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Special Fractional Curve Pairs with Fractional Calculus

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Abstract

In this study, the effect of fractional derivatives, whose application area is increasing day by day, on curve pairs is investigated. As it is known, there are not many studies on a geometric interpretation of fractional calculus. When examining the effect of fractional analysis on a curve, the conformable fractional derivative that fits the algebraic structure of differential geometry derivative is used. This effect is examined with the help of examples consistent with the theory and visualized for different values of the conformable fractional derivative. The difference of this study from others is the use of conformable fractional derivatives and integrals in calculations. Fractional calculus has applications in many fields such as physics, engineering, mathematical biology, fluid mechanics,signal processing, etc. Fractional derivatives and integrals have become an extremely important and new mathematical method in solving various problems in many sciences.

Keywords: *Fractional calculus, conformable fractional derivative, special curve pairs, curvatures.*

AMS Subject Classification (2020): *Primary: 26A33 ; Secondary: 53A045; 53A5.*

1. Introduction

The term fractional derivative is mentioned firstly in Leibniz's letter to L'Hospital in 1695, as stated in many sources. In his letter to L'Hospital, Leibniz addressed "Can integer order derivatives extended to fractional order derivatives?" question is shown as the first emergence of the consept of fractional differential. Later, the subject is given importance by many mathematicians and many studies are carried out on this subject [15, 16, 17, 6, 10, 11]. In addition, understanding that the subject can be applied to other branches of science, especially in physics and engineering clearer results, its usage area has increased day by day with.

The study of fractional derivatives and integrals by many mathematicians has led to the emergence of different definitions of fractional derivatives and integrals. Riemann-Liouville, Caputu, Cauchy and conformable fractional derivatives and integrals are just a few of these definitions. Different definitions on this subject have brought different features. For example, the derivative of zero is not constant in other fractional types, except for the Caputo and conformable fractional derivative and integral type. In addition, fractional derivative types other than conformable fractional derivatives do not have features such as the derivative of the product, the derivative of the quotient, or the chain rule, as in the classical sense [12, 1].

The most suitable derivative types for the structure of differential geometry are Caputo and conformable fractional derivatives. Because in this two fractional derivative, the derivative of the constant is zero. For this reason, Caputo fractional derivative and conformable fractional derivative are used while investigating the effect of fractional derivative on differential geometry [4, 13, 20, 21, 2, 3, 7].

In this present study, we are studied the most well-known and used curve pairs which are Bertrand curve pairs, Mannheim curve pairs and Evolute-involute curve pairs differential geometry using fractional calculus. These curve pairs have been studied by many authors in many different spaces and various studies have been carried out. The difference of this study from other studies is the use of conformable fractional derivatives and integrals in calculations. Thus, these curve pairs are considered together with this new fractional calculus in

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differential geometry and various characterizations are given about them. Finally, the effect of the fractional derivative on the curve pairs for different fractional values is observed by giving examples compatible with theory.

2. Preliminaries

2.1. Basic Definitions and Theorems of Conformable Fractional Derivative

In this section, some basic definitions and theorems of the appropriate fractionally derivative and integral are given.

Definition 2.1. Let us give a funtion $f : [0, \infty) \to \mathbb{R}$. Then the "conformable fractional derivative" for f of order α is defined by

$$D_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all each $t > 0$, $0 < \alpha < 1$. If f is α-differentiable in some $(0, \alpha)$, $\alpha > 0$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exist, then defiine $f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t)$, [12].

Theorem 2.1. *Let $f : [0, \infty) \to R$ be a function. If a function f is α-differentiable at $t_0 > 0$, $0 < \alpha < 1$, then f is continuous at t_0 [12].*

Accordingly, it is easily visible that the conformable fractional derivative provides all the properties given in the theorem below.

Theorem 2.2. *Let $f, g : [0, \infty) \to R$ be α-differentiable at for each $t > 0$, $0 < \alpha < 1$. Then*
(1) $D_\alpha(af + bg)(t) = aD_\alpha(f)(t) + bD_\alpha(g)(t)$, *for all* $a, b \in \mathbb{R}$.

(2) $D_\alpha(t^p) = pt^{p-\alpha}$, *for all* $p \in \mathbb{R}$.

(3)$D_\alpha(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$.

(4) $D_\alpha(fg)(t) = f(t)D_\alpha(g)(t) + g(t)D_\alpha(f)(t)$.

(5) $D_\alpha(\frac{f}{g})(t) = \frac{f(t)D_\alpha(g)(t) - g(t)D_\alpha(f)(t)}{g^2(t)}$.

(6) *If f it is a differentiable function, then* $D_\alpha(f)(t) = t^{1-\alpha}\frac{df(t)}{dt}$ *[12].*

Theorem 2.3. *Let $f, g : [0, \infty) \to R$ is α-differentiable at $t_0 > 0$, $0 < \alpha < 1$. If $(f \circ g)$ is $\alpha-$differentiable and for all t with $t \neq 0$ and $f(t) \neq 0$ the equation*

$$D_\alpha(f \circ g)(t) = f(t)^{\alpha-1} D_\alpha(f)(t) D_\alpha(g)(f(t)) \tag{2.1}$$

is provided, [1].

Definition 2.2. Let $f : [a, \infty) \to R$ be a function. The expression

$$I_0^\alpha f(t) = I_1^\alpha f(t^{\alpha-1} f) = \int_a^t \frac{f(x)}{x^{\alpha-1}} dx \tag{2.2}$$

is called a conformable fractional integral, where $\alpha > 0$, [1].

Theorem 2.4. *Let $f : [a, \infty) \to R$ be a function. Then for all $t > 0$ the following equation exists, [1]*

$$D_\alpha I_0^\alpha f(t) = f(t). \tag{2.3}$$

2.2. Basic Definitions and Theorems of Differential Geometry

In this section, the curves in $\mathbb{R}^3$ are introduced in a nutshell.

Definition 2.3. Let $x = x(s)$ be a regular unit speed curve in the Euclidean $3-$space where s measures its arc length. Also, let $t = x'$ be its unit tangent vector, $n = \frac{t'}{\|t'\|}$ be its principal normal vector and $b = t \times n$ be its binormal vector. The triple $\{t, n, b\}$ be the Frenet frame of the curve x. Then the Frenet formula of the curve x is given by

$$\begin{pmatrix} t'(s) \\ n'(s) \\ b'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} \tag{2.4}$$

where $\kappa(s) = \left\| \frac{d^2x}{ds^2} \right\|$ and $\tau(s) = \langle \frac{dn}{ds}, b \rangle$ are curvature and torsion of x, respectively [19].

Definition 2.4. A curve $x : I \to E^3$ with non-zero curvatures is a Bertrand curve if there is a curve $y : I_1 \to E^3$ and a bijection $\gamma : x \to x_1$ such that the principal normal vectors of $x(s)$ and $y(s_1)$ at $s \in I$, $s_1 \in I_1$ coincide. In this case, $y(s_1)$ is called the Bertrand mate of $x(s)$, [8].

Theorem 2.5. *Let $x : I \to E^3$ be a regular curve in E^3 according to Frenet frame $\{t, n, b\}$ with non-zero curvatures κ, τ, and $y : I_1 \to E^3$ be a Bertrand mate curve of x according to Frenet frame $\{t_1, n_1, b_1\}$ with non-zero curvatures κ_1, τ_1. Then the curve y can be written as [5],*

$$y(s_1) = x_1(f(s)) = x(s) + \lambda n(s). \tag{2.5}$$

Corollary 2.1. *Let $x(s)$ and $y(s_1)$ be regular curves in E^3 with the Frenet frames $\{t, n, b\}$ and $\{t_1, n_1, b_1\}$ according to arc-length parameter s and s_1, respectively. If the pair (x, y) are the Bertrand curve pair, we can write the following results*

$$\begin{aligned} t_1(s) &= \cos\theta t(s) + \sin\theta b(s) \\ n_1(s) &= n(s) \\ b_1(s) &= -\sin\theta t(s) + \cos\theta b(s) \end{aligned} \tag{2.6}$$

where θ is the angle between t and t_1, [18].

Corollary 2.2. *Let $x : I \to E^3$ be a Bertrand curve in E^3 with non-zero curvatures κ, τ and $y : I_1 \to E^3$ be a Bertrand mate curve of x in E^3. The following equations are available for the curvatures of the curve x [18],*

$$\kappa = \frac{1 - \cos\theta \frac{ds_1}{ds}}{\lambda}, \tag{2.7}$$

$$\tau = \frac{\sin\theta \frac{ds_1}{ds}}{\lambda}. \tag{2.8}$$

Definition 2.5. A curve $x : I \to E^3$ with non-zero curvatures is a Mannheim curve such that principal normal vector of $x(s)$ and binormal vector of $y(s_1)$ at $s \in I$, $s_1 \in I_1$ coincide. In this case, $y(s_1)$ is called the Mannheim mate of $x(s)$. The pair $\{x, y\}$ is said to be a Mannheim curve pair.

Theorem 2.6. *Let $x : I \to E^3$ be a regular curve in E^3 according to Frenet frame $\{t, n, b\}$ with non-zero curvatures κ, τ, and $y : I_1 \to E^3$ be a Mannheim mate curve of x according to Frenet frame $\{t_1, n_1, b_1\}$ with non-zero curvatures κ_1, τ_1. Then the curve y can be written as, [14],*

$$y(s_1) = x_1(f(s)) = x(s) + \lambda n(s). \tag{2.9}$$

Corollary 2.3. *Let $x(s)$ and $y(s_1)$ be regular curves in E^3 with the Frenet frames $\{t, n, b\}$ and $\{t_1, n_1, b_1\}$ according to arc-length parameter s and s_1, respectively. If the pair (x, y) are the Mannheim curve pair, we can write the following results*

$$\begin{aligned} t_1(s) &= \cos\theta t(s) - \sin\theta b(s) \\ n_1(s) &= \sin\theta t(s) + \cos\theta b(s) \\ b_1(s) &= n(s) \end{aligned} \tag{2.10}$$

where θ is the angle between t and t_1, [14].

Corollary 2.4. *Let $x : I \to E^3$ be a regular curve in E^3 with non-zero curvatures κ, τ and $y : I_1 \to E^3$ be a Mannheim mate curve of x in E^3. The following equations are available for the curvatures of the curve x [18],*

$$\kappa = \frac{\cos\theta\frac{ds_1}{ds} - 1}{\lambda}, \tag{2.11}$$

$$\tau = \frac{-\sin\theta\frac{ds_1}{ds}}{\lambda}. \tag{2.12}$$

Definition 2.6. Let two regular and unit speed curves be (x, y) in E^3 and also the Frenet vectors of the curves x and y be $\{t, n, b\}$ and $\{t_1, n_1, b_1\}$, respectively. If the tangent vector of x and y are the orthogonal as, $< t, t_1 >= 0$, the curve x is called the evolute curve of y and the curve y is called the involute curve of x.

Theorem 2.7. *Let (x, y) be evolute-involute partner curves. There is equilibrium indicated below between those partner curves as following*

$$y(s_1) = x(s) + \lambda(s)t(s) \tag{2.13}$$

where $\lambda(s) = c - s$ and c is a constant [8].

Corollary 2.5. *Let $x(s)$ and $y(s_1)$ be regular curves in E^3 with the Frenet frames $\{t, n, b\}$ and $\{t_1, n_1, b_1\}$ according to arc-length parameter s and s_1, respectively. If the pair (x, y) are a involute-evolute curve pair, we can write the following equations*

$$\begin{aligned} t_1(s) &= n(s) \\ n_1(s) &= \cos\theta t(s) + \sin\theta b(s) \\ b_1(s) &= -\sin\theta t(s) + \cos\theta b(s). \end{aligned} \tag{2.14}$$

In above equation, θ is angle between vectors t and n_1, [18].

2.3. Basic Definitions and Theorems of Conformable Fractional Curves

In this part of the preliminaries section, we present brief information about conformable curves using conformable fractional derivative.

Definition 2.7. Let $x = x(s)$ be a curve. If $x : (0, \infty) \to \mathbb{R}^3$ is a called α differentiable, then x is called a conformable curve in $\mathbb{R}^3$ [7].

Definition 2.8. Let $x : (0, \infty) \to \mathbb{R}^3$ be a conformable curve in $\mathbb{R}^3$. Velocity vector of x is determined by

$$\frac{D_\alpha(x)(t)}{t^{\alpha-1}}$$

for all $t \in (0, \infty)$, [7].

Definition 2.9. Let $x : (0, \infty) \to \mathbb{R}^3$ be a conformable curve in $\mathbb{R}^3$. Then the velocity function v of x is defined by

$$v(t) = \frac{\|D_\alpha(x)(t)\|}{t^{\alpha-1}}$$

for all $t \in (0, \infty)$ [7].

Definition 2.10. Let $x : (0, \infty) \to \mathbb{R}^3$ be a conformable curve in $\mathbb{R}^3$. The arc length function s of x is defined by

$$s(t) = I_\alpha^0 \|D_\alpha(x)(t)\|$$

for all $t \in (0, \infty)$. If $v(t) = 1$ for all $t \in (0, \infty)$, it is said that x has unit speed, [7].

Definition 2.11. Let x be a conformable curve. If $D_\alpha(x)(t) \neq 0$ for all $t \in (0, \infty)$, x is called a conformable regular curve, [7].

Theorem 2.8. *Let $x = x(s)$ be a regular conformable with arbitrary speed curve in the Euclidean $3-$space where s measures its arc length. Also, let $t = \frac{D_\alpha(x)(s)}{\|D_\alpha(x)(s)\|}$ be its unit tangent vector, $n = b \times t$ be its principal normal vector and*

$b = \frac{D_\alpha(x)(s) \times D_\alpha^2(x)(s)}{\|D_\alpha(x)(s) \times D_\alpha^2(x)(s)\|}$ be its binormal vector. The triple $\{t, n, b\}$ be the conformable Frenet frame of the curve x. Then the conformable Frenet formula of the curve x is given by

$$\begin{pmatrix} D_\alpha(t)(s) \\ D_\alpha(n)(s) \\ D_\alpha(b)(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_\alpha(s) v \lambda^{1-\alpha} & 0 \\ -\kappa_\alpha(s) v \lambda^{1-\alpha} & 0 & \tau_\alpha(s) v \lambda^{1-\alpha} \\ 0 & -\tau_\alpha(s) v \lambda^{1-\alpha} & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} \tag{2.15}$$

where $\lambda = \frac{t}{s(t)}$, $\kappa_\alpha(s) = \left(\frac{t}{\lambda}\right)^{1-\alpha} \frac{\|D_\alpha(x)(s) \times D_\alpha^2(x)(s)\|}{\|D_\alpha(x)(s)\|^3}$ and
$\tau_\alpha(s) = \left(\frac{t}{\lambda}\right)^{1-\alpha} \frac{\left(D_\alpha(x)(s) \times D_\alpha^2(x)(s)\right) . D_\alpha^3(x)(s)}{\|D_\alpha(x)(s) \times D_\alpha^2(x)(s)\|^2}$ are curvature and torsion of x, respectively [7].

3. Main Results

In this section, some curve pairs previously obtained with classical derivative will be obtained with conformable fractional derivative.

Definition 3.1. Let $x = x(s)$ be a regular unit speed conformable curve in the Euclidean $3-$space where s measures its arc length. Also, let $t = D_\alpha(x)(s) s^{\alpha-1}$ be its unit tangent vector, $n = \frac{D_\alpha(t)(s)}{\|D_\alpha(t)(s)\|}$ be its principal normal vector and $b = t \times n$ be its binormal vector. The triple $\{t, n, b\}$ is the conformable Frenet frame of the curve x. Then the conformable Frenet formula of the curve x is given by

$$\begin{pmatrix} D_\alpha(t)(s) \\ D_\alpha(n)(s) \\ D_\alpha(b)(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_\alpha(s) & 0 \\ -\kappa_\alpha(s) & 0 & \tau_\alpha(s) \\ 0 & -\tau_\alpha(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} \tag{3.1}$$

where $\kappa_\alpha(s) = \|D_\alpha(t)(s)\|$ and $\tau_\alpha(s) = \langle D_\alpha(n)(s),\ b \rangle$ are curvature and torsion of x, respectively.

Corollary 3.1. *Let $x = x(s)$ be a regular unit speed conformable curve in the Euclidean $3-$space where s measures its arc length. The following relation exists between the curvatures and torsion of the x curve and the conformable curvature and torsion as*

$$\kappa_\alpha = s^{1-\alpha} \kappa, \tag{3.2}$$
$$\tau_\alpha = s^{1-\alpha} \tau. \tag{3.3}$$

Conclusion 3.1. *Let $x = x(s)$ be a regular unit speed conformable curve where s measures its arc length. As can be seen from equation (3.1), when x is a unit speed curve, the conformable derivative has no effect on the Frenet frame, so the Frenet elements do not undergo any change. However, considering equations (3.2) and (3.3), the curvature and torsion of the x curve has changed under the conformable fractional derivative.*

Definition 3.2. Let $x : I \to E^3$ be a regular conformable curve in the Euclidean 3-space. The curve x with non-zero conformable curvatures is a Fractional Bertrand curve $(F - Bertrand\ curve)$ if there is a conformable curve $y : I_1 \to E^3$ and a bijection $\gamma : x \to y$ such that the principal normal vectors of $x(s)$ and $y(s_1)$ at their reciprocal points $s \in I$, $s_1 \in I_1$ coincide. In this case, $y(s_1)$ is called the Fractional Bertrand mate $(F - Bertrand\ curve\ pair)$ of $x(s)$.

Theorem 3.1. *Let $x : I \to E^3$ and $y : I_1 \to E^3$ be a conformable curves in three-dimensional Euclidean space E^3 with arc-length parameter s and s_1, respectively. If $y(s_1)$ is the F- Bertrand curve pair of $x(s)$, the following equation exists*

$$\mu \tau_\alpha + \lambda \kappa_\alpha = s^{1-\alpha}$$

where μ and λ constant functions.

Proof. Let $x : I \to E^3$ be a F-Bertrand curve in E^3 with the conformable frame $\{t,\ n,\ b\}$ and the non-zero curvatures κ_α, τ_α, and $y : I_1 \to E^3$ be a F-Bertrand mate curve of x with the conformable frame $\{t_1, n_1, b_1\}$ and the non-zero curvatures κ_1, τ_1. Then the curve y can be written as

$$y(s_1) = x_1(f(s)) = x(s) + \lambda n(s).$$

If the α-th conformable fractional derivative of both sides of the above equation with respect to s is taken, we have

$$D_\alpha y(s_1) = D_\alpha x(s) + n(s) D_\alpha \lambda + \lambda D_\alpha n(s).$$

If the necessary derivatives of the above equation are taken and equation (2.1) is considered, following equations

$$D_\alpha y(s_1) D_\alpha s_1(s)(s_1(s))^{\alpha-1} = s^{1-\alpha} t + \lambda(-\kappa_\alpha t + \tau_\alpha b)$$

and

$$s_1^{1-\alpha} t_1 D_\alpha s_1(s)(s_1(s))^{\alpha-1} = (s^{1-\alpha} - \lambda\kappa_\alpha) t + \lambda \tau_\alpha b$$

are obtained. If equation (2.6) is used in this obtained equation, we get

$$s_1^{1-\alpha} D_\alpha s_1(s)(s_1(s))^{\alpha-1} (\cos\theta t(s) - \sin\theta b(s)) = (s^{1-\alpha} - \lambda\kappa_\alpha) t + \lambda\tau_\alpha b.$$

Finally, from mutual equality, we can easily see

$$s_1^{1-\alpha} D_\alpha s_1(s)(s_1(s))^{\alpha-1} \cos\theta = s^{1-\alpha} - \lambda\kappa_\alpha \tag{3.4}$$

and

$$-s_1^{1-\alpha} D_\alpha s_1(s)(s_1(s))^{\alpha-1} \sin\theta = \lambda\tau_\alpha. \tag{3.5}$$

If the above equation are proportioned to the corresponding, we obtain following

$$-\cot\theta = \frac{s^{1-\alpha} - \lambda\kappa_\alpha}{\lambda\tau_\alpha},$$

$$-\lambda\tau_\alpha \cot\theta + \lambda\kappa_\alpha = s^{1-\alpha}.$$

If we choose $-\lambda\cot\theta = \mu$ constant function, we get

$$\mu\tau_\alpha + \lambda\kappa_\alpha = s^{1-\alpha}.$$

Thus, the theorem is proved. □

Theorem 3.2. *Let $x(s)$ be a conformable curve in three-dimensional Euclidean space E^3 with arc-length parameter s and $y(s_1)$ be the F- Bertrand curve pair of $x(s)$ with arc-length parameter s_1. In that case, the following relation exists between the classical curvatures and the conformable curvatures of the curve x, as*

$$\begin{aligned} \kappa_\alpha &= s^{1-\alpha}\left(\frac{1-\psi}{\lambda} + \psi\kappa\right), \\ \tau_\alpha &= s^{1-\alpha}\psi\tau, \end{aligned}$$

where $\psi = s^{1-\alpha}\left(s_1(s)\right)^{\alpha-1}$.

Proof. If the equations (3.4) and (3.5) obtained in the proof of the previous theorem are arranged, we can see that

$$\begin{aligned} \kappa_\alpha &= \frac{s^{1-\alpha} - s_1^{1-\alpha} D_\alpha s_1(s)(s_1(s))^{\alpha-1}\cos\theta}{\lambda}, \\ \tau_\alpha &= \frac{s_1^{1-\alpha} D_\alpha s_1(s)(s_1(s))^{\alpha-1}\sin\theta}{\lambda}. \end{aligned}$$

If the required conformable fractional derivatives are taken in the above equation, conformable curvatures are obtained as

$$\begin{aligned} \kappa_\alpha &= \frac{s^{1-\alpha} - s_1^{1-\alpha} s^{1-\alpha}(s_1(s))^{\alpha-1}\cos\theta \frac{ds_1}{ds}}{\lambda}, \\ \tau_\alpha &= \frac{s_1^{1-\alpha} s^{1-\alpha}(s_1(s))^{\alpha-1}\sin\theta \frac{ds_1}{ds}}{\lambda}. \end{aligned}$$

Considering the equations (2.7) and (2.8) on this equation and choosing the $\psi = s^{1-\alpha}\left(s_1(s)\right)^{\alpha-1}$, we get

$$\begin{aligned} \kappa_\alpha &= s^{1-\alpha}\left(\frac{1-\psi}{\lambda} + \psi\kappa\right), \\ \tau_\alpha &= s^{1-\alpha}\psi\tau. \end{aligned}$$

□

Definition 3.3. Let $x : I \to E^3$ be a regular conformable curve in the Euclidean 3-space. The curve x with non-zero conformable curvatures is a Fractional Mannheim curve $(F - Mannheim\ curve)$, if there is a conformable curve $y : I_1 \to E^3$ and a bijection $\gamma : x \to y$ such that the principal normal vector of $x(s)$ and binormal vector of $y(s_1)$ at their reciprocal points $s \in I$, $s_1 \in I_1$ coincide. In this case, $y(s_1)$ is called the Fractional Mannheim mate $(F - Mannheim\ curve\ pair)$ of $x(s)$.

Theorem 3.3. *Let $x : I \to E^3$ and $y : I_1 \to E^3$ be a conformable curves in three-dimensional Euclidean space E^3 with arc-length parameter s and s_1, respectively. If $y(s_1)$ is the F- Mannheim curve pair of $x(s)$, the following equation exists*

$$\lambda = s^{1-\alpha}\frac{\kappa_\alpha}{\tau_\alpha^2 + \kappa_\alpha^2}$$

where λ constant functions.

Proof. Let $x : I \to E^3$ be F-Mannheim curve in E^3 according to conformable frame $\{t,\, n,\, b\}$ with the non-zero curvatures κ_α, τ_α, and $y : I_1 \to E^3$ be a F-Mannheim mate curve of x according to conformable frame $\{t_1, n_1, b_1\}$ with the non-zero curvatures κ_1, τ_1. Then the curve y can be written as

$$y(s_1) = x_1(f(s)) = x(s) + \lambda n(s).$$

If the α-th conformable fractional derivative of both sides of the above equation with respect to s is taken, we get

$$D_\alpha y(s_1) = D_\alpha x(s) + n(s) D_\alpha \lambda + \lambda D_\alpha n(s).$$

If the necessary derivatives of above equation are taken and equation (2.1) is considered, following equation is obtained, as

$$s_1^{1-\alpha} t_1 D_\alpha s_1(s)(s_1(s))^{\alpha-1} = (s^{1-\alpha} - \lambda\kappa_\alpha)t + \lambda\tau_\alpha b.$$

If equation (2.10) is used in this obtained equation, we get

$$s_1^{1-\alpha} D_\alpha s_1(s)(s_1(s))^{\alpha-1}(\cos\theta t(s) - \sin\theta b(s)) = (s^{1-\alpha} - \lambda\kappa_\alpha)t + \lambda\tau_\alpha b.$$

Finally, from mutual equality

$$s_1^{1-\alpha} D_\alpha s_1(s)(s_1(s))^{\alpha-1}\cos\theta = s^{1-\alpha} - \lambda\kappa_\alpha \tag{3.6}$$

and

$$s_1^{1-\alpha} D_\alpha s_1(s)(s_1(s))^{\alpha-1}\sin\theta = -\lambda\tau_\alpha \tag{3.7}$$

are obtained. If the above equations are proportioned to the corresponding, we have

$$\cot\theta = \frac{s^{1-\alpha} - \lambda\kappa_\alpha}{-\lambda\tau_\alpha}.$$

If the equation $\frac{\tau_\alpha}{\kappa_\alpha} = -\cot\theta$ in [9] is used in the above equation, we can easily see that

$$\lambda = s^{1-\alpha}\frac{\kappa_\alpha}{\tau_\alpha^2 + \kappa_\alpha^2}.$$

□

Theorem 3.4. *Let $x(s)$ be a conformable curve in three-dimensional Euclidean space E^3 with arc-length parameter s and $y(s_1)$ be the F- Mannheim curve pair of $x(s)$ with arc-length parameter s_1. In that case, the following relation exists between the classical curvatures and the conformable curvatures of the curve x*

$$\begin{aligned} \kappa_\alpha &= -\psi\kappa + \frac{s^{1-\alpha} - \psi}{\lambda}, \\ \tau_\alpha &= \psi\tau \end{aligned}$$

where $\psi = s^{1-\alpha} s_1^{1-\alpha}(s_1(s))^{\alpha-1}$.

Proof. If the equations (3.4) and (3.5) obtained in the proof of the previous theorem are arranged, we can write as

$$\kappa_\alpha = \frac{s^{1-\alpha} - s_1^{1-\alpha} D_\alpha s_1(s)(s_1(s))^{\alpha-1}\cos\theta}{\lambda},$$
$$\tau_\alpha = \frac{s_1^{1-\alpha} D_\alpha s_1(s)(s_1(s))^{\alpha-1}\sin\theta}{\lambda}.$$

If the required conformable fractional derivatives are taken in the above equation, we obtain

$$\kappa_\alpha = \frac{s^{1-\alpha} - s_1^{1-\alpha} s^{1-\alpha}(s_1(s))^{\alpha-1}\cos\theta\frac{ds_1}{ds}}{\lambda},$$
$$\tau_\alpha = -\frac{s_1^{1-\alpha} s^{1-\alpha}(s_1(s))^{\alpha-1}\sin\theta\frac{ds_1}{ds}}{\lambda}.$$

Considering the equations (2.11) and (2.12) in above equations and if we choose the equation as $\psi = s^{1-\alpha}s_1^{1-\alpha}(s_1(s))^{\alpha-1}$, we get

$$\kappa_\alpha = -\psi\kappa + \frac{s^{1-\alpha} - \psi}{\lambda},$$
$$\tau_\alpha = \psi\tau.$$

□

Definition 3.4. Let $x : I \to E^3$ and $y : I_1 \to E^3$ be a regular conformable curve in the Euclidean 3-space with the conformable frames $\{t, n, b\}$ and $\{t_1, n_1, b_1\}$, respectively. If the tangent vector of x and y is orthogonal, the conformable curve x is called the Fractional Evolute curve $(F - Evolute\ curve)$ of the conformable curve y and the conformable curve y is called the Factional Involute curve $(F - Involute\ curve)$ of the curve x.

Theorem 3.5. *Let $x : I \to E^3$ and $y : I_1 \to E^3$ be a regular conformable curve in the Euclidean 3-space. When $x(s)$ be a F-Evolute curve in three-dimensional Euclidean space E^3 with arc-length parameter s, if $y(s_1)$ be the F-Involute curve pair of $x(s)$ with arc-length parameter s_1, the following equation exists*

$$\kappa_\alpha = \frac{s^{1-\alpha}\psi}{c - s}$$

where $\psi = D_\alpha s_1(s)(s_1(s))^{\alpha-1}$.

Proof. Let $x : I \to E^3$ be a F-Evolute curve in E^3 according to conformable frame $\{t, n, b\}$ with the non-zero curvatures κ_α, τ_α, and $y : I_1 \to E^3$ be a F-Involute mate curve of x according to conformable frame $\{t_1, n_1, b_1\}$ with the non-zero curvatures κ_1, τ_1. Then the curve y can be written as

$$y(s_1) = x(s) + (c - s)t(s).$$

If the α-th conformable fractional derivative, of both sides of the above equation is taken with respect to s, we obtain

$$D_\alpha y(s_1) = D_\alpha x(s) + t(s)D_\alpha(c - s) + (c - s)D_\alpha t(s).$$

If the necessary derivatives of above equation are taken and equation (2.1) is considered, we have

$$s_1^{1-\alpha} t_1 D_\alpha s_1(s)(s_1(s))^{\alpha-1} = s^{1-\alpha}t - s^{1-\alpha}t + (c - s)\kappa_\alpha n$$

and

$$s_1^{1-\alpha} t_1 D_\alpha s_1(s)(s_1(s))^{\alpha-1} = (c - s)\kappa_\alpha n.$$

If equation (2.14) is used in this obtained equation, we get

$$s_1^{1-\alpha} D_\alpha s_1(s)(s_1(s))^{\alpha-1} n = (c - s)\kappa_\alpha n.$$

If the above equation is arranged, we obtain curvature as

$$\kappa_\alpha = \frac{s_1^{1-\alpha} D_\alpha s_1(s)(s_1(s))^{\alpha-1}}{(c - s)}.$$

Finally, if $\psi = D_\alpha s_1(s)(s_1(s))^{\alpha-1}$ is choosen in the above equation, we can easily see that

$$\kappa_\alpha = \frac{s^{1-\alpha}\psi}{c-s}.$$

Thus, the proof is completed. $\square$

Example 3.1. Let $x : I \subset \mathbb{R} \to E^3$ be a regular with unit speed conformable curve in $\mathbb{R}^3$ parameterized by

$$x(s) = \left(\frac{-1}{\sqrt{2}} \int s^{1-\alpha} \cos s ds, \frac{-1}{\sqrt{2}} \int s^{1-\alpha} \sin s ds, \frac{1}{\sqrt{2}} \int s^{1-\alpha} ds \right).$$

The F-Bertrand curve pair of $x(s)$ is obtained as follows,

$$y(s) = \left(\frac{-1}{\sqrt{2}} \int s^{1-\alpha} \sin s ds, \frac{-1}{\sqrt{2}} \int s^{1-\alpha} \cos s ds, \frac{1}{\sqrt{2}} \int s^{1-\alpha} ds \right).$$

The view of the F-Bertrand curve pairs for different fractional α values are given below.

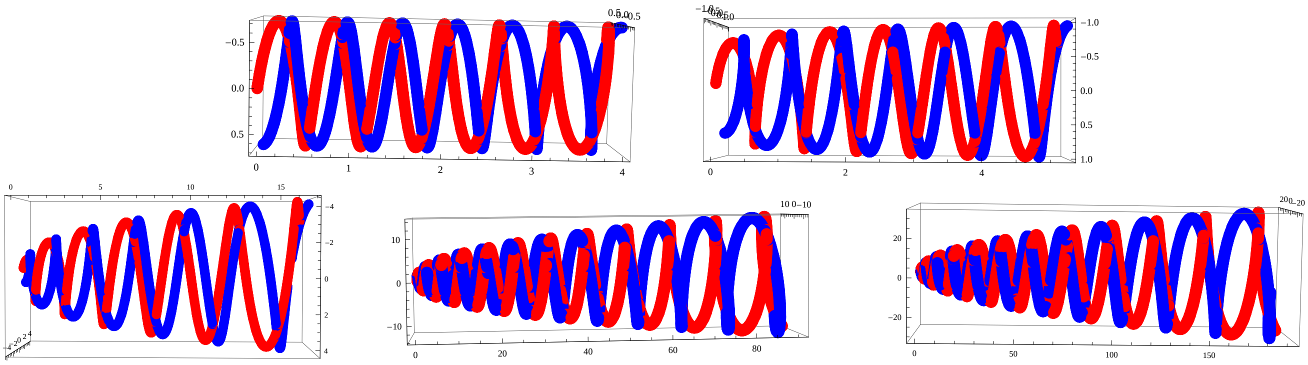

Figure 1. For $\alpha \to 1, \alpha = 0.9, \alpha = 0.5$, $\alpha = 0.3$ and $\alpha = 0.1$ F-Bertrand curve pair $x(s)(Red)$ and $y(s)(Blue)$, respectively.

Example 3.2. Let $x : I \subset \mathbb{R} \to E^3$ be a regular with arbitrary speed conformable curve in $\mathbb{R}^3$ parameterized by

$$x(s) = \left(-\frac{8}{5} \int s^{1-\alpha} \sin s ds, \frac{8}{5} \int s^{1-\alpha} \cos s ds, \frac{4}{5} \int s^{1-\alpha} ds \right).$$

The F-Manmheim curve pair of $x(s)$ is obtained as follows,

$$y(s) = \left(\frac{8}{5} \int s^{1-\alpha} (\sin s - \cos s) ds, \frac{8}{5} \int s^{1-\alpha} (\cos s - \sin s) ds, \frac{4}{5} \int s^{1-\alpha} ds \right).$$

The view of the F-Mannheim curve pairs for different fractional α values are given below.

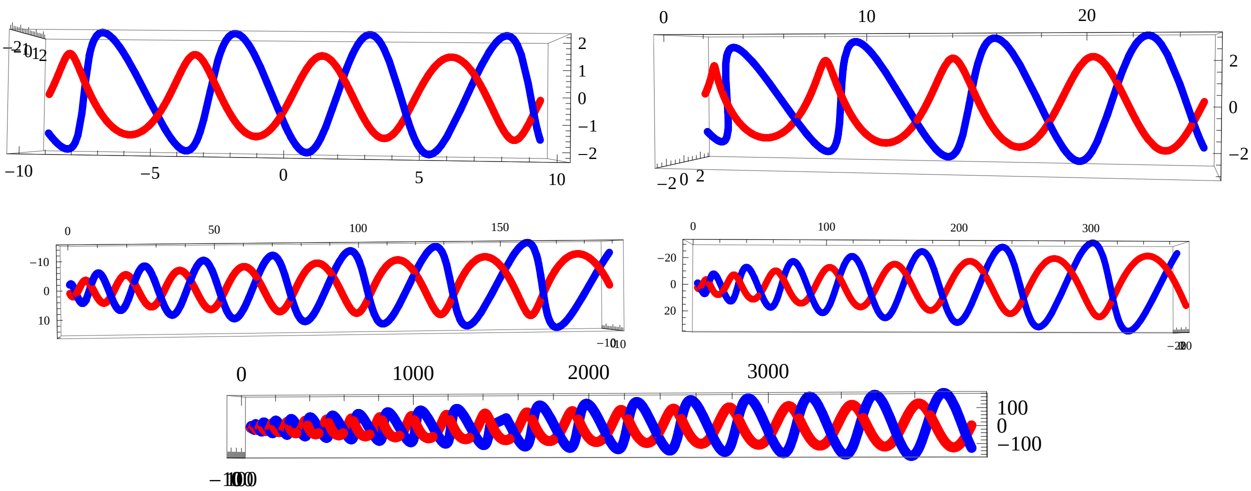

Figure 1. For $\alpha \to 1, \alpha = 0.9, \alpha = 0.5$, $\alpha = 0.3$ and $\alpha = 0.1$ F-Mannheim curve pair $x(s)(Red)$ and $y(s)(Blue)$, respectively.

Example 3.3. Let $x : I \subset \mathbb{R} \to E^3$ be a regular with arbitrary speed conformable curve in $\mathbb{R}^3$ parameterized by

$$x(s) = \left(-\frac{8}{5}\int s^{1-\alpha}\sin s ds, \frac{8}{5}\int s^{1-\alpha}\cos s ds, \frac{4}{5}\int s^{1-\alpha} ds\right).$$

The F-Involute curve pair of $x(s)$ is obtained as follows,

$$y(s) = \left(\frac{1}{5}\int s^{1-\alpha}(-8\sin s - 2\cos s + 2\sin s + 2s\cos s)ds, \frac{1}{5}\int s^{1-\alpha}(8\cos s - 2\sin s - 2\cos s + 2s\sin s)ds, \frac{3}{5}\int s^{1-\alpha} ds\right).$$

The view of the F-Involute and F-Evolute curve pairs for different fractional α values are given below.

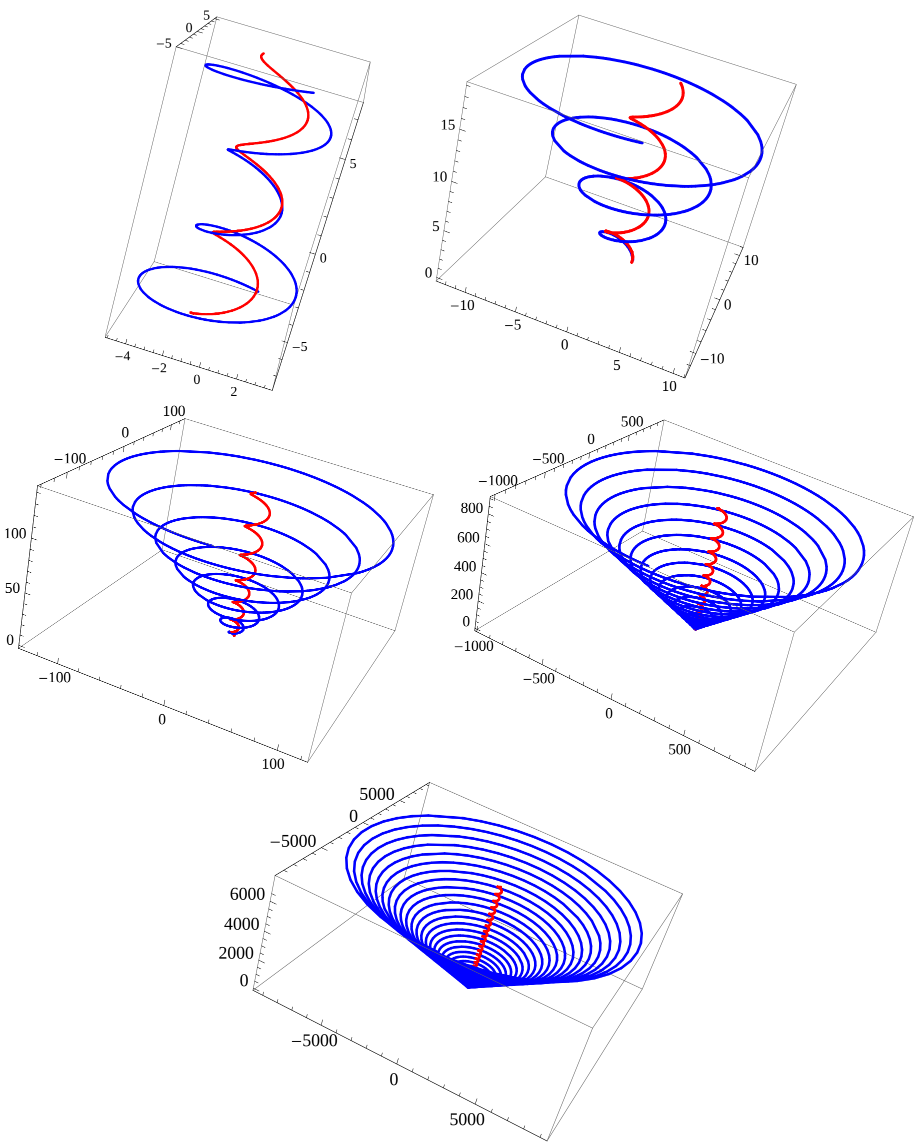

Figure 1. For $\alpha \to 1, \alpha = 0.9, \alpha = 0.5$, $\alpha = 0.3$ and $\alpha = 0.1$ F-Evolute and F-Involute curve pair $x(s)(Red)$ and $y(s)(Blue)$, respectively.

4. Conclusion

The study areas of fractional derivatives and integrals are increasing day by day. Today, fractional analysis for numerical solutions are preferred in the fields of basic sciences and engineering. The reason for this is the claim that the solutions obtained by fractional analysis give a more clearer numerical solution. Can a claim like this be considered in geometry, the question arises. Studies examining the answer to this question are still very new. For example, one of these studies examined the effect of fractional analysis on geometry [20], while another study investigated the effect of fractional analysis on electromagnetic curves in fiber optics [23]. It should be taken into account that fractional analyzes, which are claimed to offer clearer solutions numerically, can also give clearer information geometrically. For example, can the motion or characteristic of a curve be more clearly explained by fractional analysis? In this study, the answer to this question is emphasized and this effect is examined both algebraically and with examples. Future studies in this field will make many contributions to geometry.

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