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Some Results of Common Fixed Point for Compatible Mappings in Ƒ-Metric Spaces

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Abstract − Recently, Ƒ-metric space has been started, and a natural topology has been described in these spaces by Jleli and Samet. Furthermore, a new form of the Banach contraction principle has been given in the new spaces. In this work, we present some common fixed-point theorems for two weakly compatible mappings in the Ƒ-metric spaces. We also mention examples that confirm our results.

Keywords *– Common fixed point, Ƒ-metric space, weakly compatible*

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1. Introduction

Metric space is the general theory underlying various branches of mathematics. These days, generalizations of the metric space have emerged. Lately, some authors have given various generalizations of metric spaces. This situation allows authors to find new work areas. Czerwik described the concept of b–metric [1]. Khamsi and Hussain redefined the b–metric concept, and they called it metric-type [2]. Fagin et al. gave s–relaxed$_p$ metric concept [3]. Here, b–metric is more general than the s–relaxed$_p$ metric [4]. Gahler began the concept of a 2–metric [5]. This metric function is identified on the product set $X \times X \times X$. The notion of 2–metric is a generalization of the usual metric. Mustafa and Sims defined the $\mathcal{G}$–metric space concept [6]. The notion is more general than the usual metric. Branciari proposed a new extension of the notion of metric, modified the triangle inequality (iii) with a more general inequality confusing four points. Matthews defined the partial metric [7]. Jleli and Samet defined the JS–metric [8]. Currently, Jleli and Samet have given the Ƒ-metric space [9]. They compare their concepts with existing generalizations in the literature. Then, they define a natural topology $\tau_{Ƒ}$ on these spaces and examine their topological properties. Also, a new version of the Banach contraction principle is created in the tuning of Ƒ-metric spaces. They proved that their new concept is more general than the standard metric concept by showing that any metric space is an Ƒ-metric space, but the reverse is generally false. They also compared their concept with previous generalizations of metric spaces. After that, Alnaser et al. gave relation theoretic contraction and proved some generalization of fixed-point theorems in these spaces [10]. Moreover, in these generalized spaces, the coincidence points and common fixed-point

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theorems are also frequently studied [11]. In this paper, we give some common fixed-point theorems and results in Ƒ-metric spaces. Moreover, some examples that provide the theorems are presented.

2. Preliminaries

Definition 2.1. [9] Let Ƒ-be the set of function $g:(0,\infty)\to\mathbb{R}$. This function provides the below terms.

Ƒ1. g be a non-decreasing function

Ƒ2. $\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} g(a_n) = -\infty$, for every sequence $\{a_n\} \subseteq (0\ +\infty)$.

Definition 2.2. [9] Let $X \neq \emptyset$, $D: X\times X \to [0,\infty)$ is a mapping, and there exist $g \in$ Ƒ and $\gamma \in [0,\infty)$. If the following terms are satisfied, D be defined as an Ƒ-metric on X. In this case, (X,D) is defined as an Ƒ-metric space.

D_1. $(a,b) \in X\times X,\ a=b \Leftrightarrow D(a,b)=0$,

D_2. $D(a,b)=D(b,a)$ for all $a,b\in X$,

D_3. $\forall a,b\in X, n\in\mathbb{N}, n\geq 2$ and $(t_i)_{i=1}^{n} \subset X$ with $(t_1, t_{n_0}) = (a,b)$ we have

$$D(a,b)>0 \Rightarrow g(D(a,b)) \leq g(\Sigma_{i=1}^{n-1} D(t_i,t_{i+1})) + \gamma$$

Remark 2.3. [9] The metric space is an Ƒ-metric space. But the contrary of this proposition is false.

Example 2.4. [9] Let $\mathbb{N}$ be positive real numbers set. $D:\mathbb{N}\times\mathbb{N}\to[0,\infty)$ be the mapping and for all

$a,b\in\mathbb{N}$,

$$D(a,b)=\begin{cases}(a-b)^2, & a,b\in[0,3]\\ |a-b|, & a,b\notin[0,3]\end{cases}$$

Therefore, D is an Ƒ-metric with $g(a)=\ln a$, $a>0$ and $\gamma=\ln 3$.

Example 2.5. [9] Let $\mathbb{N}$ be the natural numbers set, $D:\mathbb{N}\times\mathbb{N}\to[0,\infty)$ and for all $a,b\in\mathbb{N}$,

$$D(a,b)=\begin{cases}\exp(|a-b|), & a\neq b\\ 0, & a=b\end{cases}$$

Therefore, D is an Ƒ-metric with $g(a)=-1/a$, $a>0$ and $\gamma=1$.

Definition 2.6. [9] Suppose that D be an Ƒ-metric on X, $\{a_n\}\subset X$ is a sequence.

i. If $\{a_n\}$ be convergent to an element according to Ƒ-metric D, $\{a_n\}$ is Ƒ –convergent to element a.

ii. If $\lim_{m,n\to\infty} D(a_n,a_m)=0$ then the sequence $\{a_n\}$ is Ƒ-Cauchy.

iii. If any Ƒ-Cauchy sequence be convergent, (X,D) is Ƒ-complete.

Definition 2.7. [11] Let T and S be self-maps on a set X. If $Tx = Sx = y$ for some $x\in X$, then x,y are defined as a coincident point and a coincidence point, respectively. If $x = Tx = Sx$ for some $x\in X$, x defined as a common fixed point.

Remark 2.8. [11] If T and S are weakly compatible, then the coincidence point y is the unique common fixed point.

Theorem 2.9. [9] Let (X,D) be an Ƒ-metric space, $f:X\to X$. Assume that the Ƒ-metric space (X,D) is Ƒ-complete and there exists $\alpha\in(0,1)$ such that $D(f(a),f(b))\leq\alpha D(a,b)$ for $a,b\in X$. Then f has a unique fixed point $a^*\in X$. Furthermore, the sequence $\{a_n\}\subset X$ given by $a_{n+1}=f(a_n)$, $n\in\mathbb{N}$ is Ƒ-convergent to a^*, for any $a_0\in X$.

Remark 2.10. In the theorems and results we have given throughout the article; we will think that the map $g \in Ƒ$ is surjective and $\gamma = 0$ for the proof to proceed smoothly.

3. Main Results

In this part, we give generalizations of some known fixed-point theorems in the Ƒ-metric spaces. These are coincidence points and common fixed-point theorems. Moreover, we denote some examples of the presented results.

Theorem 3.1. Let (X, D) be an Ƒ-metric. Assume that S, T$: X \to X$ provides the below conditions

i. For $\forall a, b \in X$,

$$D(T(a), T(b)) \leq kD(S(a), T(a)) + lD(S(b), T(b)) + mD(S(a), T(b)) + pD(S(b), T(a)) + tD(S(a), S(b))$$

where k, l, m, p, and t are non-negative and $k + l + m + p + t < 1$,

ii. $T(X) \subset S(X)$,

iii. $T(X)$ or $S(X)$ be an Ƒ-complete subspace.

Therefore, T and S have a unique coincidence point.

Furthermore, if T and S are weakly compatible, they have a unique common fixed point.

Proof.

Let $g \in Ƒ$, $\gamma = 0$ be such that for every $a, b \in X$ for $\forall n \in \mathbb{N}$, $n \geq 2$ and for $\forall (t_i)_{i=1}^{n} \subset X$ with $(t_1, t_{n_0}) = (a, b)$, we have

$$D(a, b) > 0 \Rightarrow g(D(a, b)) \leq g(\Sigma_{i=1}^{n-1} D((t_i, t_{i+1})) + \gamma$$

From $Ƒ_2$, for every sequence $\{a_n\} \subseteq (0, +\infty)$, there exists a $\varepsilon > 0$ such that

$$\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} g(a_n) = -\infty \text{ and } 0 < a < \varepsilon \Rightarrow g(a) < g(\varepsilon) - \gamma$$

Let a_0, $a_1 \in X$ be arbitrary and $\{a_n\} \subset X$ be the sequence defined by $Sa_{n+1} = Ta_n = b_n$ for $n \in \mathbb{N}$. We have that

$$D(b_n, b_{n+1}) \leq (k + t)D(b_{n-1}, b_n) + lD(b_n, b_{n+1}) + mD(b_{n-1}, b_{n+1})$$

and

$$D(b_{n+1}, b_n) \leq kD(b_n, b_{n+1}) + (l + m)D(b_{n-1}, b_n) + pD(b_{n-1}, b_{n+1})$$

for all n. Hence, from the above remark,

$$D(b_n, b_{n+1}) \leq \frac{k + l + t + p + 2m}{2 - (k + l + m + p)} D(b_{n-1}, b_n)$$

If we choose $c = \frac{k+l+t+p+2m}{2-(k+l+m+p)}$, then $c \in [0,1)$ and $D(b_n, b_{n+1}) \leq c\, D(b_{n-1}, b_n)$ is hold. We have

$$D(b_n, b_{n+1}) \leq c^n D(b_0, b_1)$$

Thus, for all n and z,

$$\begin{aligned} D(b_n, b_{n+z}) &\leq D(b_n, b_{n+1}) + D(b_{n+1}, b_{n+2}) + \cdots + D(b_{n+z-1}, b_{n+z}) \\ &\leq (c^n + c^{n+1} + \cdots + c^{n+z-1}) D(b_0, b_1) \\ &\leq \frac{c^n}{1 - c} D(b_0, b_1) \end{aligned}$$

holds.

Since $\lim_{n\to\infty}\frac{c^n}{1-c}D(b_0,b_1)=0$, there exists a $n_0\in\mathbb{N}$ such that $0<\frac{c^n}{1-c}D(b_0,b_1)<\varepsilon,\ n\geq n_0$. From conditions $0<b<\varepsilon\Rightarrow g(b)<g(\varepsilon)-\gamma$ and since g is non-decreasing.

$$g(\Sigma_{i=n}^{n+z-1}D(b_i,b_{i+1}))\leq g(\frac{c^n}{1-c}D(b_0,b_1))<g(\varepsilon)-\gamma,n\geq n_0\ ....*$$

Using conditions (D_3) and (*),

$$D(b_n,b_{n+z})>0,n\geq n_0\Rightarrow g(D(b_n,b_{n+z}))\leq g(\Sigma_{i=n}^{n+z-1}D(b_i,b_{i+1}))+\gamma<g(\varepsilon)$$

Thus, we obtain that $D(b_n,b_{n+z})<\varepsilon,n\geq n_0$ by (Ƒ_1). It is seen that this sequence $\{b_n\}$ is an Ƒ −Cauchy.

Because of the range of S contains the range of T and the range of at least one is Ƒ-complete, there exists a $d\in S(X)$ such that $\lim_{n\to\infty}D(Sa_n,d)=0$. Therefore there exists a sequence (x_n) in $[0,+\infty)$ and $x_n\to 0$, $D(Sa_n,d)\leq x_n$. Moreover, an $e\in X$ can be found, $Se=d$.

Now, show that $Te=d$. Suppose that $D(Te,d)>0$. From the condition (D_3),

$$g(D(Te,d))\leq g(D(Te,Tb_n))+D(Tb_n,d)+\gamma,n\in\mathbb{N}$$

Using condition of theorem and g is non-decreasing,

$$g(D(Te,d))\leq g[c(D(Te,d)+2D(Tb_n,b_n)+D(b_n,d))]+\gamma,n\in\mathbb{N}$$

Otherwise, using $\lim_{n\to\infty}b_n=0\Leftrightarrow\lim_{n\to\infty}g(b_n)=-\infty$ and $\lim_{n\to\infty}[D(Te,d)+2D(Tb_n,b_n)+D(b_n,d)]=0$, we obtain that

$$\lim_{n\to\infty}g(c[D(Te,d)+2D(Tb_n,b_n)+D(b_n,d)])+\gamma=-\infty$$

This is a contradiction. Consequently, $D(Te,d)=0$, i.e., $Te=d$ and so d is a coincidence point of T and S. If d_1 is another coincidence point, there is $e_1\in X$ with $Te_1=Se_1=d_1$. Therefore,

$$D(d,d_1)=D(Te,\text{T}e_1)\leq cD(d,d_1)$$

Hence, $D(d,d_1)=0$ that is $d=d_1$. If T and S are weakly compatible, *d* is a unique common fixed point. □

Corollary 3.2. Let (X,D) be an Ƒ-metric space. Assume S, T: $X\to X$ provides the below conditions

i. For $\forall a,b\in X$, $D(T(a),T(b))\leq cD(S(a),S(b))$ where $c<1$,

ii. $T(X)\subset$ S(X),

iii. $T(X)$ or S(X) be Ƒ-complete subspace.

Therefore, T and S have a unique coincidence point.

Besides, if T and S are weakly compatible, they have a unique common fixed point in X.

Example 3.3. Let $X=\mathbb{N}$ be the set of positive real numbers. $D:\mathbb{N}\times\mathbb{N}\to[0,\infty)$ be the mapping and for all $a,b\in\mathbb{N}$,

$$D(a,b)=\begin{cases}(a-b)^2, & a,b\in[0,10]\\ |a-b|, & a,b\notin[0,10]\end{cases}$$

If take $g(a)=\ln a$, we can show that D be an Ƒ-metric by a routine calculation. Next, define $T(a)=a^2+1$ and $S(a)=2a^2$. Then, for all $a,b\in\mathbb{N}$ we have

$$D(T(a),T(b))=\frac{1}{2}D(S(a),S(b))\leq cD(S(a),S(b))\text{ for }c<1$$

$T(X)=[1,\infty)\subset[0,\infty)=S(X)$. Moreover, $T(X)$ is an Ƒ-complete subspace. That is, all conditions of corollary are satisfied. T and S have a unique coincidence point.

If $c = 2$ be a unique coincident point, $a = 1$ and $b = -1$ be coincidence points of S and T. But since $T(S(1)) \neq S(T(1))$, S and T are not weakly compatible, so S and T have no common fixed points.

Theorem 3.4. Let (X, D) be an Ƒ-metric, S, T: $X \to X$, S^2 be a continuous, and T commute with S. Assume the below conditions is satisfied,

i. For all $a, b \in X$, $D\big(T(a), T(b)\big) \leq \alpha u(a, b)$ where $\alpha \in \left(0, \frac{1}{2}\right)$ is a constant and

$$u(a,b) \in \{D\big(S(a), S(b)\big), D\big(S(a), T(a)\big), D\big(S(b), T(b)\big), D\big(S(a), T(b)\big), D\big(S(b), T(a)\big)\}$$

ii. $TS(X) \subset S^2(\mathrm{X})$,

iii. $T(X)$ or S(X) be Ƒ-complete subspace.

Therefore, T and S have a unique common fixed point.

Proof.

Let $g \in$ Ƒ, $\gamma \in [0, \infty)$ be such that for $\forall a, b \in X$ for every $n \in \mathbb{N}, n \geq 2$ and for every $(t_i)_{i=1}^{n} \subset X, (t_1, t_{n_0}) = (a, b)$, we have

$$D(a,b) > 0 \Rightarrow g(D(a,b)) \leq g(\Sigma_{i=1}^{n-1} D((t_i, t_{i+1})) + \gamma$$

From $Ƒ_2$, for every sequence $\{a_n\} \subseteq (0, +\infty)$, there exists a $\varepsilon > 0$ such that

$$n \to \infty \lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} g(a_n) = -\infty \text{ and } 0 < a < \varepsilon \Rightarrow g(a) < g(\varepsilon) - \gamma$$

Let $a_0 \in S(X)$ be arbitrary and $\{b_n\} \subset \mathrm{S}(X)$ be the sequence defined by Sa_{n+1}=$Ta_n = b_n$ for $n \in \mathbb{N}$. Now $Sb_{n+1} = STa_{n+1}$=$TSa_{n+1} = Tb_n = c_n, n \geq 1$. We show that $\{c_n\}$ is an Ƒ-Cauchy sequence, so convergent to some c$\in X$. We denote that $S^2c = TSc$.

Since $\lim_{n\to\infty} Sb_n = \lim_{n\to\infty} STa_n$=$\lim_{n\to\infty} TSa_n = \lim_{n\to\infty} Tb_n = \lim_{n\to\infty} c_n = c$, it follows that $\lim_{n\to\infty} S^4a_n$=$\lim_{n\to\infty} S^3Ta_n = \lim_{n\to\infty} TS^3a_n = S^2c$, since S^2 is continuous. Thus, we get

$$D(S^2c, TSc) \leq D(S^2c, S^3Ta_n) + D(S^3Ta_n, TSc) \leq D(S^2c, S^3Ta_n) + \alpha u_n$$

where $u_n \in \{D(S^4a_n, S^2c), D(S^4a_n, TS^3a_n), D(S^2c, TSc), D(S^4a_n, TSc), D(S^2c, TS^3a_n)\}$.

Choose any $n_0 \in \mathbb{N}$, for all $n \geq n_0$, since $S^3Ta_n \to S^2c$ and $S^4a_n \to S^2c$, then we have $D(S^2c, S^3Ta_n) \leq x_n$ and $D(S^4a_n, S^2c) \leq y_n$, as $x_n \to 0$ and $y_n \to 0$. We have the five cases:

Case 1: $D(S^2c, TSc) \leq D(S^2c, S^3Ta_n) + \alpha D(S^4a_n, S^2c) \leq x_n + \alpha y_n$

Case 2:

$$\begin{aligned} D(S^2c, TSc) &\leq D(S^2c, S^3Ta_n) + \alpha D(S^4a_n, TS^3c) \\ &\leq D(S^2c, S^3Ta_n) + \alpha\big(D(S^4a_n, S^2c) + \alpha D(TS^3a_n, S^2c)\big) \\ &\leq x_n + \alpha(y_n + x_n) \\ &= (1+\alpha)x_n + \alpha y_n \end{aligned}$$

Case 3: $D(S^2c, TSc) \leq D(S^2c, S^3Ta_n) + \alpha D(S^2c, TSc) \leq \frac{x_n}{1-\alpha}$

Case 4:

$$\begin{aligned} D(S^2c, TSc) &\leq D(S^2c, S^3Ta_n) + \alpha D(S^4a_n, TS^3c) \\ &\leq D(S^2c, S^3Ta_n) + \alpha\big(D(S^4a_n, S^2c) + D(TSc, S^2c)\big) \\ &\leq \frac{x_n + \alpha y_n}{(1-\alpha)} \end{aligned}$$

Case 5:

$$\begin{aligned} D(S^2c, TSc) &\leq D(S^2c, S^3Ta_n) + \alpha D(S^2c, TS^3a_n) \\ &\leq x_n + \alpha x_n \\ &\leq (1+\alpha)x_n \end{aligned}$$

Therefore, $D(S^2c, TSc) = 0$ that is $S^2c = TSc$. TSc is a common fixed point for T and S. Put in the inequality $D(Ta, Tb) \leq \alpha u(a, b), a = TSc,\ b = Sc$ we get $T(TSc) = TSc$. Since $S^2c = TSc$, i.e. $S(Sc) = T(Sc)$, we have $S(TSc) = TS^2c = T(TSc) = TSc$.

4. Conclusion

In this work, we present some new common fixed-point theorems in Ƒ-metric spaces. Moreover, we give some examples that support our results. Our results extend and generalize the fixed-point theory. We hope that our research results offer a mathematical foundation. In future studies, we will explore the concrete applications of the obtained results.

Author Contributions

The author read and approved the last version of the manuscript.

Conflicts of Interest

The author declares no conflict of interest.

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