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AN APPLICATION OF CLIFFORD ALGEBRA ON THE SEMI-SYMMETRICAL ARCHIMEDEAN SOLID TRUNCATED ICOSAHEDRON

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ABSTRACT

Mathematical systems are needed to explain and prove the concepts in physics. This could be in algebra. Various algebras have been introduced and discussed by physicists and mathematicians throughout history. In this work, one of them, Clifford algebra, or in other words geometric algebra, is discussed and its application in physics is examined. Today, Clifford algebra provides convenience in many fields. It is frequently used in application areas such as robotics, quantum mechanics, and crystallography. After presenting Clifford algebra and quaternions, symmetry operations with Clifford algebra and quaternions are defined in molecular physics. Here, these symmetry operations are also applied to the fourth Archimedean solid, the truncated icosahedron. Also, the vertices of this solid presented in Cartesian coordinates are calculated.

Keywords: Clifford Algebra, Platonic Solids, Semi-symmetric molecules, Archimedean solids, Truncated icosahedron

1. INTRODUCTION

Geometric algebra utilizing geometric properties and symmetries is an optimal language for physics. In mathematical physics known as the definition of bodies and rearranging for equations of mathematics and physics, geometric algebra generates new areas of view. In classical physics, some of the algebra's power occurs from its natural spinorial formulation of rotations and Lorentz transformations. This formulation causes significant quantum-like tools to classical physics and helps break down the classical/quantum interface. It also combines Newtonian mechanics, relativity, quantum theory, and other areas of physics in a single formalism and language [1]. The quaternion algebra defined by Sir W.R. Hamilton was generalized for three-dimensional complex numbers [2]. The quaternion algebra is the Clifford algebra of the two-dimensional anti-Euclidean space. Quaternions in the three-dimensional spaces have more useful appearances for the subalgebras of Clifford algebra. Grassmann carried on the studies for multi-dimensional bodies and defined the central product, which includes both interior and exterior products. The Clifford product of vectors is Grassmann's central product. Clifford tested to compound the Grassmann's algebra and quaternions called " Application of Grassmann's Extensive Algebra" in a mathematical system [3]. Clifford algebra has a heavy hand in the explorations of the symmetry properties of systems, crystallography, molecular and solid-state physics. Spin groups and conformal groups are argued in lower dimensions. Then Clifford algebras are identified over arbitrary fields for arbitrary quadratic forms as well as for nonsymmetric linear forms [4].

2. QUATERNIONS AND SYMMETRY OPERATIONS IN $\mathbb{R}^3$

A quaternion described symbolically by **Q** is identified through the following equations

$$\mathbf{Q}=q1+Q_x\mathbf{i}+Q_y\mathbf{j}+Q_z\mathbf{k} \tag{1}$$

Or

$$\mathbf{Q}=\left[a,\vec{\mathbf{Q}}\right],\ \vec{\mathbf{Q}}=\left(Q_x,Q_y,Q_z\right) \tag{2}$$

where all q, Q_x, Q_y, Q_z coefficients are the real numbers. The unitary quaternions **i**, **j**, **k** satisfy the multiplication rules as follows:

$$\mathbf{i}\,\mathbf{j}=\mathbf{k},\ \mathbf{j}\,\mathbf{i}=-\mathbf{k},\ \mathbf{j}\,\mathbf{k}=\mathbf{i},\ \mathbf{k}\,\mathbf{j}=-\mathbf{i},\ \mathbf{i}\,\mathbf{k}=-\mathbf{j},\ \mathbf{k}\,\mathbf{i}=\mathbf{j}. \tag{3}$$

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Also i, j and k can be written as e_1, e_2 and e_3, correlatively. The vector quaternion **Q** with components [0, Q_x, Q_y, Q_z] and a vector $\vec{Q}$ of the Euclidean tridimensional space with components (Q_x, Q_y, Q_z) are reciprocally associated [5].

If **Q** and **W** quaternions are

$$\mathbf{Q} = q1 + Q_x\mathbf{e_1} + Q_y\mathbf{e_2} + Q_z\mathbf{e_3} = [q, \vec{Q}] \tag{4}$$

and

$$\mathbf{W} = w1 + W_x\mathbf{e_1} + W_y\mathbf{e_2} + W_z\mathbf{e_3} = [w, \vec{\mathbf{W}}] \tag{5}$$

the product of two quaternions, namely **Q** and **W**, is given by

$$\mathbf{QW} = [qw - \vec{Q}.\vec{\mathbf{W}}, q\vec{\mathbf{W}} + w\vec{Q} + \vec{Q}.\vec{\mathbf{W}}], \tag{6}$$

$$\mathbf{QW} = (qw - Q_xW_x - Q_yW_y - Q_zW_z) + e_1(Q_xw + qW_x + Q_yW_z - Q_zW_y) + \mathbf{e}_2(qW_y + Q_yw - Q_xW_z + Q_zW_x) + \mathbf{e}_3(qW_z + Q_zw + Q_xW_y - Q_yW_x), \tag{7}$$

where the result is a quaternion. It must be noted that the product of quaternions is not commutative, but associative.

For each quaternion **Q**, its conjugate is

$$\mathbf{Q}* = q1 - Q_x\mathbf{e}_1 - Q_y\mathbf{e}_2 - Q_z\mathbf{e}_3. \tag{8}$$

In physics, Clifford algebra is utilized for symmetry studies. There are three basic units $\mathbf{e}_i$ (i=1, 2, 3) in Clifford algebra such that [6]

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 2\delta_{ij} \tag{9}$$

which are equivalent to

$$\mathbf{e}_i\mathbf{e}_j = 1, \tag{10}$$

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i. \tag{11}$$

It is possible to describe Clifford algebra in Euclidean space. 1, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_{12}$, $\mathbf{e}_{13}$, $\mathbf{e}_{23}$, $\mathbf{e}_{123}$ form the basis of the Clifford algebra Cl_3 of the vector plane $\mathbb{R}^3$ [7, 8,9]. The Clifford algebra Cl_3 is the four-dimensional linear space and its basic elements have the multiplication table as follows:

Table 1. The basis of Clifford Algebra

The basis of Clifford Algebra	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{13}$	$\mathbf{e}_{23}$	$\mathbf{e}_{123}$
$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$\mathbf{e}_{13}$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{123}$	$\mathbf{e}_{23}$
$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$-\mathbf{e}_1$	$-\mathbf{e}_{123}$	$\mathbf{e}_3$	$-\mathbf{e}_{13}$
$\mathbf{e}_3$	$-\mathbf{e}_{13}$	$-\mathbf{e}_{123}$	1	$\mathbf{e}_{123}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_{123}$	-1	$-\mathbf{e}_{23}$	$\mathbf{e}_{13}$	$\mathbf{e}_3$
$\mathbf{e}_{13}$	$-\mathbf{e}_3$	$-\mathbf{e}_{123}$	$\mathbf{e}_1$	$\mathbf{e}_{23}$	-1	$-\mathbf{e}_{12}$	$\mathbf{e}_2$
$\mathbf{e}_{23}$	$\mathbf{e}_{123}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_{13}$	$\mathbf{e}_{12}$	-1	$-\mathbf{e}_1$
$\mathbf{e}_{123}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{13}$	$\mathbf{e}_{12}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	-1

3. AN APPLICATION OF CLIFFORD ALGEBRA ON THE SEMI-SYMMETRICAL ARCHIMEDEAN SOLID TRUNCATED ICOSAHEDRON

In solid-state physics and molecular physics, rotation operations play an important role. The symmetry operations can be easily applied to the *Platonic Solids* and *Archimedean Solids.* The Platonic solids are tetrahedron, cube, octahedron, icosahedron, and dodecahedron. Some important numbers for the Platonic solids are shown in Table 2. The Archimedean solids are truncated tetrahedron, cuboctahedron, truncated cube, truncated octahedron, rhombicuboctahedron (or small rhombicuboctahedron), truncated cuboctahedron (or great rhombicuboctahedron), snub cube (or snub hexahedron), icosidodecahedron, truncated dodecahedron, truncated icosahedron, rhombicosidodecahedron (or small rhombicosidodecahedron), truncated icosidodecahedron (or great rhombicosidodecahedron), the snub dodecahedron (or snub icosidodecahedron). Some important numbers for the Archimedean solids are shown in Table 3 [10].

Table 2. Numbers for the five Platonic solids

	Number of faces	*Number of edges*	*Number of vertices*	*Edges per face*
Tetrahedron	4	6	4	3
Cube	6	12	8	4
Octahedron	8	12	6	3
Icosahedron	20	30	12	3
Dodecahedron	12	30	20	5

Table 3. Numbers for the thirteen Archimedean solids

	Number of faces	*Number of edges*	*Number of vertices*
Truncated Tetrahedron	8 (4 triangles,4 hexagons)	18	12
Cuboctahedron	14 (8 triangles,6 squares)	24	12
Truncated Cube	14 (8 triangles,6 octagons)	36	24
Truncated Octahedron	14 6 squares,8 hexagons)	36	24
Rhombicuboctahedron	26 (8 triangles,18 squares)	48	24
Truncated Cuboctahedron	26 (12 squares,8 hexagons,6 octagons)	72	48
Snub Cube	38 (32 triangles,6 squares)	60	24
Icosidodecahedron	32 (20 triangles,12 pentagons)	60	30
Truncated Dodecahedron	32 (20 triangles,12 decagons)	90	60
Truncated Icosahedron	32 (12 pentagons,20 hexagons)	90	60
Rhombicosidodecahedron	62 (20 triangles,30 squares,12 pentagons)	120	60
Truncated Icosidodecahedron	62 (30 squares,20 hexagons,12 decagons)	180	120
Snub Dodecahedron	92 (80 triangles,12 pentagons)	150	60

In this article, the fourth of the Archimedean solids, the truncated icosahedron solid, is studied. The Clifford algebra representation of the rotation operation of this solid is shown.

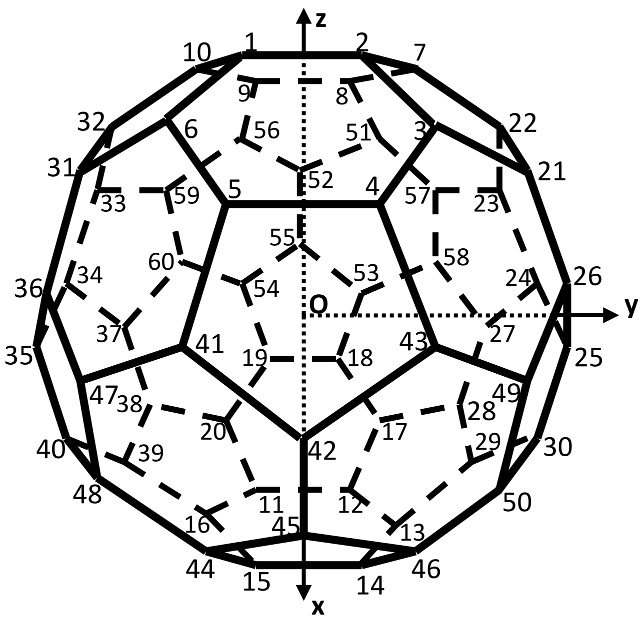

Figure 1. Truncated icosahedron

The coordinates of the numbered vertices of a normalized icosahedron, as seen in Figure 1, according to the selected coordinate system are given in Table 4.

Tablo 4. The vertices of a Truncated icosahedron

Corner	x	y	z	Corner	x	y	z
1	0	−0,5a	1	**31**	0,5a	−1,81a	1,62a
2	0	0,5a	1	**32**	−0,5a	−1,81a	1,62a
3	0,81a	a	2,12a	**33**	−a	− 2,12 a	0,81a
4	1,62a	0,5a	1,81a	**34**	−0,5a	−1	0
5	1,62a	−0,5a	1,81a	**35**	0,5a	− 1	0
6	0,81a	−a	2,12a	**36**	a	−2,12a	0,81a
7	−0,81a	a	2,12a	**37**	−a	−2,12a	−0,81a
8	−1,62a	0,5a	1,81a	**38**	−0,5a	−1,81a	−1,62a
9	−1,62a	−0,5a	1,81a	**39**	0,5a	−1,81a	−1,62a
10	−0,81a	−a	2,12a	**40**	a	−2,12a	−0,81a
11	0	−0,5a	−1	**41**	2,12a	−0,81a	a
12	0	0,5a	−1	**42**	1	0	0,5a
13	0,81a	a	−2,12a	**43**	2,12a	0,81a	a
14	1,62a	0,5a	−1,81a	**44**	2,12a	−0,81a	−a
15	1,62a	−0,5a	−1,81a	**45**	1	0	−0,5a
16	0,81a	−a	−2,12a	**46**	2,12a	0,81a	−a
17	−0,81a	a	− 2,12a	**47**	1,81a	−1,62a	0,5a
18	−1,62a	0,5a	−1,81a	**48**	1,81a	−1,62a	−0,5a
19	−1,62a	−0,5a	−1,81a	**49**	1,81a	1,62a	0,5a
20	−0,81a	−a	−2,12a	**50**	1,81a	1,62a	−0,5a
21	0,5a	1,81a	1,62a	**51**	−2,12a	0,81a	a
22	−0,5a	1,81a	1,62a	**52**	−1	0	0,5a
23	−a	2,12 a	0,81a	**53**	−2,12a	−0,81a	a
24	−0,5a	1	0	**54**	−2,12a	0,81a	−a
25	0,5a	1	0	**55**	−1	0	−0,5a
26	a	2,12a	0,81a	**56**	−2,12a	−0,81a	−a
27	−a	2,12a	−0,81a	**57**	−1,81a	1,62a	0,5a
28	−0,5a	1,81a	−1,62a	**58**	−1,81a	1,62a	−0,5a
29	0,5a	1,81a	−1,62a	**59**	−1,81a	−1,62a	0,5a
30	a	2,12a	−0,81a	**60**	−1,81a	−1,62a	−0,5a

As seen in figure 2, the length a of one side of the normalized truncated icosahedron is

$$(0{,}62 + 1{,}31 + 0{,}5)\mathrm{a} = 1, \quad \mathrm{a} \cong 0{,}41 \text{ unit.} \tag{12}$$

The surface area of the truncated icosahedron is found as

$$\mathrm{S} = 3\left(10\sqrt{3} + \sqrt{5}\sqrt{5 + 2\sqrt{5}}\right)\mathrm{a}^2 \cong 12{,}205 \text{ unit}^2 \tag{13}$$

The volume of the truncated icosahedron is found as

$$\mathrm{V} = \frac{1}{4}\left(125 + 43\sqrt{5}\right)\mathrm{a}^3 \cong 3{,}81 \text{ unit}^3 \tag{14}$$

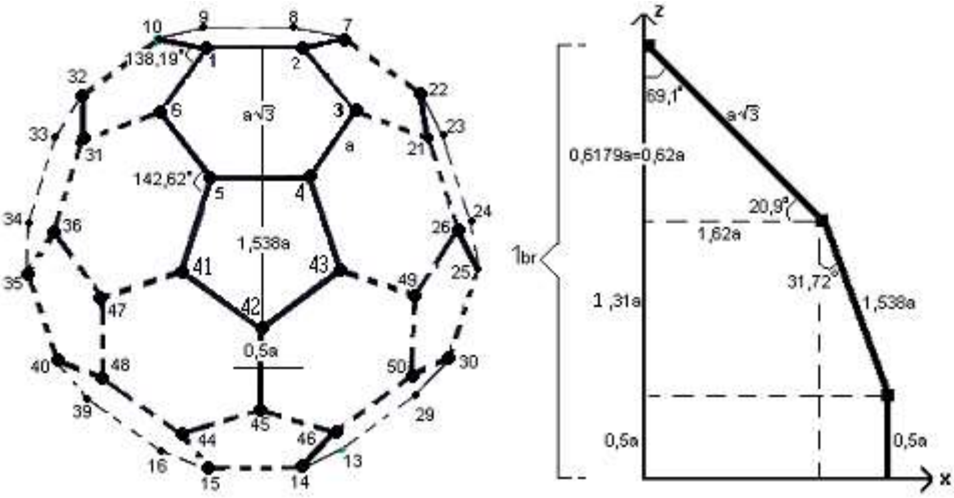

Figure 2 Front and side view of the truncated icosahedron.

I_h has full icosahedral symmetry. It has a normal subgroup I with a subindex of two. This group is isomorphic to I x C_2 or A_5 x C_2 due to rotation around three different axes, rotational reflection, and identity operations.

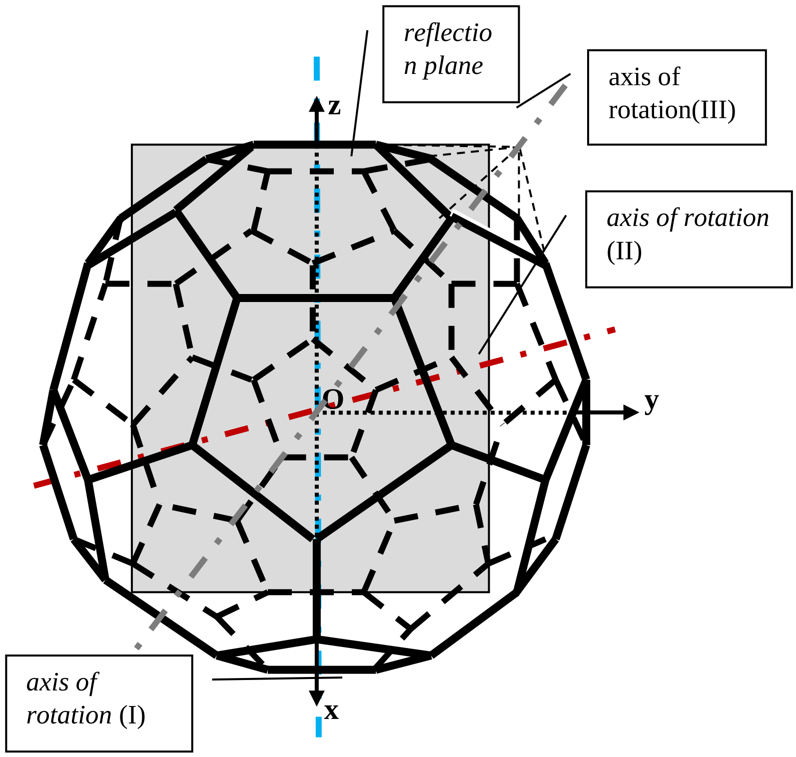

Figure 3 Reflection and rotation axes of the truncated icosahedron

The truncated icosahedron has a plane of reflection and three types of axes of rotation. In figure 3, an example of reflection plane and rotational axis types is drawn. In Table 5, a more detailed explanation is given about the rotation axes of the truncated icosahedron.

Tablo 5 Rotation operations of the Truncated Icosahedron

Truncated Icosahedron	**Describing**	**The number of vertices to which the definition will be applied**	**The number of repetitions**
rotation operation (axis of rotation-I)	The axis joining the midpoint of one edge to the midpoint of the opposite edge	15	2
rotation operation (axis of rotation-II)	The axis joining the midpoint of one hexagon surface to the midpoint of the opposite hexagonal surface	10	3
rotation operation (axis of rotation-III)	The axis joining the midpoint of a pentagon surface to the midpoint of the opposite pentagon surface	6	5

Assume that the truncated icosahedron performs a 72° rotation around the 3rd rotation axis in Figure 2. In this case, the new position of corner 2 in Figure 1 can be investigated. The vector representing corner number 2 is;

$$\vec{R}_2 = 0\mathbf{e}_1 + 0{,}5a\mathbf{e}_2 + \mathbf{e}_3 \equiv 0{,}205\mathbf{e}_2 + \mathbf{e}_3. \tag{12}$$

R rotor is expressed as

$$R_I = \cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta\vec{R}_I. \tag{13}$$

The linear cosine of $\vec{R}_I$ is found by multiplying the vector passing through the center of the pentagons by the pseudosca:

$$\vec{R}_I = \vec{R}_I I \tag{14}$$

Vector passing through the center of the pentagon

$$\vec{R}_I = 0\mathbf{e}_1 + 1{,}225a\mathbf{e}_2 + 1{,}98a\mathbf{e}_3 \equiv 0{,}50225\mathbf{e}_2 + 0{,}8118\mathbf{e}_3 .$$

For the 3rd axis of rotation, this equation is found by

$$\vec{R}_I = (0{,}50225\mathbf{e}_2 + 0{,}8118\mathbf{e}_3)\mathbf{e}_{123} = -0{,}50225\mathbf{e}_{13} + 0{,}8118\mathbf{e}_{12} \tag{15}$$

As can be seen from here, the solid will make a rotation by scanning the plane $-0{,}50225\mathbf{e}_{13} + 0{,}8118\mathbf{e}_{12}$. From this, the coordinates of the new vertex are found in the form of

$$\vec{R} = \mathbf{R_I}(\vec{R}_{13})\mathbf{R_I^*}$$

$$= \left(\cos\frac{1}{2}(72^\circ) + \sin\frac{1}{2}(72^\circ)\vec{R}_I\right)(0{,}205\mathbf{e}_2 + \mathbf{e}_3)\left(\cos\frac{1}{2}(72^\circ) + \sin\frac{1}{2}(72^\circ)\vec{R}_I\right)^*$$

$$= (0{,}81 + 0{,}59(-0{,}50225\mathbf{e}_{13} + 0{,}8118\mathbf{e}_{12}))(0{,}205\mathbf{e}_2 + \mathbf{e}_3)(0{,}81 - 0{,}59(-0{,}50225\mathbf{e}_{13} + 0{,}8118\mathbf{e}_{12}))$$

$$= (0{,}81 - 0{,}29\mathbf{e}_{13} + 0{,}48\mathbf{e}_{12})(0{,}205\mathbf{e}_2 + \mathbf{e}_3)(0{,}81 + 0{,}29\mathbf{e}_{13} - 0{,}48\mathbf{e}_{12})$$

$$\cong -0{,}32\mathbf{e}_1 + 0{,}41\mathbf{e}_2 + 0{,}86\mathbf{e}_3 \approx -0{,}81a\mathbf{e}_1 + a\mathbf{e}_2 + 2{,}12a\mathbf{e}_3 . \tag{16}$$

The result found is the vector representing the coordinates of vertex 7. When this process is repeated 5 times, the solid returns to its original state.

4. CONCLUSIONS

Clifford's Algebra was applied to the truncated icosahedron solid, which is one of the Archimedean Solids in this study. this solid is a semi-symmetrical solid. Geometric methods and matrices are also used in the analysis of symmetry operations of the symmetric solid[2]. Clifford algebras are algebras of geometries and quaternions are hypercomplex numbers [11]. In this study, Clifford algebra and quaternions are used for symmetry operations. It is clear that the calculations are easy and compact when these operations are done with Clifford algebra and quaternions. Quaternions and Clifford algebra are much simpler to apply to symmetry operations than traditional molecular symmetry methods. This method can be applied to more complex structures.

CONFLICT OF INTEREST

The author stated that there are no conflicts of interest regarding the publication of this article.

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