CRAMÉR-RAO BOUNDS FOR A SINGLE REAL SINUSOID: THE KNOWN FREQUENCY CASE

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Abstract: The Cramér-Rao bounds for estimating the amplitude and phase of a real sinusoid of known frequency in white Gaussian noise are examined as the phase is varied, and the largest and smallest bounds and the corresponding critical phases are obtained.

Key Words: Cramér-Rao bound, amplitude estimation, phase estimation, real sinusoid, low-frequency sinusoid.

Bir Reel Sinüse İlişkin Cramér-Rao Sınırları: Bilinen Frekans Durumu

Özet: Beyaz Gauss gürültü içindeki frekansı bilinen bir reel sinüsün genlik ve faz kestirimine ilişkin Cramér-Rao sınırları faz değişirken incelenmiş, ve en büyük ve en küçük sınırlar ile karşılık gelen kritik faz değerleri elde edilmiştir.

Anahtar Kelimeler: Cramér-Rao sınırı, genlik kestirimi, faz kestirimi, reel sinüs, alçak frekanslı sinüs.

1. INTRODUCTION

The Cramér-Rao (CR) bounds for estimating the parameters (the amplitude, phase and frequency) of a real sinusoid in white Gaussian noise have been considered by the author in Dilaveroğlu (1998) and Dilaveroğlu (1999) for the general case in which all the parameters of the sinusoid are assumed to be unknown. However, in some applications in signal processing and communications, the frequency of the sinusoid is known, and only the amplitude and phase of the sinusoid need to be estimated (see, e.g., the recent paper (So, 2005)). In this paper, the CR bounds are examined for the known frequency case.

2. DATA MODEL

The data model is given by

$$y(t) = \alpha \cos(\omega t + \varphi) + e(t),\ t = n, \cdots, n + N - 1, \tag{1}$$

where α is the amplitude, $\omega = 2\pi f / f_s$, f is the frequency, f_s is the sampling frequency, φ is the phase of the sinusoid, $e(t)$ is a zero-mean white Gaussian noise of variance σ^2, n is the first value of the sampling index t, and N is the total number of data samples.

We assume in (1) that the "normalized frequency" ω is known and either (i) both the amplitude α and the phase φ are unknown, or (ii) α is unknown while φ is known, or (iii) α is known while φ is unknown. The unknown parameters in each case are to be estimated from the N data samples $y(n), \cdots, y(n + N - 1)$.

3. CR BOUNDS

3.1. Unknown Amplitude and Unknown Phase

The logarithm of the probability density of $y(n), \cdots, y(n + N - 1)$ is given by

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$$\ln p\big(y(n),\cdots,y(n+N-1)\big) = -\frac{N}{2}\ln\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{t=n}^{n+N-1}\left[y(t)-\alpha\cos(\omega t+\varphi)\right]^2. \tag{2}$$

The Fisher information matrix when both the amplitude α and the phase φ are unknown, and the noise variance σ^2 is known, is given by

$$J = \begin{bmatrix} -E\left\{\dfrac{\partial^2 \ln p}{\partial\alpha^2}\right\} & -E\left\{\dfrac{\partial^2 \ln p}{\partial\alpha\partial\varphi}\right\} \\ -E\left\{\dfrac{\partial^2 \ln p}{\partial\varphi\partial\alpha}\right\} & -E\left\{\dfrac{\partial^2 \ln p}{\partial\varphi^2}\right\} \end{bmatrix}, \tag{3}$$

where $E\{\cdot\}$ denotes the expectation.

According to the CR theorem (see, e.g., Kay (1993), Ch. 3), the variance of an unbiased estimator $\hat{\alpha}$ of α and the variance of an unbiased estimator $\hat{\varphi}$ of φ are bounded below as

$$\mathrm{var}\{\hat{\alpha}\} = E\left\{(\hat{\alpha}-\alpha)^2\right\} \geq J^{1,1}, \text{ and}$$

$$\mathrm{var}\{\hat{\varphi}\} = E\left\{(\hat{\varphi}-\varphi)^2\right\} \geq J^{2,2}, \tag{4}$$

where $J^{i,i}$, $i=1,2$, denotes the (i,i)-element of J^{-1}, the inverse of the J in (3).

From (2)-(4), we get

$$J^{1,1} = \frac{2\sigma^2}{N}\frac{1-A\cos(2\varphi)+B\sin(2\varphi)}{C}, \tag{5}$$

$$J^{2,2} = \frac{2\sigma^2}{N\alpha^2}\frac{1+A\cos(2\varphi)-B\sin(2\varphi)}{C}, \tag{6}$$

where

$$A = \frac{\sin(N\omega)}{N\sin\omega}\cos[(N+2n-1)\omega],$$

$$B = \frac{\sin(N\omega)}{N\sin\omega}\sin[(N+2n-1)\omega],$$

$$C = 1-\frac{\sin^2(N\omega)}{N^2\sin^2\omega}. \tag{7}$$

The results (5) and (6) give the CR amplitude and phase bounds as simple functions of the phase φ of the sinusoid. We shall examine the bounds as the phase φ varies, and determine the largest and smallest values of the bounds and the corresponding critical values of the phase. Since the bounds are periodic in φ with a period of π it suffices to consider the bounds in the interval

$$\Phi \equiv \left\{\varphi : \varphi \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right]\right\}.$$

It can be shown (see Dilaveroğlu (1998)) that the CR amplitude bound $J^{1,1}$ has one maximum point and one minimum point in the interval Φ given by

$$\arg\max_{\varphi\in\Phi} J^{1,1} = \begin{cases} -\dfrac{[(N+2n-1)\omega]_\pi}{2} - \operatorname{sgn}\left(-[(N+2n-1)\omega]_\pi\right)\dfrac{\pi}{2}, & A \geq 0, \\ -\dfrac{[(N+2n-1)\omega]_\pi}{2}, & A < 0, \end{cases} \tag{8}$$

$$\arg\min_{\varphi\in\Phi} J^{1,1} = \begin{cases} -\dfrac{[(N+2n-1)\omega]_\pi}{2}, & A \geq 0, \\ -\dfrac{[(N+2n-1)\omega]_\pi}{2} - \operatorname{sgn}\left(-[(N+2n-1)\omega]_\pi\right)\dfrac{\pi}{2}, & A < 0, \end{cases} \tag{9}$$

where $[x]_\pi \equiv \tan^{-1}(\tan x)$ in which $\tan^{-1}(\cdot) \in [-\pi/2, \pi/2]$, and $\operatorname{sgn}(x) = 1$ if $x > 0$, and $\operatorname{sgn}(x) = -1$ if $x \leq 0$. Also, the maximum and minimum values of $J^{1,1}$ are given by

$$\max_{\varphi} J^{1,1} = \frac{2\sigma^2}{N} \frac{1}{1 - \dfrac{|\sin(N\omega)|}{N\sin\omega}}, \tag{10}$$

$$\min_{\varphi} J^{1,1} = \frac{2\sigma^2}{N} \frac{1}{1 + \dfrac{|\sin(N\omega)|}{N\sin\omega}}. \tag{11}$$

Note that *the maximum and minimum values of the CR amplitude bound* $J^{1,1}$ *are independent of the first sampling index n.*

The worst and best phase expressions (8) and (9) can be simplified greatly by considering the symmetric sampling case where N is an odd number and the first sampling index $n = -(N-1)/2$. In this case, (8) and (9) become

$$\arg\max_{\varphi\in\Phi} J^{1,1} = \begin{cases} \dfrac{\pi}{2}, & A > 0, \\ 0, & A < 0, \end{cases} \tag{12}$$

$$\arg\min_{\varphi\in\Phi} J^{1,1} = \begin{cases} 0, & A > 0, \\ \dfrac{\pi}{2}, & A < 0. \end{cases} \tag{13}$$

Similar calculations show that the maximum and minimum values of the CR phase bound $J^{2,2}$ and the corresponding critical values of the phase can be expressed in terms of those of the CR amplitude bound $J^{1,1}$ as follows:

$$\max_{\varphi} J^{2,2} = \frac{\max\limits_{\varphi} J^{1,1}}{\alpha^2}, \tag{14}$$

$$\min_{\varphi} J^{2,2} = \frac{\min\limits_{\varphi} J^{1,1}}{\alpha^2}, \tag{15}$$

$$\arg\max_{\varphi\in\Phi} \mathrm{J}^{2,2} = \arg\min_{\varphi\in\Phi} \mathrm{J}^{1,1}. \tag{16}$$

$$\arg\min_{\varphi\in\Phi} J^{2,2} = \arg\max_{\varphi\in\Phi} J^{1,1}. \tag{17}$$

Note that, as the phase φ varies, *the CR amplitude bound $J^{1,1}$ takes its largest value when the CR phase bound $J^{2,2}$ takes its smallest value and vice versa.*

3.2. Unknown Amplitude and Known Phase

If, in addition to the frequency, the phase is also known, then the CR bound for estimating the amplitude is simply given by the inverse of the (1,1)-element of the J in (3), which is

$$J_{1,1}^{-1} = \frac{2\sigma^2}{N}\frac{1}{1 + A\cos(2\varphi) - B\sin(2\varphi)}. \tag{18}$$

The maximum and minimum values of the CR amplitude bound $J_{1,1}^{-1}$ and the corresponding critical values of the phase φ are given by

$$\max_{\varphi} J_{1,1}^{-1} = \max_{\varphi} J^{1,1}, \tag{19}$$

$$\min_{\varphi} J_{1,1}^{-1} = \min_{\varphi} J^{1,1}, \tag{20}$$

$$\arg\max_{\varphi\in\Phi} J_{1,1}^{-1} = \arg\max_{\varphi\in\Phi} J^{1,1}, \tag{21}$$

$$\arg\min_{\varphi\in\Phi} J_{1,1}^{-1} = \arg\min_{\varphi\in\Phi} J^{1,1}. \tag{22}$$

Thus, *the largest and smallest values of the CR amplitude bound and the critical values of the phase for amplitude estimation are the same whether the phase is known or not*. In general, however, the bound in the known phase case is less than the bound in the unknown phase case.

3.3. Known Amplitude and Unknown Phase

If the phase is the only unknown parameter in the model (1), then the CR bound for estimating the phase equals the inverse of the (2,2)-element of the J in (3), which is given by

$$J_{2,2}^{-1} = \frac{2\sigma^2}{N\alpha^2}\frac{1}{1 - A\cos(2\varphi) + B\sin(2\varphi)}. \tag{23}$$

The maximum and minimum values of the CR phase bound $J_{2,2}^{-1}$ and the corresponding critical values of the phase can be expressed in terms of those of the CR phase bound $J^{2,2}$ as follows:

$$\max_{\varphi} J_{2,2}^{-1} = \max_{\varphi} J^{2,2}, \tag{24}$$

$$\min_{\varphi} J_{2,2}^{-1} = \min_{\varphi} J^{2,2}, \tag{25}$$

$$\arg\max_{\varphi\in\Phi} J_{2,2}^{-1} = \arg\max_{\varphi\in\Phi} J^{2,2}, \tag{26}$$

$$\arg\min_{\varphi\in\Phi} J_{2,2}^{-1} = \arg\min_{\varphi\in\Phi} J^{2,2}. \tag{27}$$

Thus, *the largest and smallest values of the CR phase bound and the critical values of the phase for phase estimation are the same whether the amplitude is known or not.*

4. DISCUSSION

If $\pi/N < \omega < (N-1)\pi/N$ in (7), then $A \cong 0$, $B \cong 0$, and $C \cong 1$. In this case, the dependence of the CR bounds upon the phase of the sinusoid may be neglected. On the other hand, if $0 < \omega < \pi/N$ or $(N-1)\pi/N < \omega < \pi$ (i.e., the "low-frequency" case), then the dependence of the bounds upon the phase becomes important. For the low-frequency case, the largest and smallest values of the bounds and the corresponding critical values of the phase derived in this paper can be used to represent the worst and best case scenarios in testing the performance of unbiased amplitude and phase estimators.

5. REFERENCES

1. Dilaveroğlu, E. (1998) Nonmatrix Cramér-Rao Bound Expressions for High-Resolution Frequency Estimators, *IEEE Transactions on Signal Processing*, 46, (2), 463-474.
2. Dilaveroğlu, E. (1999) Simple Expressions for Worst and Best Case Cramér-Rao Bounds for Amplitude and Phase Estimation of Low-Frequency Sinusoid, *Electronics Letters*, 35, (3), 206-208.
3. Kay, S.M. (1993) *Fundamentals of Statistical Signal Processing: Estimation Theory*, Prentice-Hall, New Jersey.
4. So, H.-C. (2005) Implementation of Exact Maximum Likelihood Phase Estimators for a Single Real Sinusoid in White Gaussian Noise, *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, E88-A, (7), 2019-2022.