Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (5) (2022), 1484 – 1503
DOI : 10.15672/hujms.1012056

Research Article

A new improved Liu-type estimator for Poisson regression models

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Abstract

The Poisson Regression Model (PRM) is commonly used in applied sciences such as economics and the social sciences when analyzing the count data. The maximum likelihood method is the well-known estimation technique to estimate the parameters in PRM. However, when the explanatory variables are highly intercorrelated, unstable parameter estimates can be obtained. Therefore, biased estimators are widely used to alleviate the undesirable effects of these problems. In this study, a new improved Liu-type estimator is proposed as an alternative to the other proposed biased estimators. The superiority of the new proposed estimator over the existing biased estimators is given under the asymptotic matrix mean square error criterion. Furthermore, Monte Carlo simulation studies are executed to compare the performances of the proposed biased estimators. Finally, the obtained results are illustrated in real data. Based on the set of experimental conditions which are investigated, the proposed biased estimator outperforms the other biased estimators.

Mathematics Subject Classification (2020). 62J07, 62F10

Keywords. Poisson regression, mean squared error, multicollinearity, Ridge estimator, Liu estimator

1. Introduction

In regression modeling, count responses are usually common in the social sciences, economic research, and medical fields. The number of hits recorded by the Geiger counter, the number of patient days in the hospital, and the number of goals scored in major competitions can be given as examples of count responses. In these cases, one of the standard models for explaining the relationship between the counts as the response variable and a series of explanatory variables is Poisson Regression Models (PRMs) [11].

The Maximum Likelihood Estimator (MLE) is commonly used to estimate unknown regression coefficients in the PRM. One of the disadvantages of using MLE is that the estimates of model parameters usually becomes unstable with high variance when the multicollinearity exists [4, 5, 13, 15, 20, 21, 23, 29, 31]. The multicollinearity problem, which occurs because of the approximately linear relationship between the explanatory variables, affects the estimates of model parameters in the PRMs as well as in the linear regression

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Received: 19.10.2021; Accepted: 22.06.2022

models [3, 7, 8, 16, 22, 24, 28].

On the other hand, it is known that the performance of biased estimators proposed as an alternative to MLE in PRM is affected by the selection of the biasing parameters. In general, the methods used for the selection of biasing parameters have been adapted similarly to those used in linear regression models. For example, Månsson and Shukur [24], Kibria et al. [16] and Alanaz and Algamal [1] have proposed different methods for estimation of the biasing parameter k in the Poisson Ridge Estimator (PRE). Similarly, alternative methods for the estimation of d parameter in the Poisson Liu Estimator (PLE) are given by [24] and [27].

Moreover, the use of the biased estimators with two biasing parameters has become increasingly widespread as an alternative to the PRE and PLE. As the performances of these biased estimators depend on two biasing parameters, determining the optimum performance of these estimators becomes difficult. However, Cetinkaya and Kaçranlar [8] and Asar and Genc [7] proposed iterative techniques to estimate biasing parameters in the Poisson two-parameter Estimator (PTPE). More specifically, because of some constraints on the biasing parameters, firstly the value of the d parameter is constrained so that the biasing parameter k is positive. Then, k is estimated based on the biasing parameter d. In this case, it appears that there is a functional relationship between the biasing parameters k and d. Because of this relationship of biasing parameters, new biased estimators with a biasing parameter can be developed. Therefore, our primary aim in this study is to introduce a new general biased estimator under the assumption that it depends on an approximate functional relationship between the biasing parameters in order to alleviate the multicollinearity problem in PRM.

The organization of the article is as follows: In the next section, we will briefly describe the PRM and review some of the existing biased estimators used in PRMs. In Section 2, a new biased estimator named the improved Liu-type estimator is defined and some of its properties are given. The superiority of this estimator over the other biased estimators under the matrix mean square error criteria are shown in Section 3. In Section 4, the approaches used to determine the biasing parameters for proposed biased estimators are summarized. Furthermore, several methods are proposed to determine the biasing parameters. Also, Monte Carlo simulation studies are executed in Section 5. In Section 6, a real data application is provided to illustrate the performances of the proposed biased estimators. Finally, conclusions of the study are given in Section 7.

1.1. Maximum likelihood estimator and some biased estimators for PRM

In the PRM, y_i is the response variable and follows a Poisson distribution with mean μ_i, then the probability function is defined as

$$f\left(y_i\right) = \frac{e^{-\mu_i}\mu_i^{y_i}}{y_i!},\ i = 1, 2, ..., n,\ y_i = 0, 1, 2, ... \tag{1.1}$$

where μ_i is expressed by using canonical log link function and a linear combination of explanatory variables as follows $\mu_i = \exp(x_i'\beta)$, where x_i' is the ith row of X, which is an $n \times (p+1)$ data matrix with p explanatory variables and β is a $(p+1) \times 1$ vector of coefficients. The maximum likelihood method is the well-known estimation technique to estimate the vector of coefficients β. To use the Maximum Likelihood method, firstly log likelihood function is given as follows:

$$\begin{aligned} l(\beta) &= \sum_{i=1}^{n}[y_i \log(\mu_i) - \mu_i] - \log(\prod_{i=1}^{n} y_i!) \\ &= \sum_{i=1}^{n}[y_i \log(\exp(x_i'\beta)) - \exp(x_i'\beta)] - \log(\prod_{i=1}^{n} y_i!). \end{aligned} \tag{1.2}$$

The MLE of β is obtained by maximizing the log-likelihood function, so the following equations are obtained as

$$S(\beta) = \frac{\partial l(\beta; y)}{\partial \beta} = \sum_{i=1}^{n} (y_i - \exp(x_i\prime\beta))x_i = 0. \tag{1.3}$$

Since Eq. (1.3) is nonlinear in β, the solution of $S(\beta)$ is found using the following iteratively reweighted least squares (IRLS) algorithm

$$\hat{\beta}_{MLE} = \left(X'\hat{W}X\right)^{-1} X'\hat{W}Z, \tag{1.4}$$

where Z is a vector with the ith element $z_i = \log(\hat{\mu}_i) + \frac{y_i - \hat{\mu}_i}{\hat{\mu}_i}$ and $\hat{W} = \text{diag}[\hat{\mu}_i]$. The iterations end when the difference between the old and updated values is less than a specified small value, which is usually 10^{-8} [9]. The asymptotic covariance matrix of $\hat{\beta}_{MLE}$ is $cov\left(\hat{\beta}_{MLE}\right) \approx \left(X'\hat{W}X\right)^{-1}$.

To alleviate the undesirable effects of multicollinearity, the biased estimators that are alternative to the MLE are generalized like that defined in the linear regression model. For example, Månsson and Shukur [24] proposed the PRE as follows:

$$\hat{\beta}_{PRE} = \left(X'\hat{W}X + kI\right)^{-1} X'\hat{W}X\hat{\beta}_{MLE}, \; k > 0, \tag{1.5}$$

where k is a biasing parameter. The PRE is the generalization of the Ridge estimator introduced by Hoerl and Kennard [12] for the linear regression model. Månsson et al. [22], Amin et al. [6] and Qasim et al. [27] defined the PLE as

$$\hat{\beta}_{PLE} = \left(X'\hat{W}X + I\right)^{-1} \left(X'\hat{W}X + dI\right) \hat{\beta}_{MLE}, \tag{1.6}$$

where $0 < d < 1$ is a biasing parameter. The PLE is the generalization of the Liu estimator introduced by [18] for the linear regression model.

In recent years, the estimators with two biasing parameters have been proposed as an alternative to PRE and PLE. The aim here is to encourage the use of more appropriate estimators by combining few estimators. In this context, Liu [19] introduced a new estimator which is based on the biasing parameters k and d. For the PRMs, Algamal [2] defined the Poisson Liu-type estimator (PLTE) as follows:

$$\hat{\beta}_{PLTE} = \left(X'WX + kI\right)^{-1} \left(X'WX - dI\right) \hat{\beta}_{MLE}, \tag{1.7}$$

where $k > 0$ and $d \in R$ are biasing parameters.

Moreover, Asar and Genc [7] and Cetinkaya and Kaçranlar [8] proposed another biased estimator with two biasing parameters with an expectation that the combination of two different estimators might inherit the advantages of both estimators, first defined by [26] for the linear regression models. The Poisson two-parameter Estimator (PTPE) is defined as

$$\hat{\beta}_{PTPE} = \left(X'WX + kI\right)^{-1} \left(X'WX + kdI\right) \hat{\beta}_{MLE}, \tag{1.8}$$

where $k > 0$ and $0 < d < 1$ are biasing parameters.

Following [14], Lukman et al. [20] proposed another biased estimator as follows:

$$\hat{\beta}_{PKLE} = \left(X'WX + kI\right)^{-1} \left(X'WX - kI\right) \hat{\beta}_{MLE}, \tag{1.9}$$

where k is a biasing parameter.

2. A new general biased estimator

Kurnaz and Akay [17] introduced a new general Liu-type estimator to alleviate the effects of multicollinearity in linear regression models. We can generalize this estimator to use in PRMs as follows:

$$\hat{\beta}_{ILTE} = \left(X'\hat{W}X + kI\right)^{-1}\left(X'\hat{W}X + f\left(k\right)I\right)\hat{\beta}^{*},\ k > 0, \tag{2.1}$$

where $\hat{\beta}^{*}$ is any estimator of β, k is a biasing parameter and $f(k)$ is a continuous function of the biasing parameter k. Note that k is used to control the conditioning of the $X'\hat{W}X$ matrix, while $f(k)$ is used to improve the fit and statistical property.

When we selected $f(k)$ as a linear function of k such as $f(k) = ak + b$ where $a, b \in R$, the Improved Liu-type Estimator (ILTE) becomes a general estimator which includes the other biased estimators as special cases:

- $\hat{\beta}_{ILTE} = \hat{\beta}_{MLE}$, for $\hat{\beta}^{*} = \hat{\beta}_{MLE}$ and $f(k) = k$ where $a = 1$ and $b = 0$.
- $\hat{\beta}_{ILTE} = \hat{\beta}_{PRE}$, for $\hat{\beta}^{*} = \hat{\beta}_{MLE}$ and $f(k) = 0$ where $a = 0$ and $b = 0$.
- $\hat{\beta}_{ILTE} = \hat{\beta}_{PLE}$, for $\hat{\beta}^{*} = \hat{\beta}_{MLE}$ and $f(1) = a + b$ where $a + b$ corresponds to the biasing parameter d.
- $\hat{\beta}_{ILTE} = \hat{\beta}_{PLTE}$, for $\hat{\beta}^{*} = \hat{\beta}_{MLE}$ and $f(k) = -b$ where b corresponds to the biasing parameter d.
- $\hat{\beta}_{ILTE} = \hat{\beta}_{PTPE}$, for $\hat{\beta}^{*} = \hat{\beta}_{MLE}$ and $f(k) = ak$ where a corresponds to the biasing parameter d.
- $\hat{\beta}_{ILTE} = \hat{\beta}_{PKLE}$, for $\hat{\beta}^{*} = \hat{\beta}_{MLE}$ and $f(k) = -k$ where $a = -1$ and $b = 0$.

For the suitability of comparisons, we denote $\alpha = Q'\beta$, $\Lambda = diag(\lambda_1, ..., \lambda_{p+1}) = Q'(X'\hat{W}X)Q$, where $\lambda_1 \geq \lambda_2 \geq ...\lambda_{p+1} > 0$ are the ordered eigenvalues of $X'\hat{W}X$, Q is the orthogonal matrix whose columns constitute the eigenvectors of $X'\hat{W}X$ and the ith element of $Q'\beta$ is denoted as $\alpha_j, j = 1, 2, ..., p + 1$.

The asymptotic Scalar Mean Squared Error (SMSE) and the asymptotic Matrix Mean Squared Error (MMSE) of an estimator $\hat{\beta} = Z\hat{\beta}_{MLE}$, where Z is a matrix with proper order, are defined as

$$\begin{aligned} MMSE(\hat{\beta}) =& E(\hat{\beta} - \beta)(\hat{\beta} - \beta)' = Z(\hat{\beta}_{MLE} - \beta)(\hat{\beta}_{MLE} - \beta)'Z' + (Z\beta - \beta)(Z\beta - \beta)', \\ SMSE(\hat{\beta}) =& E(\hat{\beta} - \beta)'(\hat{\beta} - \beta) = (\hat{\beta}_{MLE} - \beta)'Z'Z(\hat{\beta}_{MLE} - \beta) + (Z\beta - \beta)'(Z\beta - \beta). \end{aligned} \tag{2.2}$$

Note that there is a relationship $SMSE(\hat{\beta}) = tr(MMSE(\hat{\beta}))$ between MMSE and SMSE criteria. Because of the relation of $\alpha = Q'\beta$; $\hat{\beta}_{MLE}, \hat{\beta}_{PRE}, \hat{\beta}_{PLE}, \hat{\beta}_{PLTE}, \hat{\beta}_{PTPE}$, $\hat{\beta}_{PKLE}$ and $\hat{\beta}_{ILTE}$ have the same SMSE values as $\hat{\alpha}_{MLE}, \hat{\alpha}_{PRE}, \hat{\alpha}_{PLE}, \hat{\alpha}_{PLTE}, \hat{\alpha}_{PTPE}$, $\hat{\alpha}_{PKLE}$ and $\hat{\alpha}_{ILTE}$, respectively.

Using Eqs. (1.5), (1.6), (1.7), (1.8), (1.9) and (2.1), it is easily computed that:

$$MMSE\left(\hat{\beta}_{MLE}\right) = Q\Lambda^{-1}Q', \tag{2.3}$$

$$MMSE\left(\hat{\beta}_{PRE}\right) = Q\left((\Lambda + kI)^{-1}\Lambda(\Lambda + kI)^{-1} + k^2(\Lambda + kI)^{-1}\alpha\alpha'(\Lambda + kI)^{-1}\right)Q', \tag{2.4}$$

$$\begin{aligned} MMSE\left(\hat{\beta}_{PLE}\right) =& Q\left((\Lambda + I)^{-1}\left(\Lambda + dI\right)\Lambda^{-1}\left(\Lambda + dI\right)(\Lambda + I)^{-1}\right. \\ &\left.+(d - 1)^2(\Lambda + I)^{-1}\alpha\alpha'(\Lambda + I)^{-1}\right)Q', \end{aligned} \tag{2.5}$$

$$\begin{aligned} MMSE\left(\hat{\beta}_{PLTE}\right) =& Q\left((\Lambda + kI)^{-1}\left(\Lambda - dI\right)\Lambda^{-1}\left(\Lambda - dI\right)(\Lambda + kI)^{-1}\right. \\ &\left.+(d + k)^2(\Lambda + kI)^{-1}\alpha\alpha'(\Lambda + kI)^{-1}\right)Q', \end{aligned} \tag{2.6}$$

$$MMSE\left(\hat{\beta}_{PTPE}\right) = Q\left((\Lambda+kI)^{-1}(\Lambda+kdI)\Lambda^{-1}(\Lambda+kdI)(\Lambda+kI)^{-1} + k^2(d-1)^2(\Lambda+kI)^{-1}\alpha\alpha'(\Lambda+kI)^{-1}\right)Q', \tag{2.7}$$

$$MMSE\left(\hat{\beta}_{PKLE}\right) = Q\left((\Lambda+kI)^{-1}(\Lambda-kI)\Lambda^{-1}(\Lambda-kI)(\Lambda+kI)^{-1} + 4k^2(\Lambda+kI)^{-1}\alpha\alpha'(\Lambda+kI)^{-1}\right)Q', \tag{2.8}$$

$$MMSE\left(\hat{\beta}_{ILTE}\right) = Q\left((\Lambda+kI)^{-1}(\Lambda+f(k)I)\Lambda^{-1}(\Lambda+f(k)I)(\Lambda+kI)^{-1} + (f(k)-k)^2(\Lambda+kI)^{-1}\alpha\alpha'(\Lambda+kI)^{-1}\right)Q'. \tag{2.9}$$

Moreover, we compute the SMSE functions of the biased estimators explicitly as follows:

$$SMSE(\hat{\beta}_{PRE}) = \sum_{j=1}^{p+1}\frac{\lambda_j}{(\lambda_j+k)^2} + \sum_{j=1}^{p+1}\frac{k^2\alpha_j^2}{(\lambda_j+k)^2}, \tag{2.10}$$

$$SMSE(\hat{\beta}_{PLE}) = \sum_{j=1}^{p+1}\frac{(\lambda_j+d)^2}{\lambda_j(\lambda_j+1)^2} + \sum_{j=1}^{p+1}\frac{(d-1)^2\alpha_j^2}{(\lambda_j+1)^2}, \tag{2.11}$$

$$SMSE(\hat{\beta}_{PLTE}) = \sum_{j=1}^{p+1}\frac{(\lambda_j-d)^2}{\lambda_j(\lambda_j+k)^2} + \sum_{j=1}^{p+1}\frac{(d+k)^2\alpha_j^2}{(\lambda_j+k)^2}, \tag{2.12}$$

$$SMSE(\hat{\beta}_{PTPE}) = \sum_{j=1}^{p+1}\frac{(\lambda_j+kd)^2}{\lambda_j(\lambda_j+k)^2} + \sum_{j=1}^{p+1}\frac{k^2(1-d)^2\alpha_j^2}{(\lambda_j+k)^2}, \tag{2.13}$$

$$SMSE(\hat{\beta}_{PKLE}) = \sum_{j=1}^{p+1}\frac{(\lambda_j-k)^2}{\lambda_j(\lambda_j+k)^2} + \sum_{j=1}^{p+1}\frac{4k^2\alpha_j^2}{(\lambda_j+k)^2}, \tag{2.14}$$

$$SMSE(\hat{\beta}_{ILTE}) = \sum_{j=1}^{p+1}\frac{(\lambda_j+f(k))^2}{\lambda_j(\lambda_j+k)^2} + \sum_{j=1}^{p+1}\frac{(f(k)-k)^2\alpha_j^2}{(\lambda_j+k)^2}, \tag{2.15}$$

where the first term is the asymptotic variance and the second term is the squared bias.

Let $\hat{\beta}_1$ and $\hat{\beta}_2$ be any two estimators of β. Then, $\hat{\beta}_2$ is superior to $\hat{\beta}_1$ with respect to the MMSE criterion if and only if $MMSE(\hat{\beta}_1) - MMSE(\hat{\beta}_2)$ is a positive definite (pd) matrix. If $MMSE(\hat{\beta}_1) - MMSE(\hat{\beta}_2)$ is a non-negative definite (nnd) matrix, then $SMSE(\hat{\beta}_1) - SMSE(\hat{\beta}_2) \geq 0$. But, the reverse is not always true [30].

We use the following theorem to compare the above-biased estimators in terms of MMSE sense.

Theorem 2.1 ([10]). *Let A be a positive definite matrix, namely $A > 0$, and c nonzero vector. Then, $A - cc'$ is a positive definite matrix iff $c'A^{-1}c \leq 1$.*

3. The superiority of the new improved Liu-type estimator in PRMs

In this section, we compare the ILTE with the MLE, PRE, PLE, PLTE, PTPE and PKLE according to the MMSE criterion.

The following theorem is given to show the superiority of ILTE over MLE.

Theorem 3.1. *Let be $k > 0$ and $-2\lambda_j - k < f(k) < k$. Then, $MMSE(\hat{\beta}_{MLE}) - MMSE(\hat{\beta}_{ILTE}) > 0$ iff*

$$bias(\hat{\beta}_{ILTE})'Q(\Lambda^{-1} - (\Lambda+kI)^{-1}(\Lambda+f(k)I)\Lambda^{-1}(\Lambda+kI)^{-1}(\Lambda+f(k)I))^{-1} \times Q'bias(\hat{\beta}_{ILTE}) < 1, \tag{3.1}$$

where $bias(\hat{\beta}_{ILTE}) = (f(k) - k)Q(\Lambda + kI)^{-1}\alpha$.

Proof. Using Eqs. (2.3) and (2.9), we obtain

$$\begin{aligned} MMSE(\hat{\beta}_{MLE}) - MMSE(\hat{\beta}_{ILTE}) &= Q\Lambda^{-1}Q' - Q(\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + f(k)I) \\ &\quad \times (\Lambda + kI)^{-1}Q' - bias(\hat{\beta}_{ILTE})bias(\hat{\beta}_{ILTE})' \\ &= Qdiag\left\{\frac{1}{\lambda_j} - \frac{(\lambda_j + f(k))^2}{\lambda_j(\lambda_j + k)^2}\right\}_{j=1}^{p+1} Q' - bias(\hat{\beta}_{ILTE})bias(\hat{\beta}_{ILTE})'. \end{aligned}$$

In this case, we set $A = Q(\Lambda^{-1} - (\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + kI)^{-1}(\Lambda + f(k)I))Q'$ according to Theorem 2.1. The matrix A, that is $\Lambda^{-1} - (\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + kI)^{-1}(\Lambda + f(k)I)$, is the pd matrix if $(\lambda_j + k)^2 - (\lambda_j + f(k))^2 > 0$, which is equivalent to $(k - f(k))(2\lambda_j + k + f(k)) > 0$ where $j = 1, 2, ..., p + 1$. $(k - f(k))(2\lambda_j + k + f(k)) > 0$ is equivalent to $-2\lambda_j - k < f(k) < k$ and $k > 0$. Thus $\Lambda^{-1} - (\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + kI)^{-1}(\Lambda + f(k)I)$ is the pd matrix if $-2\lambda_j - k < f(k) < k$ and $k > 0$. By Theorem 2.1, the proof is completed. □

To show the superiority of the estimator ILTE over PRE, the following theorem is given.

Theorem 3.2. *Let be* $k > 0$ *and* $-2\lambda_j < f(k) < 0$. *Then,* $MMSE(\hat{\beta}_{PRE}) - MMSE(\hat{\beta}_{ILTE}) > 0$ *iff*

$$bias(\hat{\beta}_{ILTE})'(MMSE(\hat{\beta}_{PRE}) - Q(\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + kI)^{-1}(\Lambda + f(k)I)Q')^{-1} \times bias(\hat{\beta}_{ILTE}) < 1, \tag{3.2}$$

where $bias(\hat{\beta}_{ILTE}) = (f(k) - k)Q(\Lambda + kI)^{-1}\alpha$.

Proof. Using Eqs. (2.4) and (2.9), we obtain

$$\begin{aligned} MMSE(\hat{\beta}_{PRE}) - MMSE(\hat{\beta}_{ILTE}) &= Q((\Lambda + kI)^{-1}\Lambda(\Lambda + kI)^{-1} - (\Lambda + kI)^{-1}(\Lambda + f(k)I) \\ &\quad \times \Lambda^{-1}(\Lambda + kI)^{-1}(\Lambda + f(k)I))Q' + bias(\hat{\beta}_{PRE})bias(\hat{\beta}_{PRE})' \\ &\quad - bias(\hat{\beta}_{ILTE})bias(\hat{\beta}_{ILTE})' \\ &= Qdiag\left\{\frac{\lambda_j}{(\lambda_j + k)^2} - \frac{(\lambda_j + f(k))^2}{\lambda_j(\lambda_j + k)^2}\right\}_{j=1}^{p+1} Q' \\ &\quad + bias\left(\hat{\beta}_{PRE}\right) bias\left(\hat{\beta}_{PRE}\right)' - bias\left(\hat{\beta}_{ILTE}\right) bias\left(\hat{\beta}_{ILTE}\right)'. \end{aligned}$$

$(\Lambda + kI)^{-1}\Lambda(\Lambda + kI)^{-1} - (\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + kI)^{-1}(\Lambda + f(k)I)$ is the pd matrix if $\lambda_j^2 - (\lambda_j + f(k))^2 > 0$, which is equivalent to $f(k)(2\lambda_j + f(k)) < 0$ where $j = 1, 2, ..., p+1$. Thus $(\Lambda + kI)^{-1}\Lambda(\Lambda + kI)^{-1} - (\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + kI)^{-1}(\Lambda + f(k)I)$ is the pd matrix if $-2\lambda_j < f(k) < 0$ and $k > 0$. By Theorem 2.1, the proof is completed. □

The following theorem is given to show the superiority of ILTE over PLE.

Theorem 3.3. *Let be* $k > 0$ *and* $0 < d < 1$. $MMSE(\hat{\beta}_{PLE}) - MMSE(\hat{\beta}_{ILTE}) > 0$ *iff*

$$bias(\hat{\beta}_{ILTE})'(MMSE(\hat{\beta}_{PLE}) - Q(\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + kI)^{-1}(\Lambda + f(k)I)Q')^{-1} \times bias(\hat{\beta}_{ILTE}) < 1, \tag{3.3}$$

where $-\lambda_j - \frac{(\lambda_j+k)(\lambda_j+d)}{(\lambda_j+1)} < f(k) < -\lambda_j + \frac{(\lambda_j+k)(\lambda_j+d)}{(\lambda_j+1)}$.

Proof. Using Eqs. (2.5) and (2.9), we obtain

$$\begin{aligned}MMSE(\hat{\beta}_{PLE}) - MMSE(\hat{\beta}_{ILTE}) &= Q((\Lambda+I)^{-1}(\Lambda+dI)\Lambda^{-1}(\Lambda+dI)(\Lambda+I)^{-1} \\ &- (\Lambda+kI)^{-1}(\Lambda+f(k)I)\Lambda^{-1}(\Lambda+kI)^{-1}(\Lambda+f(k)I))Q' \\ &+ bias(\hat{\beta}_{PLE})bias(\hat{\beta}_{PLE})' - bias(\hat{\beta}_{ILTE})bias(\hat{\beta}_{ILTE})' \\ &= Qdiag\left\{\frac{(\lambda_j+d)^2}{(\lambda_j+1)^2\lambda_j} - \frac{(\lambda_j+f(k))^2}{\lambda_j(\lambda_j+k)^2}\right\}_{j=1}^{p+1} Q' \\ &+ bias\left(\hat{\beta}_{PLE}\right) bias\left(\hat{\beta}_{PLE}\right)' - bias\left(\hat{\beta}_{ILTE}\right) bias\left(\hat{\beta}_{ILTE}\right)'.\end{aligned}$$

$(\Lambda+I)^{-1}(\Lambda+dI)\Lambda^{-1}(\Lambda+dI)(\Lambda+I)^{-1} - (\Lambda+kI)^{-1}(\Lambda+f(k)I)\Lambda^{-1}(\Lambda+kI)^{-1}(\Lambda+f(k)I)$ is the pd matrix if $(\frac{\lambda_j+d}{\lambda_j+1} - \frac{\lambda_j+f(k)}{\lambda_j+k}) > 0$, which is equivalent to $f(k) < -\lambda_j + \frac{(\lambda_j+k)(\lambda_j+d)}{(\lambda_j+1)}$, and $(\frac{\lambda_j+d}{\lambda_j+1} + \frac{\lambda_j+f(k)}{\lambda_j+k}) > 0$, which is equivalent to $-\lambda_j - \frac{(\lambda_j+k)(\lambda_j+d)}{(\lambda_j+1)} < f(k)$ where $j = 1, 2, ..., p+1$. Thus, $(\Lambda+I)^{-1}(\Lambda+dI)\Lambda^{-1}(\Lambda+dI)(\Lambda+I)^{-1} - (\Lambda+kI)^{-1}(\Lambda+f(k)I)\Lambda^{-1}(\Lambda+kI)^{-1}(\Lambda+f(k)I)$ is the pd matrix if $-\lambda_j - \frac{(\lambda_j+k)(\lambda_j+d)}{(\lambda_j+1)} < f(k) < -\lambda_j + \frac{(\lambda_j+k)(\lambda_j+d)}{(\lambda_j+1)}$ where $k > 0$, $0 < d < 1$ and $j = 1, 2, ..., p+1$. By Theorem 2.1, the proof is completed. □

To show the superiority of the estimator ILTE over PLTE, the following theorem is given.

Theorem 3.4. *Let us consider $d - 2\lambda_j < f(k) < -d$ or $-d < f(k) < d - 2\lambda_j$ where $k > 0$ and $d \in R$. Then, $MMSE(\hat{\beta}_{PLTE}) - MMSE(\hat{\beta}_{ILTE}) > 0$ iff*

$$\begin{aligned}bias(\hat{\beta}_{ILTE})'(MMSE(\hat{\beta}_{PLTE}) - Q(\Lambda+kI)^{-1}(\Lambda+f(k)I)\Lambda^{-1}(\Lambda+kI)^{-1}(\Lambda+f(k)I)Q')^{-1} \\ \times bias(\hat{\beta}_{ILTE}) < 1, \qquad (3.4)\end{aligned}$$

where $bias(\hat{\beta}_{ILTE}) = (f(k) - k)Q(\Lambda+kI)^{-1}\alpha$.

Proof. Using Eqs. (2.6) and (2.9), we obtain

$$\begin{aligned}MMSE(\hat{\beta}_{PLTE}) - MMSE(\hat{\beta}_{ILTE}) &= Q((\Lambda+kI)^{-1}(\Lambda-dI)\Lambda^{-1}(\Lambda-dI)(\Lambda+kI)^{-1} \\ &- (\Lambda+kI)^{-1}(\Lambda+f(k)I)\Lambda^{-1}(\Lambda+kI)^{-1}(\Lambda+f(k)I))Q' \\ &+ bias(\hat{\beta}_{PLTE})bias(\hat{\beta}_{PLTE})' - bias(\hat{\beta}_{ILTE})bias(\hat{\beta}_{ILTE})' \\ &= Qdiag\left\{\frac{(\lambda_j-d)^2}{\lambda_j(\lambda_j+k)^2} - \frac{(\lambda_j+f(k))^2}{\lambda_j(\lambda_j+k)^2}\right\}_{j=1}^{p+1} Q' \\ &+ bias(\hat{\beta}_{PLTE})bias(\hat{\beta}_{PLTE})' - bias(\hat{\beta}_{ILTE})bias(\hat{\beta}_{ILTE})'\end{aligned}$$

$(\Lambda+kI)^{-1}(\Lambda-dI)\Lambda^{-1}(\Lambda-dI)(\Lambda+kI)^{-1} - (\Lambda+kI)^{-1}(\Lambda+f(k)I)\Lambda^{-1}(\Lambda+kI)^{-1}(\Lambda+f(k)I)$ is the pd matrix if $d \leq 0$ and $d - 2\lambda_j < f(k) < -d$ or $0 < \lambda_i < d$ and $-d < f(k) < d - 2\lambda_j$ or $0 < d < \lambda_i$ and $d - 2\lambda_j < f(k) < -d$ where $k > 0$ and $j = 1, 2, ..., p+1$. Then, the proof is completed using Theorem 2.1. □

To show the superiority of the estimator ILTE over PTPE, the following theorem is given.

Theorem 3.5. *Let us consider $-2\lambda_j - kd < f(k) < kd$ where $k > 0$ and $0 < d < 1$. Then, $MMSE(\hat{\beta}_{PTPE}) - MMSE(\hat{\beta}_{ILTE}) > 0$ iff*

$$\begin{aligned}bias(\hat{\beta}_{ILTE})'(MMSE(\hat{\beta}_{PTPE}) - Q(\Lambda+kI)^{-1}(\Lambda+f(k)I)\Lambda^{-1}(\Lambda+kI)^{-1}(\Lambda+f(k)I)Q')^{-1} \\ \times bias(\hat{\beta}_{ILTE}) < 1, \qquad (3.5)\end{aligned}$$

where $bias(\hat{\beta}_{ILTE}) = (f(k) - k)Q(\Lambda+kI)^{-1}\alpha$.

Proof. Using Eqs. (2.7) and (2.9), we obtain

$$\begin{aligned} MMSE(\hat{\beta}_{PTPE}) - MMSE(\hat{\beta}_{ILTE}) &= Q((\Lambda + kI)^{-1}(\Lambda + kdI)\Lambda^{-1}(\Lambda + kdI)(\Lambda + kI)^{-1} \\ &- (\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + kI)^{-1}(\Lambda + f(k)I))Q' \\ &+ bias(\hat{\beta}_{PTPE})bias(\hat{\beta}_{PTPE})' - bias(\hat{\beta}_{ILTE})bias(\hat{\beta}_{ILTE})' \\ &= Qdiag\left\{ \frac{(\lambda_j + kd)^2}{\lambda_j(\lambda_j + k)^2} - \frac{(\lambda_j + f(k))^2}{\lambda_j(\lambda_j + k)^2} \right\}_{j=1}^{p+1} Q' \\ &+ bias(\hat{\beta}_{PTPE})bias(\hat{\beta}_{PTPE})' - bias(\hat{\beta}_{ILTE})bias(\hat{\beta}_{ILTE})'. \end{aligned}$$

$(\Lambda + kI)^{-1}(\Lambda + kdI)\Lambda^{-1}(\Lambda + kdI)(\Lambda + kI)^{-1} - (\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + kI)^{-1}(\Lambda + f(k)I)$ is the pd matrix if $(kd - f(k)) > 0$, which is equivalent to $f(k) < kd$, and $(2\lambda_j + kd + f(k)) > 0$, which is equivalent to $-2\lambda_j - kd < f(k)$ where $j = 1, 2, ..., p + 1$. Thus $(\Lambda + kI)^{-1}(\Lambda + kdI)\Lambda^{-1}(\Lambda + kdI)(\Lambda + kI)^{-1} - (\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + kI)^{-1}(\Lambda + f(k)I)$ is the pd matrix if $-2\lambda_j - kd < f(k) < kd$ where $k > 0$, $0 < d < 1$ and $j = 1, 2, ..., p + 1$. By Theorem 2.1, the proof is completed. □

To show the superiority of the estimator ILTE over PKLE, the following theorem is given.

Theorem 3.6. *Let us consider $k - 2\lambda_j < f(k) < -k$ or $-k < f(k) < k - 2\lambda_j$ where $k > 0$ and $j = 1, 2, ..., p + 1$. Then, $MMSE(\hat{\beta}_{PKLE}) - MMSE(\hat{\beta}_{ILTE}) > 0$ iff*

$$bias(\hat{\beta}_{ILTE})'(MMSE(\hat{\beta}_{PKLE}) - Q(\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + kI)^{-1}(\Lambda + f(k)I)Q')^{-1} \times bias(\hat{\beta}_{ILTE}) < 1, \quad (3.6)$$

where $bias(\hat{\beta}_{ILTE}) = (f(k) - k)Q(\Lambda + kI)^{-1}\alpha$.

Proof. Using Eqs. (2.8) and (2.9), we obtain

$$\begin{aligned} MMSE(\hat{\beta}_{PKLE}) - MMSE(\hat{\beta}_{ILTE}) &= Q((\Lambda + kI)^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - kI)(\Lambda + kI)^{-1} \\ &- (\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + kI)^{-1}(\Lambda + f(k)I))Q' \\ &+ bias(\hat{\beta}_{PKLE})bias(\hat{\beta}_{PKLE})' - bias(\hat{\beta}_{ILTE})bias(\hat{\beta}_{ILTE})' \\ &= Qdiag\left\{ \frac{(\lambda_j - k)^2}{\lambda_j(\lambda_j + k)^2} - \frac{(\lambda_j + f(k))^2}{\lambda_j(\lambda_j + k)^2} \right\}_{j=1}^{p+1} Q' \\ &+ bias\left(\hat{\beta}_{PKLE}\right) bias\left(\hat{\beta}_{PKLE}\right)' - bias\left(\hat{\beta}_{ILTE}\right) bias\left(\hat{\beta}_{ILTE}\right)'. \end{aligned}$$

$(\Lambda + kI)^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - kI)(\Lambda + kI)^{-1} - (\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + kI)^{-1}(\Lambda + f(k)I)$ is the pd matrix if $k - 2\lambda_j < f(k) < -k$ or $-k < f(k) < k - 2\lambda_j$ where $k > 0$ and $j = 1, 2, ..., p + 1$. Thus, the proof is completed by using Theorem 2.1. □

4. Determination of $f(k)$ function

Since the performances of biased estimators depend on the estimates of biasing parameters, it is an important problem to find the optimal biasing parameters for these biased estimators. To estimate the biasing parameters in PRE, PLE, PLTE, PTPE and PKLE methods used in the linear regression models were adapted. Generally, the estimates of the biasing parameters are obtained in such a way that the SMSEs are minimized. Note that the SMSEs given by Eqs. (2.9) to (2.13) are a function of the biasing parameters and the unknown parameter α. These functions are sometimes quadratic, sometimes nonlinear functions of the biasing parameters. In some cases, for the estimates of the biasing parameters, approximate methods have been proposed because of the SMSE is not a linear function of the biasing parameter. This situation becomes even more complicated for the estimators with two biasing parameters.

Note that different approaches have been proposed for the selection of the biasing parameter in biased estimators with two biasing parameters. In general, the biasing parameter

k is considered to be a constant, and then parameter d is estimated or vice versa. More specifically, since the biasing parameter k is positive, firstly the value of the parameter d is constrained, and then the parameter k is estimated by using arithmetic mean or geometric mean, or harmonic mean. Besides, iterative techniques have been developed for the estimation of biasing parameters. In this case, iterative techniques are also not successful because of the constraints on the biasing parameters.

The main advantage of the proposed biased estimator over the estimators with two biasing parameters is based on the prior knowledge of an approximate functional relationship between the biasing parameters. The performance of the proposed ILTE is based on the $f(k)$ function, and therefore has only the biasing parameter k. The proper choice of $f(k)$ function result in different biased estimators. We may give a method to find the optimal $f(k)$ function minimizing $SMSE\left(\hat{\beta}_{ILTE}\right)$ according to k parameter. Remember that $SMSE\left(\hat{\beta}_{ILTE}\right)$ is a nonlinear function of k parameter. So, writing $h(k) = SMSE\left(\hat{\beta}_{ILTE}\right)$, we have

$$h(k) = \sum_{j=1}^{p+1} \frac{(\lambda_j + f(k))^2}{\lambda_j(\lambda_j + k)^2} + \sum_{j=1}^{p+1} \frac{(f(k) - k)^2 \alpha_j^2}{(\lambda_j + k)^2}. \tag{4.1}$$

Then, we find $h'(k)$ as follows differentiating $h(k)$ with respect to k,

$$h'(k) = \sum_{j=1}^{p+1} \frac{(f'(k)(\lambda_j + k) - (f(k) + \lambda_j))\left(2(\lambda_j + f(k)) + 2\lambda_j \alpha_j^2 (f(k) - k)\right)}{\lambda_j(\lambda_j + k)^3}. \tag{4.2}$$

When it is accepted $h'(k) = 0$, we have two facts as follows:

Fact 1. $f'(k)(\lambda_j + k) - (f(k) + \lambda_j) = 0$. From this equation we obtain

$$f(k) = c_1 k + (c_1 - 1)\lambda_j, \quad j = 1, 2, ..., p+1 \tag{4.3}$$

where c_1 is the constant of integration.

Fact 2. $(\lambda_j + f(k)) + \lambda_j \alpha_j^2 (f(k) - k) = 0$. From this equation we obtain

$$f(k) = \frac{\lambda_j \alpha_j^2}{1 + \lambda_j \alpha_j^2} k + \left(\frac{\lambda_j \alpha_j^2}{1 + \lambda_j \alpha_j^2} - 1\right)\lambda_j, \; j = 1, 2, ..., p+1$$

or

$$f(k) = \frac{\lambda_j \alpha_j^2}{1 + \lambda_j \alpha_j^2} k - \frac{\lambda_j}{1 + \lambda_j \alpha_j^2}, \; j = 1, 2, ..., p+1. \tag{4.4}$$

According to Fact 1 and Fact 2, the selection of $f(k) = ak + b$ where $a, b \in R$ as a linear function of the biasing parameter k is appropriate. Note that, $f(k)$ which is given in Fact 1 is a solution of the differential equation, which is obtained in Fact 2. Also, depending on the functions obtained in Fact 1 and Fact 2, we can make the following generalizations. Firstly, note that $f(k)$ given in Eqs. (4.3) and (4.4) makes the $SMSE\left(\hat{\beta}_{ILTE}\right)$ function approximately minimum for a j value. Also, the function $f(k)$ depends on the eigenvalues of $X'WX$, the unknown parameter α and the parameter k. In order to determine this function, we can propose several function approximations by using Eq. (4.4). In this paper, we used the following functions for the determination of $f(k)$ as follows:

$$f_1(k) = ck + (c-1)\lambda_{\min} \text{ where } c \in (0,1), \tag{4.5}$$

$$f_2(k) = \frac{\lambda_{\min}\alpha^2_{\min}}{p + \lambda_{\max}\alpha^2_{\max}}k + \left(\frac{\lambda_{\min}\alpha^2_{\min}}{p + \lambda_{\max}\alpha^2_{\max}} - 1\right)\lambda_{\min}, \tag{4.6}$$

$$f_3(k) = \frac{\lambda_{\min}\alpha^2_{\min}}{1 + \lambda_{\max}\alpha^2_{\max}}k + \left(\frac{\lambda_{\min}\alpha^2_{\min}}{1 + \lambda_{\max}\alpha^2_{\max}} - 1\right)\lambda_{\min}, \tag{4.7}$$

$$f_4(k) = \frac{\lambda_{\min}\alpha^2_{\min}}{n(1 + \lambda_{\max}\alpha^2_{\max})}k + \left(\frac{\lambda_{\min}\alpha^2_{\min}}{n(1 + \lambda_{\max}\alpha^2_{\max})} - 1\right)\lambda_{\min}, \tag{4.8}$$

$$f_5(k) = \frac{\lambda_{\min}\alpha^2_{\min}}{1 + \lambda_{\max}\alpha^2_{\max}}k - \frac{\lambda_{\min}}{1 + \lambda_{\min}\alpha^2_{\min}}, \tag{4.9}$$

where $\alpha^2_{\min}$ and $\alpha^2_{\max}$ is defined as the minimum and maximum value of $\alpha^2_j, j = 1, 2, ..., p+1$. Similarly, $\lambda_{\min}$ and $\lambda_{\max}$ is defined as the minimum and maximum eigenvalue of $X'\hat{W}X$, respectively.

In this paper, for the determination of $f(k)$ function, we will examine only the first degree polynomial functions given in Eqs. (4.5) to (4.9). However, it is clear that the function $f(k)$ can be selected as any continuous function of the biasing parameter k. Since the proposed estimator will depend on a single biasing parameter k, the suitable estimates of k can be used [16]. Based on the simulation studies, we can used the following estimators to estimate k in the ILTEs,

$$\hat{k}_{ILTE} = \frac{\lambda_{\max} - 4\lambda_{\min}}{3}, \tag{4.10}$$

$$\hat{k}_{ILTE} = \frac{\lambda_{\max} - 3\lambda_{\min}}{2}, \tag{4.11}$$

$$\hat{k}_{ILTE} = \frac{\lambda_{\max} - \lambda_{\min}}{p}, \tag{4.12}$$

where p is number of explanatory variables. We should note that k in the ILTEs must be estimated in such a way as to control the conditioning of the $X'\hat{W}X$ matrix.

5. The Monte Carlo simulation study

Many authors executed several simulation studies to compare the performances of the proposed biased estimators in PRMs in the presence of multicollinearity. Similarly, we will design a simulation study to compare the performance of the proposed biased estimator with respect to other proposed biased estimators. We will investigate the effects of sample size (n), the degree of the collinearity (ρ) and the number of the explanatory variables (p) on the comparison of the biased estimators.

The dependent variable of the PRM is generated using pseudo-random numbers from the $Poisson(\mu_i)$ distribution, where

$$\mu_i = \exp(\beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip}), i = 1, 2, \ldots n, j = 1, 2, \ldots p.$$

Similarly, we generate the explanatory variables by following [7] and [8] as $x_{ij} = (1-\rho^2)^{1/2} w_{ij} + \rho w_{ip+1},\ i = 1, 2, ..., n,\ j = 1, 2, ..., p$ where w_{ij} are independent standard normal pseudo-random numbers and ρ is specified so that the correlation between any two variables is given by ρ^2. Four different sets of correlations are investigated corresponding to $\rho = 0.8, 0.9, 0.99$ and 0.999.

The explanatory variables are then standardized by using unit length scaling so that $X'X$ is a matrix of correlations. Number of explanatory variables is chosen as $p = 2, p = 4$

and $p = 8$. The sample sizes are taken as $n = 25, 50, 100$ and 200. For each set of explanatory variables, β is chosen as the normalized eigenvector corresponding to the largest eigenvalue of $X'X$ so that $\beta'\beta = 1$. In estimating the model parameters, we use glm() algorithm in R with the convergence criterion as default epsilon $=10^{-8}$ [9]. We also set the intercept equals 0.

The best estimation of the biasing parameter for the PRE, PLE, PLTE, PTPE and PKLE in the simulation and application sections is defined based on [1,7,8,16,21,22,24,27].

To estimate the biasing parameter k in PRE, we used the best estimate of k as $\hat{k}_{PRE} = \max\left(\frac{1}{m_j}\right)$ where $m_j = \sqrt{\frac{\hat{\sigma}^2}{\hat{\alpha}_j^2}}, j = 1, 2, ..., p$ and $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(y_i-\hat{\mu}_i)^2}{n-p-1}$ which is recommended by [16].

Based on the results given by [27], we use the best estimation of d in PLE as $\hat{d}_{PLE} = \max\left(0, \min\left(\frac{\hat{\alpha}_j^2-1}{\max\left(\frac{1}{\lambda_j}\right)+\hat{\alpha}_{\max}^2}\right)\right)$. For PLTE, the biasing parameters k and d were estimated by grouping them in five different ways as follows;

PLTE I: $\hat{k}_{PLTE} = \max\left(\frac{1}{m_j}\right)$ where $m_j = \sqrt{\frac{\hat{\sigma}^2}{\hat{\alpha}_j^2}}$, $j = 1, 2, ..., p$ and $\hat{d}_{PLTE} = \frac{\sum_{j=1}^{p}\frac{1-\hat{k}_{PLTE}\hat{\alpha}_j^2}{\left(\lambda_j+\hat{k}_{PLTE}\right)^2}}{\sum_{j=1}^{p}\frac{1+\lambda_j\hat{\alpha}_j^2}{\lambda_j\left(\lambda_j+\hat{k}_{PLTE}\right)^2}}$.

PLTE II: $\hat{k}_{PLTE} = \frac{\lambda_1-100\lambda_p}{99}$ and $\hat{d}_{PLTE} = \frac{\sum_{j=1}^{p}\frac{1-\hat{k}_{PLTE}\hat{\alpha}_j^2}{\left(\lambda_j+\hat{k}_{PLTE}\right)^2}}{\sum_{j=1}^{p}\frac{1+\lambda_j\hat{\alpha}_j^2}{\lambda_j\left(\lambda_j+\hat{k}_{PLTE}\right)^2}}$.

PLTE III: $\hat{d}_{PLTE} = \frac{1}{2}\min\left\{\frac{\lambda_j}{1+\lambda_j\hat{\alpha}_j^2}\right\}$, $j = 1, 2, ..., p$ and $\hat{k}_{PLTE} = \frac{1}{p}\sum_{j=1}^{p}\frac{\lambda_j-\hat{d}^*_{PLTE}\left(1+\lambda_j\hat{\alpha}_j^2\right)}{\lambda_j\hat{\alpha}_j^2}$

PLTE IV: $\hat{d}_{PLTE} = \frac{1}{2}\min\left\{\frac{\lambda_j}{1+\lambda_j\hat{\alpha}_j^2}\right\}$, $j = 1, 2, ..., p$ and
$\hat{k}_{PLTE} = \left(\prod_{j=1}^{p}\left(\frac{\lambda_j-\hat{d}^*_{LLTE}\left(1+\lambda_j\hat{\alpha}_j^2\right)}{\lambda_j\hat{\alpha}_j^2}\right)\right)^{1/p}$

PLTE V: $\hat{d}_{PLTE} = \frac{1}{2}\min\left\{\frac{\lambda_j}{1+\lambda_j\hat{\alpha}_j^2}\right\}$, $j = 1, 2, ..., p$ and $\hat{k}_{PLTE} = \frac{p}{\sum_{j=1}^{p}\left(\frac{\lambda_j^2\hat{\alpha}_j^2}{\lambda_j-\hat{d}^*_{LLTE}\left(1+\lambda_j^2\hat{\alpha}_j^2\right)}\right)}$

For the PTPE, the iterative method used by [8] was used. For the iterative method proposed by [8], the pair of the biasing parameters k and d are grouped in three different ways. In these case, the estimates of the biasing parameters for three PTPEs are defined as follows:

PTPE I: $\hat{k}_{PTPE} = \frac{1}{p}\sum_{j=1}^{p}\frac{\hat{\sigma}^2}{\left[\hat{\alpha}_j^2-\hat{d}^*_{LTPE}\left(\frac{\hat{\sigma}^2}{\hat{\lambda}_j}+\hat{\alpha}_j^2\right)\right]}$ and $\hat{d}_{PTPE} = \frac{\sum_{j=1}^{p}\frac{\left(\hat{k}_{PTPE}\hat{\alpha}_j^2-\hat{\sigma}^2\right)}{\left(\hat{\lambda}_j+\hat{k}_{PTPE}\right)^2}}{\sum_{j=1}^{p}\frac{\hat{k}_{PTPE}\left(\hat{\sigma}^2+\hat{\alpha}_j^2\hat{\lambda}_j\right)}{\hat{\lambda}_j\left(\hat{\lambda}_j+\hat{k}_{PTPE}\right)^2}}$.

PTPE II: $\hat{k}_{PTPE} = \frac{\hat{\sigma}^2}{\prod_{i=1}^{p}\left[\hat{\alpha}_j^2-\hat{d}^*_{LTPE}\left(\frac{\hat{\sigma}^2}{\hat{\lambda}_j}+\hat{\alpha}_j^2\right)\right]^{1/p}}$ and $\hat{d}_{PTPE} = \frac{\sum_{j=1}^{p}\frac{\left(\hat{k}_{PTPE}\hat{\alpha}_j^2-\hat{\sigma}^2\right)}{\left(\hat{\lambda}_j+\hat{k}_{PTPE}\right)^2}}{\sum_{j=1}^{p}\frac{\hat{k}_{PTPE}\left(\hat{\sigma}^2+\hat{\alpha}_j^2\hat{\lambda}_j\right)}{\hat{\lambda}_j\left(\hat{\lambda}_j+\hat{k}_{PTPE}\right)^2}}$.

PTPE III: $\hat{k}_{PTPE} = \frac{p\hat{\sigma}^2}{\sum_{j=1}^{p}\left[\hat{\alpha}_j^2 - \hat{d}^*_{LTPE}\left(\frac{\hat{\sigma}^2}{\hat{\lambda}_j}+\hat{\alpha}_j^2\right)\right]}$ and $\hat{d}_{PTPE} = \frac{\sum_{j=1}^{p}\frac{\left(\hat{k}_{PTPE}\hat{\alpha}_j^2-\hat{\sigma}^2\right)}{\left(\hat{\lambda}_j+\hat{k}_{PTPE}\right)^2}}{\sum_{j=1}^{p}\frac{\hat{k}_{PTPE}\left(\hat{\sigma}^2+\hat{\alpha}_j^2\hat{\lambda}_j\right)}{\hat{\lambda}_j\left(\hat{\lambda}_j+\hat{k}_{PTPE}\right)^2}}$.

where $\hat{d}^*_{PTPE} = \frac{1}{2}\min\left\{\frac{\hat{\alpha}_j^2}{\frac{\hat{\sigma}^2}{\hat{\lambda}_j}+\hat{\alpha}_j^2}\right\}$ and $j = 1, 2, ..., p$. Also, if $\hat{d}_{LTPE}$ is negative, $\hat{d}_{LTPE} = \hat{d}^*_{LTPE}$ [8].

Asar and Genc [7] suggested to use the following choice of biasing parameters d and k as a best option which gives the lowest asymptotic MSE value of PTPE as follows:

PTPE IV: $\hat{d}_{PTPE} = \frac{1}{2}\min\left(\frac{\lambda_j\hat{\alpha}_j^2}{1+\lambda_j\hat{\alpha}_j^2}\right)$, $\quad \hat{k}_{PTPE} = \max\left(\frac{\lambda_j}{\lambda_j\alpha_j^2\left(1-\hat{d}_{PTPE}\right)-\hat{d}_{PTPE}}\right)$, $j = 1, 2, ..., p$.

For the PKLE, we use the following estimates of the biasing parameter k; $\hat{k}_{PKLE} = \sqrt{\max\left(0, \min\left(\frac{\lambda_i}{1+2\lambda_i\alpha_i^2}\right)\right)}$ $j = 1, 2, ..., p$, as suggested in [20].

The obtained results are reported in Tables 1 to 4, together with the following estimates of k and $f(k)$ functions.

ILTE I: $\hat{k}_{PLTE} = \frac{\lambda_{\max}-4\lambda_{\min}}{3}$ and $f(k) = \frac{\lambda_{\min}\alpha_{\min}^2}{p+\lambda_{\max}\alpha_{\max}^2}k + \left(\frac{\lambda_{\min}\alpha_{\min}^2}{p+\lambda_{\max}\alpha_{\max}^2} - 1\right)\lambda_{\min}$

ILTE II: $\hat{k}_{PLTE} = \frac{\lambda_{\max}-3\lambda_{\min}}{2}$ and $f(k) = \frac{\lambda_{\min}\alpha_{\min}^2}{n(1+p\lambda_{\max}\alpha_{\max}^2)}k + \left(\frac{\lambda_{\min}\alpha_{\min}^2}{n(1+p\lambda_{\max}\alpha_{\max}^2)} - 1\right)\lambda_{\min}$

ILTE III: $\hat{k}_{PLTE} = \frac{\lambda_{\max}-\lambda_{\min}}{p}$ and $f(k) = \frac{\lambda_{\min}\alpha_{\min}^2}{n(1+\lambda_{\max}\alpha_{\max}^2)}k + \left(\frac{\lambda_{\min}\alpha_{\min}^2}{n(1+\lambda_{\max}\alpha_{\max}^2)} - 1\right)\lambda_{\min}$

The performance of the estimated MSEs (EMSEs) is used as basis for comparing the proposed estimators which are calculated for an estimator $\hat{\beta}$ of β as

$$EMSE\left(\hat{\beta}\right) = \frac{1}{2000}\sum_{r=1}^{2000}\sum_{j=1}^{p}\left(\hat{\beta}_{rj} - \beta_j\right)^2,$$

where $\hat{\beta}_{rj}$ denotes the estimate of the jth parameter in rth replication and β_j is the true parameter values. For each case of n, p and ρ, the experiment was replicated 2000 times by generating response variable.

The bold numbers in the tables show the estimators with the smallest EMSE values, and in addition, the signs (**) and (***) represent the second and third smallest EMSE values, respectively.

The EMSE values for the different n, p, and ρ are given in Table 1-4. According to the values in the Tables, the results are summarized as follows:

i. As the degree of rho correlation ρ increases, regardless of the n and p values, EMSE values of MLE, PRE, PLTE1, PLTE2, PLTE4, PLTE5, PTPE1, PTPE2, PTPE3, PTPE4, PKLE increase while EMSE values of PLE, PLTE3, ILTE1, ILTE3, and ILTE4 decrease.

ii. Regardless of n and ρ, in the case where the number of variables p is 2, it was observed that ILTE II has the smallest EMSE values in all cases except for a few cases.

iii. In the case where ρ is 0.8 and 0.9, it was observed that ILTE II performs best as it has the smallest overall EMSE value.

iv. In the case where ρ is 0.99 and 0.999, and the number of variables p is 4, 8 and 12, ILTE III outperformed other estimators in all cases considered.

Table 1. The EMSE values of the estimators for the model when $p = 2$.

p	n	ρ	**MLE**	**PRE**	**PLE**	**PLTE1**	**PLTE2**	**PLTE3**	**PLTE4**	**PLTE5**	**PTPE1**	**PTPE2**	**PTPE3**	**PTPE4**	**PKLE**	**ILTE1**	**ILTE2**	**ILTE3**
2	25	0.8	2.1906	0.5964	0.7671	1.1713	1.3154	0.9074	0.9740	1.1673	1.4936	1.4867	1.5152	0.9539	0.6572	0.4400***	**0.3841***	0.4107**
2	25	0.9	6.7558	0.4936	0.5296	3.2310	3.2977	0.8182	1.4023	2.6612	4.3166	4.3164	4.3217	1.2431	0.5937	**0.3181***	0.3185**	0.3252***
2	25	0.99	50.9805	0.6094	0.3805	23.8110	23.8503	0.6251	3.0243	18.4141	30.8905	30.8989	30.9062	4.4957	13.4626	0.3302***	**0.3259***	0.3263**
2	25	0.999	365.3614	0.7269	0.3371	166.9668	166.8949	0.4620	4.2653	127.4819	226.6360	226.6387	226.6389	27.4771	219.3860	**0.3155***	0.3174**	0.3175***
2	50	0.8	2.6760	0.5792	0.7472	1.3797	1.5745	0.9613	1.0785	1.3537	1.8166	1.8097	1.8476	1.0023	0.6383	0.4848***	0.4030**	**0.3944***
2	50	0.9	6.5857	0.5067	0.5780	3.2776	3.3824	0.9394	1.5338	2.7615	4.3625	4.3654	4.3749	1.3231	0.6008	0.3424***	**0.3323***	0.3363**
2	50	0.99	45.9337	0.6033	0.3668	21.3185	21.3221	0.5904	2.7523	16.3888	28.1352	28.1399	28.1430	4.2213	11.7559	**0.3184***	0.3203**	0.3210***
2	50	0.999	607.5476	0.7964	0.3505	289.5969	289.5340	0.4971	6.2355	223.7744	385.2196	385.2297	385.2339	47.6157	400.8927	0.3432***	**0.3374***	0.3374**
2	100	0.8	3.1180	0.5457	0.6873	1.5812	1.7231	0.9316	1.1004	1.4674	2.0701	2.0693	2.0939	1.0208	0.5648	0.3947***	**0.3593***	0.3678**
2	100	0.9	4.6898	0.5334	0.6387	2.3261	2.4575	0.9344	1.2772	2.0247	3.1025	3.1066	3.1192	1.1405	0.5459	0.3770***	**0.3578***	0.3624**
2	100	0.99	51.6504	0.6270	0.3860	24.8723	24.9322	0.7001	3.5518	19.4659	32.7033	32.7076	32.7134	4.8645	13.2279	0.3517***	**0.3408***	0.3408**
2	100	0.999	389.7609	0.7653	0.3414***	185.1483	185.1317	0.5313	5.7399	142.7121	250.5518	250.5659	250.5754	30.7377	234.2662	0.3419	0.3295**	**0.3295***
2	200	0.8	2.7874	0.5856	0.7302	1.4818	1.6640	0.9423	1.0885	1.4238	1.8880	1.8827	1.9186	0.9948	0.6425	0.4502***	**0.4008***	0.4090**
2	200	0.9	5.5311	0.5386	0.6317	2.7217	2.8650	0.9613	1.3560	2.3338	3.5814	3.5854	3.6088	1.2213	0.5460	0.3817***	**0.3590***	0.3602**
2	200	0.99	46.8409	0.6234	0.3867	22.2652	22.3220	0.7013	3.4286	17.3702	28.4967	28.5025	28.5111	4.5943	11.6414	0.3481***	**0.3385***	0.3386**
2	200	0.999	530.0244	0.7797	0.3321	249.2439	249.1938	0.4961	5.0000	191.3571	332.5028	332.5172	332.5239	38.8103	346.0227	0.3310***	**0.3249***	0.3250**

Table 2. The EMSE values of the estimators for the model when $p = 4$.

p	n	ρ	**MLE**	**PRE**	**PLE**	**PLTE1**	**PLTE2**	**PLTE3**	**PLTE4**	**PLTE5**	**PTPE1**	**PTPE2**	**PTPE3**	**PTPE4**	**PKLE**	**ILTE1**	**ILTE2**	**ILTE3**
4	25	0.8	8.6660	0.5435	0.9286	3.1525	5.0153	1.0101	1.5983	3.1864	4.7766	4.7377	4.7996	0.9648	0.9426	0.4127**	**0.3471***	0.4134***
4	25	0.9	12.4273	0.4812	0.8335	4.5605	6.8358	1.1579	2.0725	4.5716	6.7607	6.7258	6.8195	1.0706	0.9773	0.3276**	**0.2982***	0.3469***
4	25	0.99	148.5465	0.5006	0.2475	49.3663	49.8092	0.9800	11.0515	49.4836	77.3315	77.3501	78.3900	3.0236	35.2005	0.1665**	0.1909***	**0.1616***
4	25	0.999	2245.7780	0.7774	0.1992***	738.7080	738.5543	0.3380	71.6367	721.9778	1113.5773	1113.3824	1131.6640	39.4298	1539.1446	0.1899**	0.2152	**0.1826***
4	50	0.8	8.2338	0.5426	0.9483	3.0653	5.2790	1.0397	1.5783	3.0983	4.5755	4.5352	4.6112	0.9832	0.9343	0.4099***	**0.3357***	0.4039**
4	50	0.9	18.0101	0.4500	0.6620	6.4130	8.2938	1.2253	2.5698	6.3809	9.8761	9.8411	9.9244	1.1423	0.8872	0.2517**	**0.2439***	0.2638***
4	50	0.99	158.4902	0.5639	0.2550	53.2787	53.5509	0.9502	11.3701	52.7688	85.6270	85.5622	85.9978	3.1042	38.7039	0.1826**	0.2047***	**0.1788***
4	50	0.999	1269.8213	0.7700	0.2139***	444.4797	444.4731	0.4699	47.5843	433.9908	695.0887	695.2617	701.1869	21.3974	751.4041	0.1986**	0.2175	**0.1958***
4	100	0.8	8.8893	0.5467	0.8823	3.2901	5.2819	1.0223	1.6168	3.3212	5.1121	5.0638	5.0877	0.9694	0.8624	0.3327***	**0.2979***	0.3228**
4	100	0.9	16.9114	0.4747	0.6817	6.07924	7.4231	1.1449	2.3572	6.0478	9.4202	9.3761	9.4020	1.1066	0.7566	**0.2460***	0.2503***	0.2491**
4	100	0.99	141.0306	0.5942	0.2871	49.7427	50.9807	1.1119	10.8962	48.3787	77.0070	77.0830	77.8167	3.1288	30.3061	**0.2067***	0.2213***	0.2089**
4	100	0.999	1435.9947	0.7926	0.2031***	510.7612	510.4139	0.4446	53.1711	503.9287	804.6926	804.3209	805.5580	23.4794	861.7974	0.1923**	0.2115	**0.1893***
4	200	0.8	8.7040	0.5331	0.8648	3.2206	4.7682	0.9902	1.5412	3.2726	5.0904	5.0380	5.0451	0.9469	0.8470	0.2974***	**0.2846***	0.2897**
4	200	0.9	14.9728	0.4717	0.7276	5.4235	6.8454	1.2083	2.3179	5.4085	8.1315	8.0840	8.0993	1.1097	0.7323	0.2545**	**0.2525***	0.2566***
4	200	0.99	128.6683	0.5850	0.2960	45.2398	45.3863	1.1393	10.1378	44.3910	70.6161	70.5936	70.6722	2.8582	24.7105	0.2057**	0.2219***	**0.2052***
4	200	0.999	1611.1355	0.8005	0.2040***	566.1079	565.8659	0.4021	54.8741	552.3307	896.3838	896.1955	897.1313	26.7709	1008.8865	0.1967**	0.2172	**0.1928***

Table 3. The EMSE values of the estimators for the model when $p = 8$.

p	n	ρ	**MLE**	**PRE**	**PLE**	**PLTE1**	**PLTE2**	**PLTE3**	**PLTE4**	**PLTE5**	**PTPE1**	**PTPE2**	**PTPE3**	**PTPE4**	**PKLE**	**ILTE1**	**ILTE2**	**ILTE3**
8	25	0.8	19.9604	0.7002***	1.7117	6.0787	10.5962	0.9563	2.1777	6.3790	8.2742	8.2829	8.6449	0.8612	2.4137	0.4823**	**0.3653***	1.0670
8	25	0.9	50.3732	0.4010***	1.2357	15.24389	20.0306	1.2535	4.0239	15.0817	21.2448	21.5647	22.6219	0.8435	5.1302	0.2714**	**0.2425***	0.6702
8	25	0.99	678.7506	0.4648	0.2648	209.1801	214.0220	2.5675	34.7650	195.6696	292.0214	298.1667	323.4471	1.9962	259.8824	**0.1324***	0.1613**	0.1747***
8	25	0.999	4797.9148	0.6398	0.1190**	1429.5596	1430.7209	1.2640	211.9912	1435.6574	2036.6562	2066.2588	2193.8417	11.4756	3088.6060	0.1201***	0.1565	**0.0981***
8	50	0.8	23.1118	0.5276***	1.5568	6.8909	11.6862	1.0024	2.5944	7.6557	10.9605	10.9489	11.3150	0.8368	2.2327	0.3037**	**0.2577***	0.7111
8	50	0.9	36.7291	0.4187***	1.2531	10.7539	16.0530	1.2587	3.8106	11.8446	17.1040	17.1687	17.7393	0.8492	2.3882	0.2658**	**0.2281***	0.6647
8	50	0.99	223.7915	0.3749	0.3763	64.5415	66.8170	2.1322	16.3142	70.4098	104.4372	105.3266	108.7627	1.0632	35.1382	**0.1251***	0.1556***	0.1553**
8	50	0.999	4347.4910	0.6440	0.0977**	1260.8569	1260.8368	0.8039	224.0822	1389.2884	2005.0925	2015.4493	2088.1672	9.5767	2699.2838	0.1108***	0.1499	**0.0803***
8	100	0.8	17.7779	0.6010	1.6950	5.4219	8.9292	0.9364	2.2290	6.1144	8.9076	8.8171	8.9501	0.8277	2.0538	0.2771**	**0.2449***	0.5035***
8	100	0.9	39.4905	0.3900***	1.1444	11.8371	15.6376	1.2733	4.0712	13.0443	20.0703	20.0842	20.5145	0.8346	1.9978	0.1972**	**0.1968***	0.4066
8	100	0.99	381.6767	0.4677	0.2611	112.7645	113.2014	1.9397	27.7767	126.0296	188.0628	188.4003	191.3872	1.3365	76.3400	0.1276**	0.1598***	**0.1234***
8	100	0.999	3627.0487	0.7005	0.1225***	1075.1131	1075.0786	0.9647	187.6869	1180.9691	1783.6746	1788.2760	1831.7104	8.5713	2143.6282	0.1201**	0.1530	**0.1037***
8	200	0.8	17.2762	0.5981	1.6679	5.2985	8.5759	0.9661	2.1501	5.9332	9.1431	9.0523	9.1458	0.8637	1.9797	0.2640**	**0.2382***	0.4304***
8	200	0.9	31.7980	0.4217	1.2216	9.3928	12.4060	1.1264	3.2557	10.4256	16.4409	16.3758	16.5625	0.8319	1.5650	**0.1971***	0.2034**	0.3402***
8	200	0.99	336.0953	0.4916	0.2686	98.7287	98.9368	1.9056	23.2179	109.8654	170.9257	171.0178	172.1852	1.2656	59.5865	0.1295**	0.1622***	**0.1204***
8	200	0.999	3491.9374	0.7161	0.1212**	1029.0573	1028.7909	0.8387	168.9759	1140.7771	1781.7882	1783.2158	1799.4035	6.0556	2052.4230	0.1263***	0.1615	**0.1031***

Table 4. The EMSE values of the estimators for the model when $p = 12$.

p	n	ρ	MLE	PRE	PLE	PLTE1	PLTE2	PLTE3	PLTE4	PLTE5	PTPE1	PTPE2	PTPE3	PTPE4	PKLE	ILTE1	ILTE2	ILTE3
12	25	0.8	53.1765	0.8472	2.4497	16.2334	25.1813	0.9144	3.0091	13.6027	16.7910	17.1776	19.0736	0.8467***	10.2881	0.5868**	**0.4082***	2.6388
12	25	0.9	87.6276	0.4898***	1.9322	24.6530	34.3489	1.2548	5.5933	23.9929	27.5965	28.2992	31.0229	0.8022	13.1589	0.3595**	**0.2719***	1.9631
12	25	0.99	819.0660	0.3558	0.4522	222.3210	228.4877	4.0131	43.1098	225.4499	257.7814	264.7171	288.7300	1.1491	278.9719	**0.1279***	0.1517**	0.3463***
12	25	0.999	9851.5942	0.4857	0.0895***	2789.6461	2791.0096	2.9127	358.9169	2552.2604	2993.3973	3072.6351	3418.4177	5.9664	6658.4281	0.0859**	0.1278	**0.0551***
12	50	0.8	26.9191	0.8941	2.6674	7.6994	13.6900	0.8875	2.7022	8.6470	11.1284	11.1524	11.6293	0.8469***	3.8505	0.4381**	**0.3176***	1.7131
12	50	0.9	91.3918	0.3120***	1.4870	25.0002	30.3065	1.4289	7.1250	27.5596	38.8585	39.4416	41.2478	0.7126	7.5691	**0.1516***	0.1600**	0.6570
12	50	0.99	772.5703	0.3300	0.2990	211.0052	212.4866	3.6254	50.7461	235.9541	320.3124	325.6473	343.0424	0.8299	204.3173	**0.0934***	0.1318***	0.1137**
12	50	0.999	4789.7109	0.5543	0.1099***	1297.1881	1297.8816	2.8586	258.4862	1425.8389	1994.4136	2025.3488	2136.6223	3.7638	2676.1617	0.0941**	0.1327	**0.0759***
12	100	0.8	32.2855	0.6009***	2.3594	8.9524	13.8731	0.8733	3.0662	10.3035	14.6705	14.6619	15.1176	0.8012	3.5371	0.2539**	**0.2207***	1.0590
12	100	0.9	55.2432	0.3776***	1.8157	15.1558	19.0122	1.1482	5.1554	17.7482	25.2651	25.3088	25.9966	0.7289	3.3406	**0.1598***	0.1684**	0.6059
12	100	0.99	676.9560	0.3642	0.2901	187.1011	187.8777	3.2900	46.7527	214.3195	306.0885	308.5362	318.9352	0.9292	146.6335	**0.0953***	0.1330***	0.1074**
12	100	0.999	5789.9572	0.6301	0.1024***	1567.3124	1567.5022	2.0956	309.5873	1797.0425	2615.4288	2636.1645	2733.8814	3.5567	3358.4564	0.1001**	0.1374	**0.0795***
12	200	0.8	32.1463	0.5653***	2.3225	9.0554	12.9833	0.9328	3.0872	10.4879	15.6700	15.5960	15.8533	0.8359	3.3523	0.2152**	**0.2014***	0.7412
12	200	0.9	64.2974	0.3527	1.5438	17.7186	20.0955	1.1666	5.5290	20.6331	31.7778	31.7425	32.1616	0.7467	2.5125	**0.1331***	0.1560**	0.3437***
12	200	0.99	563.1001	0.4144	0.3164	157.1794	157.5916	2.9385	38.7535	180.4213	280.6969	281.4347	285.3453	0.8721	97.8548	**0.1025***	0.1377***	0.1111**
12	200	0.999	6092.4526	0.6762	0.0998**	1678.9439	1678.6590	1.7194	330.9140	1951.5203	3049.1844	3051.5974	3091.7909	3.2394	3566.3584	0.1067***	0.1453	**0.0772***

v. In the case where ρ is 0.999, regardless of the n and p, it was observed that ILTE III has the smallest EMSE values in all cases considered. All the estimators we suggested showed superiority over other estimators in all 64 scenarios in the simulation study. In general, it has been observed that the behavior of the proposed estimators depends on the correlation ρ between variables rather than the number of observations n, or the number of variables p. This shows that the performance of the proposed estimators is affected due to the multicollinearity problem. Finally, ILTE II provided superiority in lower correlation, while ILTE I and ILTE III provided superiority in high correlation.

6. Numerical example: the aircraft damage data

In this section, the aircraft damage data, examined by [5, 7, 20, 21, 25] reanalyzed to illustrate the benefits of the proposed estimator. There are 30 observations in the data with three explanatory variables. The first explanatory variable (x_1) is a dichotomous variable showing the type of the aircraft. The explanatory variables (x_2) and (x_3) are bomb load in tons and total months of aircrew experience, respectively. The count variable y is the number of locations where damage was inflicted on the aircraft.

Myers et al. [25] indicated the presence of severe multicollinearity in the data set. Asar and Genc [7] and Amin et al. [5] made investigations using the following model $\mu = \exp\left(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3\right)$. The eigenvalues of the data matrix X are 2085.2251, 374.8961 and 4.3333. Thus, the condition number is 219.3654, indicates there is multicollinearity problem among the explanatory variables. Also, the eigenvalues of the matrix $X'WX$ are obtained as $\lambda_1 = 283543.5$, $\lambda_2 = 789.85$, $\lambda_3 = 4.2887$ and $\lambda_4 = 1.2585$. The condition number is 474.653 which is considerably larger than 30, indicating that MLE is still affected due to multicollinearity.

In addition, the dispersion parameter ϕ can be estimated by dividing the Deviance or Pearson Chi-square statistics by the degrees of freedom. According to the model under consideration, the dispersion parameter is estimated as 0.9981 and 0.9207, respectively, using these statistics. Therefore the estimated dispersion parameter is approximately 1, which shows us the considered model does not affect under/over-dispersion.

The parameter values and the estimated variance values corresponding to k, d and $f\left(k\right)$ functions are given in Table 5. As a result of the comparison of estimated variance values in Table 5, ILTEs have smaller variance values than MLE and the other biased estimators. This result is also compatible with simulation results.

To illustrate the theoretical results, the $f\left(k\right)$ function is set to $f\left(k\right) = 7.6486 \times 10^{-10} k - 1.2585$ using ILTE I. In computing the MMSE values, $\hat{\alpha}_{MLE}$ is used in place of the unknown parameter α.

For Theorem 3.1, $\operatorname{cov}\left(\hat{\beta}_{MLE}\right) - \operatorname{cov}\left(\hat{\beta}_{ILTE}\right)$ is the pd matrix for $k > 0$. The k values satisfying (3.1) criterion are $0 < k < 4.1078$. Consequently, $MMSE\left(\hat{\beta}_{MLE}\right) - MMSE\left(\hat{\beta}_{ILTE}\right)$ is the pd matrix where $0 < k < 4.1078$.

For Theorem 3.2, $\operatorname{cov}\left(\hat{\beta}_{PRE}\right) - \operatorname{cov}\left(\hat{\beta}_{ILTE}\right)$ is the pd matrix for $0 < k < 1.6455 \times 10^9$. Also, k values which provide (3.2) criterion are $0 < k < 0.8665$. Therefore, $MMSE\left(\hat{\beta}_{PRE}\right) - MMSE\left(\hat{\beta}_{ILTE}\right)$ is the pd matrix where $0 < k < 0.8665$.

To illustrate Theorem 3.3, lets take $\hat{d}_{PLE} = 0$. In this case, $\operatorname{cov}\left(\hat{\beta}_{PLE}\right) - \operatorname{cov}\left(\hat{\beta}_{ILTE}\right)$ is the pd matrix for $k > 0$. Also, k values which provide (3.3) criterion are $0 < k < 0.5859$. Therefore, $MMSE\left(\hat{\beta}_{PLE}\right) - MMSE\left(\hat{\beta}_{ILTE}\right)$ is the pd matrix where$0 < k < 0.58586$.

Lets take $\hat{d}_{PLTE} = 1.10037$ for Theorem 3.4. In this case, $\operatorname{cov}\left(\hat{\beta}_{PLTE}\right) - \operatorname{cov}\left(\hat{\beta}_{ILTE}\right)$

is the pd matrix for $0 < k < 1.64546 \times 10^9$. But, the criterion (3.4) given in Theorem 3.4 is not held.

Table 5. The estimated parameter values and the estimated variance values.

	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	variance
$\hat{\beta}_{MLE}$	–0.4060	0.5689	0.1654	–0.0135	1.0290
$\hat{\beta}_{PRE}$ $\left(\hat{k}_{PRE} = 0.5392\right)$	–0.3062	0.5180	0.1654	–0.0143	0.5747
$\hat{\beta}_{PLE}$ $\left(\hat{d}_{PLE} = 0\right)$	–0.2555	0.4789	0.1665	–0.0147	0.4013
$\hat{\beta}_{PLTE\ \mathrm{I}}$ $\left(\hat{k} = 0.5392,\ \hat{d} = 1.1004\right)$	–0.1024	0.4142	0.1656	–0.0158	0.1091
$\hat{\beta}_{PLTE\ \mathrm{II}}$ $\left(\hat{k} = 2862.8050,\ \hat{d} = -935.6250\right)$	–0.1318	0.1887	0.0827	–0.0026	0.1104
$\hat{\beta}_{PLTE\ \mathrm{III}}$ $\left(\hat{k} = 3699.495279,\ \hat{d} = 0.568440\right)$	0.0012	0.0033	0.0348	0.0037	0.00004
$\hat{\beta}_{PLTE\ \mathrm{IV}}$ $\left(\hat{k} = 43.8027,\ \hat{d} = 0.5684\right)$	–0.0074	0.0608	0.1798	–0.0153	0.0027
$\hat{\beta}_{PLTE\ \mathrm{V}}$ $\left(\hat{k} = 4.8840,\ \hat{d} = 0.5684\right)$	–0.0768	0.2568	0.1760	–0.0159	0.0496
$\hat{\beta}_{PTPE\ \mathrm{I}}$ $\left(\hat{k} = 5211.5079,\ \hat{d} = 0.2665\right)$	–0.1076	0.1535	0.0633	–0.0001	0.8981
$\hat{\beta}_{PTPE\ \mathrm{II}}$ $\left(\hat{k} = 97.525582,\ \hat{d} = 0.234583\right)$	–0.0972	0.1629	0.1692	–0.0138	1.0110
$\hat{\beta}_{PTPE\ \mathrm{III}}$ $\left(\hat{k} = 12.1237,\ \hat{d} = 0.0986\right)$	–0.0880	0.2088	0.1789	–0.0157	1.0287
$\hat{\beta}_{PTPE\ \mathrm{IV}}$ $\left(\hat{k} = 15357.8575,\ \hat{d} = 0.0483\right)$	–0.0193	0.0284	0.0179	0.0056	0.0024
$\hat{\beta}_{PKLE}$ $\left(\hat{k} = 0.9905\right)$	–0.1068	0.3906	0.1675	–0.0158	0.1036
$\hat{\beta}_{PLTE\ \mathrm{I}}$ $\left(f(k) = 7.6486 \times 10^{-10}k - 1.2585\right)$ $\hat{k}_{PLTE\ \mathrm{I}} = 94512.8372$	0.0001	0.0002	0.0024	0.0059	0.0000021
$\hat{\beta}_{PLTE\ \mathrm{II}}$ $\left(f(k) = 6.3740 \times 10^{-12}k - 1.2585\right)$ $\hat{k}_{PLTE\ \mathrm{II}} = 141769.8851$	0.0001	0.0002	0.0018	0.0053	0.0000016
$\hat{\beta}_{PLTE\ \mathrm{III}}$ $\left(f(k) = 2.5496 \times 10^{-11}k - 1.2585\right)$ $\hat{k}_{PLTE\ \mathrm{III}} = 70885.5718$	0.0002	0.0003	0.0030	0.0063	0.0000024

To illustrate Theorem 3.5, lets take $\hat{d}_{PTPE} = 0.0367$. In this case, $\mathrm{cov}\left(\hat{\beta}_{PTPE}\right) - \mathrm{cov}\left(\hat{\beta}_{ILTE}\right)$ is the pd matrix for $k > 0$. Also, k values which provide (3.5) criterion are $0 < k < 0.8974$. Therefore, $MMSE\left(\hat{\beta}_{PLE}\right) - MMSE\left(\hat{\beta}_{ILTE}\right)$ is the pd matrix where $0 < k < 0.5859$.

For Theorem 3.6, $\mathrm{cov}\left(\hat{\beta}_{PKLE}\right) - \mathrm{cov}\left(\hat{\beta}_{ILTE}\right)$ is the pd matrix for $0 < k \leq 1.2585$ and $k > 567085.8336$. The k values satisfying (3.6) criterion are $0 < k < 0.4294$ and $k > 567085.8336$. Consequently, $MMSE\left(\hat{\beta}_{PKLE}\right) - MMSE\left(\hat{\beta}_{ILTE}\right)$ is the pd matrix where $0 < k < 0.4294$ and $k > 567085.8336$.

Finally, based on the above results, we have shown that the theoretical conditions given in Theorems 3.1 to 3.6 hold for this data set. Therefore, we can say that ILTEs can outperform other biased estimators when we use $f(k)$ as an appropriate linear function of k.

7. Conclusion

In this article, we proposed a new biased estimator named ILTE as an alternative to MLE and the other biased estimators in the presence of multicollinearity for the PRM. The ILTE is a general estimator which includes other biased estimators, such as PRE, PLE, PLTE, PTPE and PKLE as special cases. Also, we investigated several function for the determination function. These functions were used with different k estimates. The results obtained with the simulation study show that our proposed estimator performs best in both low and high correlation between explanatory variables. Especially, ILTE II provided superiority in lower correlation, while ILTE I and ILTE III provided superiority in high correlation. Finally, an empirical application is conducted for the PRM and its results reveal the same results of the simulation study. Therefore, the ILTEs are recommended to the practitioners when there is multicollinearity in the PRMs.

Acknowledgment. The authors would like to thank the editors and the reviewers for various suggestions that help improve the quality of the presentation.

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