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Research Article

New Analytical Method For Solution of Second Order Ordinary Differential Equations With Variable Coefficients

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Abstract

In this study, we propose four methods for the analytical solution of second order ordinary non-homogeneous differential equations with variable coefficients. In the first case, the method directly gives the solution in explicit or integral form. In the second method, the solution of the problem reduces to the solutions of adjoined second order ordinary differential equation of homogeneous type. As long as the analytical solution of two adjoined equations can be solved, the analytical solution can always be found. In the third method, the differential equation is transformed into Riccati equation. Riccati equation is solved by means of a method recently developed. In order to solve non-homogeneous differential equation, the fourth method is developed. The strategy is different but the solution is again based on the solution of Riccati equation.

Keywords: Second order differential equations, analytical method, Riccati equation.

İkinci Mertebeden Değişken Katsayılı Diferansiyel Denklemlerin Analitik Çözümü Üzerine Yeni Bir Yöntem

Öz

Bu çalışmada değişken katsayılı ikinci mertebe non-homojen diferansiyel denklemlerin analitik çözümü için dört yöntem geliştirilmektedir. İlk yöntem çözümü doğrudan açık ya da kapalı formda vermektedir. İkinci yöntemde problemin çözümü homojen tipte ek ikinci mertebe denklemlerin çözümüne indirgenmektedir. Ek denklemler çözüldüğü sürece analitik çözüm daima bulunabilmektedir. Üçüncü yöntemde diferansiyel denklem Riccati denklemine dönüştürülmektedir. Riccati denkleminin çözümünde en son geliştirilen bir yöntem kullanılmaktadır. Dördüncü yöntem de strateji farklı olup, yöntem yine Riccati denklemin çözümüne indirgenmektedir.

Anahtar Kelimeler: İkinci mertebe diferensiyel denklem, analitik yöntem, Riccati denklemi.

1. Introduction

The second order non-homogeneous differential equations of general type are commonly encountered in engineering, physics, biology, astrophysics, etc. Therefore, their analytical solutions are extremely important for researchers in applied sciences. There have been many studies on the analytical solution of second order ordinary differential equations with variable

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coefficients [2 - 6, 10, 11, 13 – 16]. When the solution methods developed are carefully checked, it can be seen that most of the analytical methods can be classified into three groups. In the first group, the equation is transformed into Riccati equation which is a first order nonlinear ordinary differential equation. As long as the analytical solution of Riccati differential equation is known, this method can be valid and logical. However, in the most general case, Riccati equation cannot be solved. The methods available require one proper solution of the equation. However, for an equation of general type, the proper solution cannot be found or proposed very easily. Therefore, this approach mostly remains inconclusive [1, 8 - 10, 12]. Other methods are of limited usage since they involve restrictions on the functions involved. To that end, Pala and Ertaş have recently developed a general method for solving Riccati equations of general type [7]. However, in the second part of this work, Riccati equation is again transformed into a second order ordinary equation of the simplest form. This simplest form reveals whether Riccati equation leads to an analytical or an exact solution. In the second group, the differential equation given is transformed into a solvable form or a second order differential equation with constant coefficients. This approach is far away from being general since the method brings several limitations on the functions involved in the differential equations. In the third group, the method of operator factorization is used to solve the second order equations with variable coefficients. Although this method seems to be more general compared to the other conventional methods, this method also includes several limitations and depends on the availability of the solution of Riccati equations [10]. Consequently, new analytical methods for solving this type of equations are always welcome.

2. Material and Methods

2.1. First Method of Solution

As a first method, we start with a known relatively simple method. Let us consider the second order non-homogeneous differential equation with variable coefficients

$$y'' + P(x)y' + Q(x)y = R(x) \tag{1}$$

where $P(x)$, $Q(x)$ and $R(x)$ are arbitrary smooth functions of x. In order to solve Eq. (1), a linear transformation in the following form is introduced:

$$\bar{y} = y' + P(x)y \tag{2}$$

By differentiating Eq. (2), we have

$$\bar{y}' = y'' + Py' + P'y \tag{3}$$

The right hand side of Eq. (3) will be the same as the left hand side of Eq. (1) if we assume $P' = Q$. Thus, if we assume $\bar{y}' = R(x)$, then Eq. (3) becomes identical to Eq. (1). We take care that this already known method brings a restriction of the form $P' = Q$ on the analytical solution of Eq. (1). This restriction will be removed later. The solution procedure after this point is as follows. First, we solve the transformed equation $\bar{y}' = R(x)$. Second, we substitute the result obtained into Eq. (2) to obtain $y(x)$.

Example 1. Let us consider the equation

$$y'' - xy' - y = 1 \tag{4}$$

The condition $P' = Q$ is satisfied here. Thus, integrating the equation $\bar{y}' = 1$ yields $\bar{y} = x + c_1$, where c_1 is a constant. Inserting this result into Eq. (2), we have

$$y' - xy = x + c_1$$

This is a first order ordinary equation which can readily be solved. The solution has the form

$$y(x) = c_2 e^{\frac{x^2}{2}} - 1 + c_1 e^{\frac{x^2}{2}} \int e^{-\frac{x^2}{2}}\, dx \tag{5}$$

Here, c_2 is a constant. Eq. (5) is the general solution of Eq. (4). Recall here that the term $c_1 e^{\frac{x^2}{2}} \int e^{-\frac{x^2}{2}}\, dx$ and $c_2 e^{\frac{x^2}{2}}$ are two different solutions of the homogeneous part of Eq. (4).

Example 2. Let us consider the equation

$$y'' - (\cos x)y' + (\sin x)y = -\sin x \tag{6}$$

Solving the equation $\bar{y}' = -\sin x$ gives $\bar{y} = \cos x + c_1$, where c_1 is a constant. Eq. (2) reads

$$y' - (\cos x)y = \cos x + c_1 \tag{7}$$

The Solution of Eq. (7) is obtained in the form

$$y(x) = c_2 e^{\sin x} - 1 + c_1 e^{\sin x} \int e^{-\sin x}\, \mathrm{d}x$$

Recall that the term $c_2 e^{\sin x} - 1$ satisfies Eq. (6).

Example 3. In order to show the effectiveness of the method, let us now consider a simple second order equation with constant coefficient in the form

$$y'' - 5y' = e^x \tag{8}$$

The solution of $\bar{y}' = e^x$ yields $\bar{y} = e^x + c_1$. Inserting this result into Eq. (2) gives

$$y' - 5y = e^x + c_1 \tag{9}$$

The solution of Eq. (9) is obtained as

$$y(x) = c_2 e^{5x} - \frac{1}{4} e^x - \frac{c_1}{5} \tag{10}$$

Eq. (10) is the analytical solution of Eq. (8).

2.2. A More General Method of Solution

In the first section, the method of solution was valid provided that the condition $P' = Q$ is satisfied. It is obvious that this condition is not satisfied every time. Therefore, we need to develop a more general method for the solution of Eq. (1).

Now, we propose a new transformation of the form

$$\bar{y} = Ny' + My \tag{11}$$

where $N(x)$ and $M(x)$ are unknown functions to be (hopefully) determined. We differentiate Eq. (11):

$$\bar{y}' = N'y' + Ny'' + M'y + My'$$

or, arranging

$$y'' + \left(\frac{N'}{N} + \frac{M}{N}\right) y' + \frac{M'}{N} y = \frac{\bar{y}'}{N}$$

If we take

$$\frac{N'}{N} + \frac{M}{N} = P(x) \tag{12a}$$

$$\frac{M'}{N} = Q(x) \tag{12b}$$

$$\frac{\bar{y}'}{N} = R(x) \tag{12c}$$

we then obtain Eq. (1). Thus, if we can solve Eqs. (12) for $N(x)$ and $M(x)$ by any means, then we can obtain $y(x)$ from Eq. (11). The first two equations in Eqs. (12) can be combined in the following way:

$$N'' - PN' + (Q - P')N = 0 \tag{13}$$

or

$$M'' - \left(P + \frac{Q'}{Q}\right) M' + QM = 0 \tag{14}$$

If Eq. (13) or Eq. (14) is a Euler-Cauchy type equation, they can readily be solved. However, in the most general case, the analytical solutions of Eqs. (13) and (14) are still not known. Therefore, we consider various cases for which we can obtain the analytical solutions of Eqs. (13) and (14) to determine $N(x)$ or $M(x)$.

Although, Eqs. (13) and (14) cannot be solved in the most general case, they can give analytical solutions in some cases. In order to show this, we consider the equation proposed by Urdaletova and Kydyraliev [14].

$$y'' - 4xy' + (4x^2 - 2)y = 6xe^{x^2}$$

Comparing this equation with Eq. (1), we find

$$P = -4x\,,\ \ Q = 4x^2 - 2\,,\ \ R = 6xe^{x^2}$$

If we take $N = e^{-x^2}$, then Eq. (13) is satisfied. Using Eq. (12a), we have $M = -2xe^{-x^2}$. Inserting these functions into Eq. (12c) yields

$$\frac{\bar{y}'}{e^{-x^2}} = 6xe^{x^2}$$

from which we obtain

$$\bar{y} = 3x^2 + c_1$$

Putting now this result into Eq. (11), we have

$$e^{-x^2}y' - 2xe^{-x^2}y = 3x^2 + c_1 \tag{15}$$

The solution of Eq. (15) yields

$$y(x) = e^{x^2}(x^3 + c_1x + c_2)$$

This result is the same as that given by Urdaletova and Kydyraliev [14].

For the solution of Eq. (13) or Eq. (14), we can consider the following cases for which we obtain the analytical solution of the problem.

Case 1: Let us assume that $P = (ax + b)$, $Q = cx + d$ in Eq. (13). Here *a*, *b*, *c*, *d* are constants. In this case, Eq. (13) reads

$$N'' - (ax + b)N' + (cx + d - a)N = 0 \tag{16}$$

In addition, we assume $N = e^{sx}$ where *s* is a constant to be determined. Inserting this form into Eq. (16) and equating mutual terms, we have

$$-as + c = 0 \tag{17a}$$

$$s^2 - bs + d - a = 0 \tag{17b}$$

We have five unknowns. However, we can choose *a*, *b*, *c* and *s*. The constant *d* can be found from Eq. (17b). As an example, we choose $a = 1, c = 3,\ b = 1,\ s = 3$. Eq. (17b) gives $d = -5$. Thus, Eq. (13) takes the form

$$N'' - (x + 1)N' + (3x - 6)N = 0$$

whose solution is given by $N = e^{sx} = e^{3x}$. From Eq. (12) we have

$M' = NQ = (3x-5)e^{3x} \rightarrow$

$$M = (x-2)e^{3x} + c_1 \tag{18}$$

After we determine M and N in this way, for example, we can solve the equation

$y'' + (x+1)y' + (3x-5)y = 5$

Using Eq. (12c) gives

$\frac{\bar{y}'}{N} = R(x)$

$$\bar{y}' = 5e^{3x} \;\rightarrow\; \bar{y} = \frac{5}{3}e^{3x} + c_2 \tag{19}$$

Now, inserting Eq. (19) into Eq. (11), we have

$$e^{3x}y' + (x-2)e^{3x}y = \frac{5}{3}e^{3x} + c_2 \tag{20}$$

Here, without loss of generality, we have omitted c_1 in Eq. (18). We simplify Eq. (20) as

$$y' + (x-2)y = \frac{5}{3} + c_2e^{-3x} \tag{21}$$

The solution of Eq. (21) can readily be written as

$y = c_3e^{2x-\frac{x^2}{2}} + e^{2x-\frac{x^2}{2}} \int e^{\frac{x^2}{2}-2x}\left(\frac{5}{3} + c_2e^{-3x}\right)dx$

Case 2: We assume that $N = 1$. Then, Eq. (13) yields

$Q = P'$

This case is identical to that in Section 1.

Case 3: We assume that $N = e^{ax}$, where a is constant. Then, Eq. (13) yields

$$a^2 - Pa + (Q - P') = 0 \tag{22}$$

When P is arbitrarily chosen, then Q must be determined from Eq. (22).

Example 4: Let us take $a = 1,\ P = x^2$. Eq. (22) gives $Q = 2x + x^2 - 1$. This means that we can solve the equation

$$y'' + x^2y' + (2x + x^2 - 1)y = x \tag{23}$$

Taking $N = e^x$, we find from Eq. (12b) that

$M' = NQ = e^x(2x + x^2 - 1) \;\rightarrow\; M = e^x(x^2 - 1) + c_0$

Eq. (12c) gives

$\bar{y}' = NR = xe^x \;\rightarrow\; \bar{y} = e^x(x-1) + c_1$

Thus, we can write Eq. (11) as

$$\bar{y} = Ny' + My = y' + (x^2-1)y = (x-1) + \frac{C_1}{e^x} \quad (24)$$

The solution of Eq. (24) is in the form

$$y(x) = e^{\left(x-\frac{x^3}{3}\right)}\left(c_2 + \int e^{\left(\frac{x^3}{3}-x\right)}\left[(x-1) + c_1 e^{-x}\right]dx\right)$$

Case 4 : This time, we assume that $M = e^{ax}$ and use it in Eq. (14):

$$a^2 - a\left(P + \frac{Q'}{Q}\right) + Q = 0 \quad (25)$$

where a is an arbitrary constant. If P is given, Q can be determined from this equation.

Example 5 : Let us assume $a = 1, P = 0$. Then, Eq. (25) gives

$Q' - Q^2 = Q$

whose solution is given by

$Q = \frac{1}{C_1 e^{-x} - 1}$

Thus, taking

$M = e^x,\; a = 1,\; P = 0,\; N = \frac{M'}{Q} = c_1 - e^x$

we can solve the equation

$y'' + \left(\frac{e^x}{c_1 - e^x}\right)y = \mathrm{R}(x) = e^{-2x}$

Here, $R(x) = e^{-2x}$ is arbitrarily chosen. Eq. (12c) gives

$\frac{\bar{y}'}{N} = e^{-2x} \;\rightarrow\; \bar{y}' = e^{-2x}(c_1 - e^x)$

$\bar{y} = -\frac{c_1}{2}e^{-2x} + e^{-x} + c_2$

Here, without loss of generality, we can assume $c_2 = 0$. Using Eq. (11), we have

$\bar{y} = -\frac{C_1}{2}e^{-2x} + e^{-x} = (c_1 - e^x)y' + e^x y$

or

$$y' + \left(\frac{e^x}{c_1 - e^x}\right) y = \frac{-\frac{c_1}{2} e^{-2x} + e^{-x}}{c_1 - e^x}$$

The solution for $y(x)$ is

$$y = (c_1 - e^x)\left[c_3 + \frac{c_1^2 e^{-2x} + \frac{2e^x}{(c_1 - e^x)} - 2\ln(1 - c_1 e^{-x})}{4{c_1}^3}\right]$$

Case 5 : Let us consider the case when $P(x) = 0$ in Eq. (1). In this case, Eq. (1) reads

$$y'' + Q(x)y = R(x)$$

On the other hand, Eq. (13) takes the form

$$N'' + QN = 0 \quad (26)$$

There are many situations for which we can obtain a proper solution of Eq. (26). In this case, by obtaining N and M, we can solve the original equation given. For the present case, we have prepared Table 1 so as to solve some problems readily.

Example 6. We try to solve the following non-homogeneous differential equation

$$y'' - (2a + 4a^2x^2)y = x \quad (27)$$

where a is a constant. The homogeneous part of Eq. (27) is Weber equation in the form

$$y'' - (2a + 4a^2x^2)y = 0 \quad \text{or} \quad y'' - Qy = 0$$

where $Q = 2a + 4a^2x^2$. A proper solution to Eq. (27) is $s = e^{ax^2}$ (See, Table 1). We assume a solution $y = su$ for Eq. (27). Here, u is a function to be determined. Recalling that $y' = s'u + su'$, $y'' = s''u + 2u's' + su''$ and inserting these results into Eq. (27), after arrangement, we have

$$u'' + \frac{2s'}{s}u' + \left(\frac{s''}{s} - Q\right)u = \frac{x}{s} \quad (28)$$

However, we can readily show that the last term $(s''/s - Q)$ is zero. Then, Eq. (28) reads

$$u'' + 4axu' = xe^{-ax^2} \quad (29)$$

Putting $u' = v$, the solution of Eq. (29) can be obtained as

$$v = c_1 e^{-2ax^2} + \frac{1}{2a} e^{-ax^2}$$

Reminding $u' = v$ and integrating this equation, we have

$$u = c_1 \int e^{-2ax^2} \mathrm{d}x + \frac{1}{2a} \int e^{-ax^2} \mathrm{d}x \; + c_2$$

Thus, the general solution to Eq. (27) is given by

$$y = su = e^{ax^2} \left[c_1 \int e^{-2ax^2} \mathrm{d}x + \frac{1}{2a} \int e^{-ax^2} \mathrm{d}x + c_2 \right] \tag{30}$$

It can be verified that Eq. (30) is the solution of Eq. (27). The form given by Eq. (30) is perhaps the most suitable form to be used in initial or boundary value problems.

Using this approach, the general solutions of the non-homogeneous second order differential equations, whose proper solutions of their homogeneous parts are given in Table 1, can readily be obtained.

2.3. Third method of Solution

In the previous method, except for some special cases, Eq. (13) or Eq. (14) cannot be solved for arbitrary forms of $P(x)$ and $Q(x)$. Therefore, different new methods that can solve the equation for which the second method doesn't become successful. However, it is already known that a second order ordinary homogeneous differential equation with variable coefficients can be transformed into a Riccati equation. This method was not quite meaningful since the general solution to Riccati equation could not be found in most cases. However, when we follow the recently published paper by Pala and Ertas [7], this method can now be applied to the analytical solution of second order homogeneous equations. In this way, many equations that cannot be solved by means of the second method can be solved.

Under the transformation $y = e^{\int v dx}$, the second order equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{31}$$

is transformed into a non-linear Riccati equation of the form

$$v' + P(x)v + v^2 = -Q(x) \tag{32}$$

In [7], by utilizing the transformation $\bar{y} = fe^{\int gy dx}$, Riccati equation of the form

$$y' + P(x)y + Qy^2 = R(x) \tag{33}$$

is transformed into

$$y' + \left[\frac{2f'}{f} + \frac{g'}{g} \right] y + gy^2 + \frac{f''}{fg} = \frac{1}{fg} \bar{y}'' e^{-\int gy dx} \tag{34}$$

Here, f, g and $\bar{y}$ are functions to be determined. We take care that the left hand side of Eq. (34) has the form of Riccati equation. Comparing Eq. (33) and Eq. (34), we have

$$\left[\frac{2f'}{f}+\frac{g'}{g}\right]=P(x), \tag{35a}$$

$$g(x)=Q(x), \tag{35b}$$

$$s(x)-\frac{f''}{fg}=R(x) \tag{35c}$$

Here, $s(x)$ is a function to be determined later. If we wish to solve an equation of the form

$$y'+\left[\frac{2f'}{f}+\frac{g'}{g}\right]y+gy^2+\frac{f''}{fg}=0,$$

then, by Eqs. (35), in the first place, we can assume

$$\bar{y}''=0\rightarrow\bar{y}=ax+b$$

The functions f and g are to be found such that Eqs. (35) are satisfied. Using the inverse transform, we obtain

$$y=\frac{1}{g}\frac{d}{dx}\left[\ln\frac{\bar{y}}{f}\right] \tag{36}$$

as the solution of Eq. (33). As an illustration of the method, we consider the following example.

Example 7. We try to solve the equation

$$y''+5xy'+\left(\frac{25}{4}x^2+\frac{5}{2}\right)y=0 \tag{37}$$

The transformation $y=e^{\int v dx}$ reduces Eq. (37) into

$$v'+5xv+v^2+\left(\frac{25}{4}x^2+\frac{5}{2}\right)=0 \tag{38}$$

Eq. (38) is a Riccati equation that can solved by the new general method presented in (Pala and Ertas, 2017). Eqs. (35) give

$$\left[\frac{2f'}{f}+\frac{g'}{g}\right]=5x, \tag{39a}$$

$$g(x)=1, \tag{39b}$$

$$\frac{f''}{fg}=\left(\frac{25}{4}x^2+\frac{5}{2}\right) \tag{39c}$$

Simultaneous solution of Eqs. (39a) and (39b) gives $f=c_0e^{\frac{5}{4}x^2}$, where c_0 is a constant. Now, using Eq. (36), we obtain

$$v=\left(\frac{1}{x+c}-\frac{5}{2}x\right),\qquad c=\text{constant}$$

Remembering the transformation $y = e^{\int v dx}$,we finally obtain

$$y = c_2(x+c)e^{-\frac{5}{4}x^2}, \qquad c, c_2 \text{ are contants}$$

Example 8. We try to solve the equation

$$y'' + 8xy' + (16x^2 + 4)y = 0 \tag{40}$$

The transformation $y = e^{\int v dx}$ reduces Eq. (40) into

$$v' + 8xv + v^2 + (16x^2 + 4) = 0 \tag{41}$$

Comparing Eq. (41) and Eqs. (35), we have

$$\left[\frac{2f'}{f} + \frac{g'}{g}\right] = 8x, \tag{42a}$$

$$g(x) = 1, \tag{42b}$$

$$\frac{f''}{fg} = (16x^2 + 4) \tag{42c}$$

The first one of Eqs. (42) gives $f = c_1 e^{2x^2}$, where $c_1$is a constant. Eq. (42c) is also satisfied. Thus, Using Eq. (36), we have

$$v = \left(\frac{1}{x+c} - 4x\right), \qquad c = \text{constant}$$

Remembering the transformation $y = e^{\int v dx}$, we finally obtain

$$y = c_2(x+c)e^{-2x^2}, \qquad c, c_2 \text{ are contants} \tag{43}$$

Example 9. In the buckling theory of tapered beams, we encounter the second order equations. Let us consider a beam as in Fig. (1). The equilibrium equation for the beam is given by

$$x^2 \frac{d^2y}{dx^2} + \frac{Fa^2}{EI} y = 0, \qquad y(a) = y'(L+a) = 0$$

Here, E is the modulus of elasticity and I is the moment of inertia. Under the transformation $y = e^{\int v dx}$, Eq. (32) can be written as

$$v' + v^2 + \frac{k}{x^2} = 0, \qquad k = \frac{Fa^2}{EI} \tag{44}$$

The solution of (44) can be shown to be

$$v(x) = 1 - \frac{\sqrt{1-4k}\left(\frac{2c_1}{c_1 + x^{\sqrt{1-4k}}} - 1\right)}{2x}$$

Thus, we have for $y(x)$ that

$$y(x) = c_2\sqrt{x}\left(1 + \frac{c_1}{x^{\frac{\sqrt{1-4k}}{2}}}\right)$$

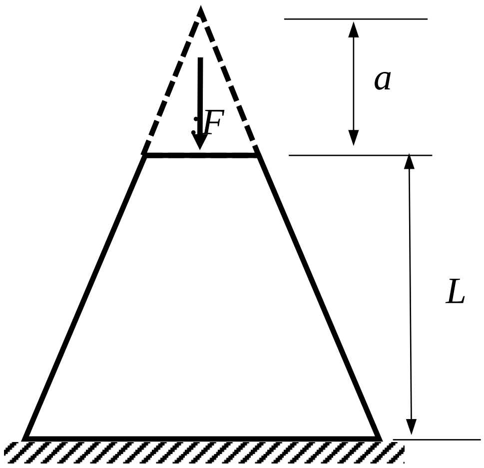

Figure 1 Buckling model of a tapered column.

Table 1. Proper solutions of some second order ordinary differential equations

Differential Equation	Proper Solution
$y'' - (2a + 4a^2x^2)y = 0$	$s = e^{ax^2}$, a is a constant
$y'' - (6a + 4a^2x^2)y = 0$	$s = xe^{ax^2}$
$y'' - (2x^{-2} + 10a + 4a^2x^2)y = 0$	$s = x^2e^{ax^2}$
$y'' - (12ax^2 + 16a^2x^6)y = 0$	$s = e^{ax^4}$
$y'' - (2a + (2ax + b)^2)y = 0$	$s = e^{ax^2+bx+c}$, a ,b,c are constants
$y'' - [n(n-1)x^{-2} + 2an + 2a(n+1) + 4a^2x^2]y = 0$	$s = x^ne^{ax^2}$, a ,n are constants
$y'' - \left[\frac{4adx}{dx+e} + 2a + 4a^2x^2\right]y = 0$	$s = (dx + e)e^{ax^2}$, a ,d,e are constants

Table 1. Continued

$y'' - \left[\frac{2(2ax+b)d}{dx+e} + 2a + (2ax+b)^2\right]y = 0$	$s = (dx+e)e^{ax^2+bx+c}$, *a,b,c,d,e* are constants
$y'' - \left[an(n-1)x^{n-2} + a^2n^2x^{2(n-1)}\right]y = 0$	$s = e^{ax^n}$
$y'' - 2axy' - 2ay = 0$	$s = e^{ax^2}$
$y'' - (2ax+b)y' - 2ay = 0$	$s = e^{ax^2+bx+c}$ or $s = e^{ax^2+bx}$
$y'' - 3ax^2y' - 6axy = 0$	$s = e^{ax^3}$
$y'' - 4ax^3y' - 12ax^2y = 0$	$s = e^{ax^4}$
$y'' - anx^{n-1}y' - an(n-1)x^{n-2}y = 0$	$s = e^{ax^n}$
$y'' - \left[\frac{dm(m-1)x^{m-2} + es(s-1)x^{s-2}}{dx^m + ex^s} + \frac{2dmx^{m-1} + esx^{s-1}}{dx^m + ex^s}(na)x^{n-1} + na(n-1)x^{n-2} + n^2a^2x^{2(n-1)}\right]y = 0$	$s = (dx^m + ex^s)e^{ax^n}$ *d,e,a,m,n* are constants
$y'' - ax^{n-2}(ax^n + n + 1)y = 0$	$s = xe^{ax^n/n}$
$y'' + ay' + b(-bx^2 + ax + 1)y = 0$	$s = e^{-\frac{b}{2}x^2}$
$y'' + (x^2 - b^2)y' - (x+b)y = 0$	$s = x - b$
$y'' + (ax^2 + bx)y' + c(ax^2 + bx - c)y = 0$	$s = e^{-cx}$
$y'' + (ax^n + bx^m)y' - (ax^{n-1} + x^{m-1})y = 0$	$s = x$
$y'' + (ax^2 + 2b)y' + (abx^2 - ax + b^2)y = 0$	$s = xe^{-bx}$

2.4. Alternative method of Solution

Now, as a new method, we propose the transformation $y = ae^{k\int b dx}$ where *k* is a constant while *a*(*x*) and *b*(*x*) are functions to be determined. Taking derivatives y', y'', and substituting into Eq. (1), one obtains

$$a''e^{k\int bdx} + (2kb + P)a'e^{k\int bdx} + (kab' + k^2ab^2 + Pkab + Qa)e^{k\int bdx} = R(x) \tag{45}$$

Now, we choose $b(x)$ such that

$$b' + Pb + kb^2 + \frac{Q}{k} = 0 \tag{46}$$

Eq. (46) is a Riccati equation. Under this assumption, Eq. (45) can be written in the form of

$$a'' + Aa' = \bar{R} \tag{47}$$

where $\bar{R} = Re^{-k\int bdx}$ and $A = 2kb + P$. Putting $a' = u$ in Eq. (47) one obtains

$$u' + Au = \bar{R} \tag{48}$$

It is already known that the solution of Eq. (48) is given by

$$u = e^{-\int Adx}\left[c_4 + \int \bar{R}(x)e^{\int Adx}dx\right]$$

Example 10. The non-homogeneous equation of the form

$$y'' + 5xy' + \left(\frac{25}{4}x^2 + \frac{5}{2}\right)y = R(x) \tag{49}$$

Here, $R(x)$ is an arbitrary function. After taking $k = 1$, Eq. (46) takes the form of

$$b' + 5xb + b^2 + \left(\frac{25}{4}x^2 + \frac{5}{2}\right) = 0$$

The solution of the equation above is

$$b = \frac{1}{x+c} - \frac{5}{2}x$$

Inserting $b(x)$ into Eq. (48) yields

$$u' + \frac{2}{x+c}u = e^{\left[-\ln(x+c) + \frac{5}{4}x^2\right]}R(x)$$

As an example, let us assume that $R(x) = e^{-5x^2/4}$. In this case, u is obtained as

$$u = \frac{c_2}{(x+c)^2} + \frac{1}{2}$$

Remembering that $a' = u$ and $y = ae^{\int bdx}$, we find

$$a = -\frac{c_2}{x+c} + \frac{x}{2} + c_3$$

and

$$y = \left[-c_2 + \frac{x(x+c)}{2} + c_3(x+c)\right]\left(e^{-\frac{5}{4}x^2}\right) \tag{50}$$

Eq. (50) can be written as follow

$$y = c_2\left[-1 + \frac{x\left(\frac{x}{c_2}+\frac{c}{c_2}\right)}{2} + \frac{c_3}{c_2}(x+c)\right]\left(e^{-\frac{5}{4}x^2}\right)$$

Since Eq. (49) is linear, it can be taken $c_2 = 1$. Thus, the solution of Eq. (49) is obtained as

$$y = \left[-1 + \frac{x(x+c)}{2} + c_3(x+c)\right]\left(e^{-\frac{5}{4}x^2}\right)$$

Example 11. Let us consider the equation

$$y'' + 8xy' + (16x^2 - 12)y = R(x)$$

Here, $R(x)$ is a function that will be chosen arbitrarily later. Taking $k = 4$ and using the transformation equation $y = ae^{4\int b dx}$, one finds

$$b' + 8xb + 4b^2 + 4x^2 - 3 = 0 \tag{51}$$

and

$$a'' + Aa' = \overline{R}$$

where $\overline{R} = Re^{-4\int b dx}$ and $A = 2kb + P(x) = 8b + 8x$. To solve Eq. (51), the method proposed by Pala and Ertas [7] is used. Eqs. (35) are applied to this example as follows

$$g = 4 \tag{52a}$$

$$\left[\frac{2f'}{f} + \frac{g'}{g}\right] = 8x \tag{52b}$$

$$s(x) = \frac{f''}{fg} - (4x^2 - 3) \tag{52c}$$

Eq. (52a) and Eq. (52b) give $f(x) = ce^{2x^2}$. Substituting $f(x)$ into Eq. (52c) gives $s(x) = 4$. Then, the transformation equation $\overline{y}'' - gs(x)\overline{y} = 0$ yields

$$\overline{y}'' - 16\overline{y} = 0$$

whose solution is given by

$$\overline{y} = c_1e^{4x} + c_2e^{-4x}$$

Now, using Eq. (36), we obtain

$$b(x) = \frac{1}{g}\frac{d}{dx}\left(\ln\frac{\overline{y}}{f}\right) = \frac{c_1e^{4x} - c_2e^{-4x}}{c_1e^{4x} + c_2e^{-4x}} - x$$

or

$$b(x) = \frac{e^{8x} - \overline{c}}{e^{8x} + \overline{c}} - x$$

where $\overline{c} = c_2/c_1$. Putting $a' = u$ yields

$$u' + Au = \overline{R} \tag{53}$$

The solution of Eq. (53) is expressed as

$$u = e^{-\int A dx}\left[c_4 + \int e^{\int A dx}\overline{R}\, dx\right]$$

Remembering $A = 8b + 8x$ and putting into the equation above, we finally obtain

$$a' = u = \frac{e^{8x}}{(\overline{c} + e^{8x})^2}\left[c_4 + \int (\overline{c} + e^{8x})^2 e^{-8x}\overline{R}(x)dx\right]$$

$$a = \int \frac{e^{8x}}{(\overline{c} + e^{8x})^2}\left[c_4 + \int (\overline{c} + e^{8x})^2 e^{-8x}\overline{R}(x)dx\right]dx + c_5$$

and

$$y = \left(\int \frac{e^{8x}}{(\overline{c} + e^{8x})^2}\left[c_4 + \int (\overline{c} + e^{8x})^2 e^{-8x}\overline{R}(x)dx\right] dx\right)(\overline{c} + e^{8x})e^{-2x(x+2)}$$

3. Conclusion

In this study, we have proposed new approaches for the analytical solution of second order ordinary differential equations of general type. Four methods have been given. The first one is limited in that it requires $P' = Q$ while the second method is general. However, the second method requires the solution of adjoined equations (13) or (14) of second order homogeneous type. Different cases have been considered for the solution of Eqs. (13) and (14). In order that the analytical solution of either Eq. (13) or (14) be obtained, we have included a table in which proper solutions of some important equations are given. The method of transforming into Riccati equation has also been studied and a different method has been proposed based on a new method developed in [7]. According to this method, the transformed equation takes the simplest form whatever the original equation is. The transformed equation reveals whether the problem is solved in terms of standard or special mathematical functions. In the fourth method, the approach is different, but the solution is also based on the solution of Riccati equation. Therefore, the last two methods utilize the recently developed method for Riccati equation.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

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