931471806	-0.7115027294	0.01835554879
-0.4	-0.6683710291	-0.6745608344	6.189805276 $E-3$
-0.3	-0.5927836007	-0.5942329067	1.449306045 $E-3$
-0.2	-0.4623401221	-0.4625117131	1.715909749 $E-4$
-0.1	-0.2692764696	-0.2692801896	3.720014378 $E-6$
0	0	0	0
0.1	0.3696400494	0.369632039	8.010350236 $E-6$
0.2	0.8862108331	0.885393821	8.170120638 $E-4$
0.3	1.655570831	1.639230083	0.01634074779
0.4	3.01708904	2.819871789	0.1972172514

EXAMPLE 4

Solve $y''(x) + \pi^2 e^{-y(x)} = 0$ (52)

$y(0) = 0 \quad y'(0) = \pi$ (53)

Exact solution is $y(x) = In(1 + \sin(\pi x))$

SOLUTION

$$y''(x) + \pi^2 e^{-y(x)} = 0 \tag{54}$$

Take the Aboodh transform

$$A[y''(x)] + A\left[\pi^2 e^{-y(x)}\right] = A[0]$$

$$v^2K(v) - \frac{f'(0)}{v} - f(0) + A\left[\pi^2 e^{-y(x)}\right] = 0 \tag{55}$$

Applying the initial conditions,

$$v^2K(v) - \frac{\pi}{v} + A\left[\pi^2 e^{-y(x)}\right] = 0$$

$$v^2K(v) = \frac{\pi}{v} - A\left[\pi^2 e^{-y(x)}\right]$$

Take the inverse of the Aboodh transform to obtain:

$$y(x) = A^{-1}\left[\frac{\pi}{v^3} - \frac{\pi^2}{v^2}A\left(e^{-y(x)}\right)\right] \tag{56}$$

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \qquad and \qquad e^{-y(x)} = \sum_{n=0}^{\infty} A_n \tag{57}$$

To obtain the values of A_n, we expand $e^{-y(x)}$ about y_0

$$e^{-y(x)} = A_0 + A_1 + A_2 + \cdots ..$$

Substitute eqn. (57) into Eqn. (56)

$$A\left[\sum_{n=0}^{\infty} y_n(x)\right] = \frac{\pi}{v^3} - \frac{\pi^2}{v^2} A\left[\sum_{n=0}^{\infty} A_n\right] \tag{58}$$

using the linearity property of the Aboodh transform,

$$A\left[\sum_{n=0}^{\infty} y_n(x)\right] = \frac{\pi}{v^3} - \frac{\pi^2}{v^2} \left[\sum_{n=0}^{\infty} A(A_n)\right] \tag{59}$$

where n=0, $y_0(x) = \pi x$, $A^0 = e^{-\pi x}$

n=1, $y_1(x) = 1 - \pi x - e^{-\pi x}$, $A^1 = -e^{-\pi x} + \pi x e^{-\pi x} + e^{-2\pi x}$

n=2, $y_2(x) = \frac{9}{4} - \frac{\pi x}{2} - 2e^{-\pi x} - \pi x e^{-\pi x} - \frac{e^{-2\pi x}}{4}$

$$y(x) = y_1(x) + y_2(x) + y_3(x) + \cdots.$$

$$y(x) = (\pi x) + (-1 - \pi x + e^{-\pi x}) + \left(-\frac{5}{4} - \frac{\pi x}{2} + e^{-\pi x} - \pi x e^{-\pi x} + \frac{e^{-2\pi x}}{4}\right)$$

$$y(x) = -\frac{9}{4} - \frac{\pi x}{2} + 2e^{-\pi x} - \pi x e^{-\pi x} + \frac{e^{-2\pi x}}{4} \tag{60}$$

Table 4.4. Numerical results

x	Exact	Approximate	Absolute Error
-0.4	-3.01708904	-2.819871789	0.1972172514
-0.3	-1.655570831	-1.639230083	0.01634074779
-0.2	-0.8862108331	-0.885393821	$8.170120638\ E-4$
-0.1	-0.3696400494	-0.369632039	$8.010350236\ E-6$
0	0	0	0
0.1	0.2692764696	0.2692801896	$3.720014378\ E-6$
0.2	0.4623401221	0.4625117131	$1.715909749\ E-4$
0.3	0.5927836007	0.5942329067	$1.449306045\ E-3$
0.4	0.6683710291	0.6745608344	$6.189805276\ E-3$
0.5	0.6931471806	0.7115027294	0.01835554879

6. Conclusion

This study uses Aboodh Adomian Decomposition Method (AADM) to examine Bratu equations. The method is used directly without any linearization or discretization. Numerical results show that the procedure is accurate and effective in obtaining analytical and numerical solutions for a broad class of linear and nonlinear differential equations. Results gotten by the presented method compare favorably to those obtained by other known methods. However, the most vital point of this method is that it involves less rigorous computation, unlike other methods for the same class of equations. Therefore, the presented technique is an alternative to overcome the demerit of complex calculation.

In the future, we will apply Aboodh Adomian Decomposition Method to a wider range of differential equations to access its versatility further.

Declaration of Competing Interest

No conflict of interest was declared by the authors.

Authorship Contribution Statement

Taiwo Abass: Draft preparation, Writing, Reviewing, Analysis and Results Interpretation
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Odetunde Olutunde: Methodology, Reviewing, Supervision, Investigation and Results Interpretation
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References

[1] A. Wazwaz, "The Successive differentiation method for solving Bratu equation and Bratu-type equations," *Rom. Journ. Phys.*, vol. 61, no. 5-6, pp. 774-783, 2016.

[2] L. H. Koudahoun, J. Akande, D. K. Adjai, Y. F. Kpomahou, and M.D. Monsia, "On the general solution to Bratu and generalized Bratu equations," *Journal of Mathematics and Statistics*, vol. 14, pp. 193-200, 2018.

[3] M. Abukhaled, S. Khuri, and A. Sayfy, "Spline-based numerical treatments of Bratu-type equations," *Palestine Journal of Mathematics*, vol. 1, pp. 63-70, 2012.

[4] M. I. Syam, and A. Hamdan, "An efficient method for solving Bratu equations," *Applied Mathematics and Computation*, vol. 176, pp. 704- 713, 2006.

[5] S. H. Kashkari, and S. Z. Abbas, "Solution of initial value problem of Bratu-type equation using modifications of homotopy perturbation method," *International Journal of Computer Applications,* vol. 162, no. 5, 2017.

[6] A. Batiha, "Numerical solution of Bratu-type equations by the variational iteration method," *Journal of Mathematics and Statistics,* vol. 39, no. 1, pp. 23-29, 2010.

[7] N. Das, R. Singh, A. Wazwaz, and J. Kumar, "An algorithm based on the variational iteration technique for the Bratu-type and the Lane-Emden problems," *Journal of Mathematical Chemistry,* vol. 54, pp. 527-551, 2015.

[8] X. Feng, Y. He, and J. Meng, "Application of homotopy perturbation method to the Bratu-type equation," *Journal of Julius Schauder Center,* vol. 31, pp. 243-252, 2008.

[9] K. H. Sedeeg, and M. M. A. Mahgoub, "Aboodh transform homotopy perturbation method for solving system of nonlinear partial differential equations," *Mathematical Theory and Modeling,* vol. 6, no. 8, 2016.

[10] J. Ahmad, and J. Taric, "Application of Aboodh differential transform method on some higher order problems," *Journal of Science and Arts,* no. 1, no. 42, pp. 5-18, 2018.

[11] K. S. Aboodh, R. A. Farah, I. A. Almardy, and A. K. Osman, "Solving delay differential equations by Aboodh transformation method," *International Academy of Science, Engineering and Technology*, vol. 7, pp. 55-64, 2018.

[12] S. Qureshi, M. S. Chandio, A. A. Shaikh, and R. A. Memon, "On the Use of Aboodh Transform for Solving Non-integer Order Dynamical Systems," *Sindh University Research Journal (Social Series),* vol. 51, no. 1, pp. 53-58, 2019.

[13] D. Baleanu, S. Qureshi, A. Soomro, and A. A. Shaikh, "Variable Stepsize Construction of a Two-step Optimized Hybrid Block Method with a Relative Stability," *Open Physics,* vol. 20, no. 1, pp. 1112-1126, 2022.

[14] S. Qureshi, A. Soomro, E. Hincal, J. R. Lee, C. Park, and M. S. Osman, "An Efficient Variable Stepsize Rational Method for Stiff, Singular and Singularly Perturbed Problems," *Alexandria University Journal*, vol. 61, pp. 10953-10963, 2022.

[15] S. Qureshi, A. Soomro, S. E. Fadugba, and E. Hincal, "A New Family of L-stable Block Methods with Relative Measure of Stability," *International Journal of Applied nonlinear Science*, vol. 3, no. 3, pp. 197-222, 2022.