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Comparison Criteria for Three-interval Sturm-Liouville Equations

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Abstract. This study devoted to the investigation of comparison properties for periodic Sturm-Liouville problems, defined on three disjoint intervals together with additional transfer conditions across the common endpoint of these intervals, so-called transmission conditions. The results obtained generalize the corresponding classical results of Sturm's comparison and oscillation theory.

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1. Introduction

The existence and location of the zeros of the solutions of Sturm-liouville type equations are very important in physical applications. Accordingly a large literature on this subject has arisen during the past century. After Sturm's significant work [20] in 1836, Sturmian comparison and oscillation theorems have been derived for differential equations of various types. In order to obtain comparison theorems for Sturm-liouville equation Picone [18] established an important identity, known as the Picone identity. In [9] have been derived Picone-type inequalities for elliptic equations. Jaros and Kusano developed Sturmian theory for both forced and unforced half-linear equations based on Picone identity [10, 11]. The Sturm-Picone theorem and much of the related theory should allow generalization to certain partial differential equations. There are many papers and books dealing with Sturm comparison and oscillation results for a pair of elliptic type differential operators. We refer to Kreith [13, 14], Swanson [19] for Sturmian comparison theorems for linear elliptic equations and to Allegretto [7], Dunninger [8], Yoshida [24, 25] for Picone identities, Sturmian comparison and oscillation theorems for various type partial differential equations.

Taking into account that numerous vibrating physical systems admit mathematical descriptions as boundary-value problems with periodic boundary conditions, Karlin and Lee [12] investigated the total positivity properties of a class of periodic boundary-value problems for Polya type differential equations.

The main goal of study is to consider some spectral properties of the Sturm-Liouville equation,

$$\Theta f := -f''(x) + q(x)f(x) = \lambda f(x) \tag{1.1}$$

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defined on the three-interval $\cup_{i=1}^{3}\Omega_i := [-1,\xi_1) \cup (\xi_1,\xi_2) \cup (\xi_2,1]$, subject to periodic boundary conditions at the end points $x = \pm 1$ given by

$$f(-1) - f(1) = 0, \tag{1.2}$$

$$f'(-1) - f'(1) = 0, \tag{1.3}$$

and subject to the four transmission conditions at the points of discontinuity $x = \xi_1$ and $x = \xi_2$ given by

$$f(\xi_1-) - f(\xi_1+) = 0, \tag{1.4}$$

$$f'(\xi_1-) - f'(\xi_1+) = 0, \tag{1.5}$$

and

$$f(\xi_2-) - f(\xi_2+) = 0, \tag{1.6}$$

$$f'(\xi_2-) - f'(\xi_2+) = 0, \tag{1.7}$$

respectively, where $q(x)$ is a given real-valued functions, which is continuous in $\Omega_1 := [-1,\xi_1)$, $\Omega_2 := (\xi_1,\xi_2)$ and $\Omega_3 := (\xi_2,1]$ and has a finite limits $q(\xi_1\mp) = \lim_{x\to\xi_1\mp} q(x)$, $q(\xi_2\mp) = \lim_{x\to\xi_2\mp} q(x)$; λ is a complex eigenvalue parameter.

We know that transmission problems appear frequently in various fields of physics and technics. For example, in electrostatics and magnetostatics the model problem which describes the heat transfer through an infinitely conductive layer is a transmission problem (see, [17] and the references listed therein).

In recent years, Sturm-Liouville problems with transmission conditions have been an important research topic in theorretical and applied mathematics physics [1–6, 15, 16, 21–23]. This study investigates some properties of eigenvalues and corresponding eigenfunctions as well as comparison properties of the periodic Sturm-Liouville boundary value problems on three disjoint intervals with additional transmission conditions at the common endpoints of these intervals.

2. Eigenvalues and Eigenfunctions of the Problem

Let us define a new the Hilbert space $L_2^{\cup_{i=1}^{3}\Omega_i} := L_2(\Omega_1) \oplus L_2(\Omega_2) \oplus L_2(\Omega_3)$ consisting of all function f defined on $\cup_{i=1}^{3}\Omega_i$ and beloning to $L_2(\Omega_i)$ on Ω_i for $i = 1,2,3$ with the inner product

$$< f,g >_{L_2^{\Omega}} = \int_{-1}^{\xi_1-} f(x)\overline{g(x)}dx + \int_{\xi_1+}^{\xi_2-} f(x)\overline{g(x)}dx + \int_{\xi_2+}^{1} f(x)\overline{g(x)}dx.$$

In the space $L_2^{\cup_{i=1}^{3}\Omega_i}$, we define a linear operator $\Pi : L_2^{\cup_{i=1}^{3}\Omega_i} \to L_2^{\cup_{i=1}^{3}\Omega_i}$ by the domain of definition

$$\begin{aligned} dom(\Pi) \;:=\; & \Big\{ f(x) \in L_2^{\cup_{i=1}^{3}\Omega_i} : f(x), f'(x) \text{ are absolutely continuous in } \cup_{i=1}^{3}\Omega_i, \\ & \text{and has a finite limits } f(\xi_i \mp 0) \text{ and } f'(\xi_i \mp 0), i = 1,2,3,\ \Theta f \in L_2^{\cup_{i=1}^{3}\Omega_i}, \\ & f(-1) - f(1) = 0,\ f'(-1) - f'(1) = 0, f(\xi_i-) - f(\xi_i+) = 0, \\ & f'(\xi_i-) - f'(\xi_i+) = 0, i = 1,2 \Big\} \end{aligned}$$

in the Hilbert space $L_2^{\cup_{i=1}^{3}\Omega_i}$. The considered problem (1.1) − (1.7) can be rewritten in operator form as

$$\Pi f = \lambda f, \quad f \in dom(\Pi).$$

That is, the problem (1.1) − (1.7) can be considered as the eigenvalue problem for the differential operator Π.

Theorem 2.1. *The operator Π is symmetric in the Hilbert space $L_2^{\cup_{i=1}^{3}\Omega_i}$.*

Proof. Let $f, g \in dom(\Pi)$. Integration by parts twice yields the equality

$$\begin{aligned} < \Pi f, g >_{L_2^{\cup_{i=1}^{3}\Omega_i}} - < f, \Pi g >_{L_2^{\cup_{i=1}^{3}\Omega_i}} \;=\; & [W(f,\overline{g};\xi_1-) - W(f,\overline{g};-1)] \\ & +[W(f,\overline{g};\xi_2-) - W(f,\overline{g};\xi_1+)] \\ & +[W(f,\overline{g};1) - W(f,\overline{g};\xi_2+)], \end{aligned} \tag{2.1}$$

where, as usual

$$W(f,g;x) = f(x)g'(x) - f'(x)g(x)$$

is the Wronskians of the functions f and g. Since f and $\overline{g}$ satisfies the boundary conditions (1.2) − (1.3) we have

$$W(f,\overline{g};-1) = +W(f,\overline{g};1). \tag{2.2}$$

Similarly, by using the transmission conditions (1.4) − (1.7) we can show that

$$W(f,\overline{g};\xi_1-) = W(f,\overline{g};\xi_1+) \tag{2.3}$$

and

$$W(f,\overline{g};\xi_2-) = W(f,\overline{g};\xi_2+). \tag{2.4}$$

Now, substituting (2.2), (2.3) and (2.4) in (2.1) yields the required equality

$$< \Pi f, g >_{L_2^{\cup_{i=1}^3 \Omega_i}} = < f, \Pi g >_{L_2^{\cup_{i=1}^3 \Omega_i}} .$$

□

Corollary 2.2. *All eigenvalues of the boundary-value-transmission problem (1.1) − (1.7) are real.*

Taking into account the Corollary 2.2, from now on we can assume that all eigenfunctions of the problem (1.1)−(1.7) are real-valued.

Corollary 2.3. *If λ_1 and λ_2 are two different eigenvalues of the problem (1.1) − (1.7), with the corresponding eigenfunctions f and g, then*

$$\int_{-1}^{\xi_1-} f(x)g(x)dx + \int_{\xi_1+}^{\xi_2-} f(x)g(x)dx + \int_{\xi_2+}^{1} f(x)g(x)dx = 0.$$

In fact, this formula means the orthogonality of the eigenfunctions f and g in the Hilbert space $L_2^{\cup_{i=1}^3 \Omega_i}$.

3. Comparison Theorems

Here, we establish a generalization of the classical Sturm comparison theorems in the following form.

Theorem 3.1. *If $f(x)$ and $g(x)$ are linearly independent solutions of*

$$y'' + q(x)y = 0, \quad x \in \cup_{i=1}^3 \Omega_i \tag{3.1}$$

satisfying the transmission conditions

$$y(\xi_i-) - y(\xi_i+) = 0, \; y'(\xi_i-) - y'(\xi_i+) = 0, \; i = 1,2$$

for $i = 1,2$, then all zeros of the solutions $f(x)$ and $g(x)$ are simple, and between any to zeros of the solution $f(x)$ there is precisely one zero of the solution $g(x)$.

Proof. Let $x_0 \in \cup_{i=1}^3 \Omega_i$ be any zero of the solution $f(x)$ (or $g(x)$). Suppose, it possible, that x_0 is not simple zero of $f(x)$, that is $f(x_0) = f'(x_0) = 0$. Then, by well-known existence and uniqueness theorem of linear differential equations theory this functions would be a trivial solution and hence we obtain a contraction. Let α and β ($\alpha < \beta$) be two consecutive zeros of $f(x)$ and suppose, if possible, that $g(x) \neq 0$ on the interval (α,β). Without loss of generality take $f(x) > 0$ and $g(x) > 0$ on (α,β). Then, we have

$$W(f,g;\alpha) = f(\alpha)g'(\alpha) - f'(\alpha)g(\alpha) = -f'(\alpha)g(\alpha) \tag{3.2}$$

and

$$W(f,g;\beta) = f(\beta)g'(\beta) - f'(\beta)g(\beta) = -f'(\beta)g(\beta). \tag{3.3}$$

Since $f(\alpha) = f(\beta) = 0$ and $f(x) > 0$ for all $x \in (\alpha,\beta)$ we have

$$f'(x) = \lim_{x \to \alpha^+} \frac{f(x)}{x-\alpha} \geq 0. \tag{3.4}$$

Since $f(x)$ is the nontrivial solution of the differential equation (3.1) $f'(\alpha)$ can not be equal to zero. Then, from (3.4) it follows that $f'(\alpha) > 0$. Similarly we can show that $f'(\beta) < 0$. Therefore from (3.2) and (3.3) we conclude that $W(f, g; \alpha) < 0$ and $W(f, g; \beta) > 0$. It is well-known that the Wronskian of linearly independent solutions f, g cannot change sign. Thus, we get a contradiction which completes the proof. □

Remark 3.2. If we change the roles of the solutions $f(x)$ and $g(x)$, we see that between two consecutive zeros of the solutions $g(x)$ there must be a zero of the solution $f(x)$. Hence the zeros of $f(x)$ and $g(x)$ must interface.

Theorem 3.3. *Let $f(x)$ be the solution of the equation*

$$f''(x) + q_1(x)f(x) = 0, \quad x \in \cup_{i=1}^{3}\Omega_i$$

satisfying the transmission conditions (1.4) – (1.7) *and let $g(x)$ be the solution of the equation*

$$g''(x) + q_2(x)g(x) = 0, \quad x \in \cup_{i=1}^{3}\Omega_i$$

satisfying the same transmission conditions. If $q_1(x) > q_2(x)$ on $\cup_{i=1}^{3}\Omega_i$, then between any to consecutive zeros of $f(x)$ there is at least one zero of $g(x)$.

Proof. Multiplying the first equation by $g(x)$ and the second equation by $f(x)$ and then substracting we find

$$(f'(x)g(x) - f(x)g'(x))' = (q_2(x) - q_1(x))fg.$$

Let α and β with $\alpha < \beta$ be consecutive zeros of f. Integrating the last equality by parts over (α, β), we obtain

$$(f'(x)g(x) - g'(x)f(x))|_{\alpha}^{\beta} = \int_{\alpha}^{\beta} (q_2(x) - q_1(x))f(x)g(x)dx.$$

Since $f(\alpha) = f(\beta) = 0$,

$$f'(\beta)g(\beta) - f'(\beta)g(\beta) = \int_{\alpha}^{\beta} (q_2(x) - q_1(x))f(x)g(x)dx. \tag{3.5}$$

Suppose, it possible that, g does not have a zero between α and β. Consider the case $-1 \leq \alpha < \beta < \xi_1$. Without loss of generality we can suppose that $f(x) > 0$ and $g(x) > 0$ over (α, β). These conditions ensure that the integral on the right of (3.5) is negative. Since $f(\alpha) = f(\beta) = 0$ and $f(x) > 0$ for all $x \in (\alpha, \beta)$, we have $f'(\alpha) > 0$ and $f'(\beta) < 0$. Thus $g(x)$ is vanish at least once between the zeros of $f(x)$. The proofs for the cases $-1 \leq \alpha < \xi_1 < \beta < \xi_2$, $-1 \leq \alpha < \xi_1 < \xi_2 < \beta \leq 1$, $\xi_1 < \alpha < \beta < \xi_2$, $\xi_1 < \alpha < \xi_2 < \beta \leq 1$, $\xi_2 < \alpha < \beta \leq 1$ are totally similar.

□

Theorem 3.4. *Suppose that $p(x)$ and $q(x)$ are continuous in $\cup_{i=1}^{3}\Omega_i$ and have finite limits $p(\xi_i\pm)$ and $q(\xi_i\pm)$. Let $g(x)$ be any nontrivial solution of the equation*

$$g''(x) + q(x)g(x) = 0 \tag{3.6}$$

on three disjoint intervals Ω_1, Ω_2 and Ω_3 satisfying the periodic boundary conditions $g(1) = g(-1) = 0$, where discontinuity in g and g' at interior singular points $x = \xi_i (i = 1, 2)$ are prescribed by transmission conditions

$$g(\xi_i-) - g(\xi_i+) = 0, \; g'(\xi_i-) - g'(\xi_i+) = 0, i = 1, 2.$$

If

$$\int_{-1}^{\xi_1} (p - q)g^2 dx + \int_{\xi_1+}^{\xi_2-} (p - q)g^2 dx + \int_{\xi_2+}^{1} (p - q)g^2 dx \geq 0,$$

then a nontrivial solution of the boundary value transmission problem

$$f''(x) + p(x)f(x) = 0 \quad x \in \Omega, \tag{3.7}$$

$$f(1) = 0, \quad f(\xi_i-) - f(\xi_i+) = 0, f'(\xi_i-) - f'(\xi_i+) = 0, i = 1, 2 \tag{3.8}$$

has a zero belonging to $\cup_{i=1}^{3}\Omega_i$.

Proof. Let $f(x)$ be any nontrivial solution of the problem (3.7)-(3.8). Suppose, if possible, $f(x) \neq 0$ for $x \in \cup_{i=1}^{3}\Omega_i$. Then, using the equations (3.6),(3.7) we obtain

$$\frac{g(x)}{f(x)}(f(x)g'(x) - g(x)f'(x))' = g^2(x)(p(x) - q(x)). \tag{3.9}$$

Integrating the equality (3.9) on each of the intervals $[-1, \xi_1)$, (ξ_1, ξ_2) and $(\xi_2, 1]$ we have

$$\begin{aligned}
&\lim_{\epsilon \to 0^+} \frac{g(x)}{f(x)}(f(x)g'(x) - g(x)f'(x))|_{-1+\epsilon}^{\xi_1-} - \lim_{\epsilon \to 0^+} \int_{-1+\epsilon}^{\xi_1-} (\frac{g(x)}{f(x)})'(f(x)g'(x) - g(x)f'(x))dx \\
&+ \lim_{\epsilon \to 0^+} \frac{g(x)}{f(x)}(f(x)g'(x) - g(x)f'(x))|_{\xi_1+\epsilon}^{\xi_2-} - \lim_{\epsilon \to 0^+} \int_{\xi_1+\epsilon}^{\xi_2-} (\frac{g(x)}{f(x)})'(f(x)g'(x) - g(x)f'(x))dx \\
&+ \lim_{\epsilon \to 0^+} \frac{g(x)}{f(x)}(f(x)g'(x) - g(x)f'(x))|_{\xi_2+\epsilon}^{1} - \lim_{\epsilon \to 0^+} \int_{\xi_2+\epsilon}^{1} (\frac{g(x)}{f(x)})'(f(x)g'(x) - g(x)f'(x))dx \\
&= \int_{-1}^{\xi_1-} (p(x) - q(x))g^2(x)dx + \int_{\xi_1+}^{\xi_2-} (p(x) - q(x))g^2(x)dx + \int_{\xi_2+}^{1} (p(x) - q(x))g^2(x)dx \geq 0.
\end{aligned}$$

Using the transmission conditions (3.8) and $f(-1) = g(-1) = 0$ we get

$$\begin{aligned}
&-\int_{-1}^{\xi_1-} \frac{(f(x)g'(x) - g(x)f'(x))^2}{f^2(x)}dx - \int_{\xi_1+}^{\xi_2-} \frac{(f(x)g'(x) - g(x)f'(x))^2}{f^2(x)}dx \\
&-\int_{\xi_2+}^{1} \frac{(f(x)g'(x) - g(x)f'(x))^2}{f^2(x)}dx = \int_{-1}^{\xi_1-} (p(x) - q(x))g^2(x)dx \\
&+\int_{\xi_1+}^{\xi_2-} (p(x) - q(x))g^2(x)dx + \int_{\xi_2+}^{1} (p(x) - q(x))g^2(x)dx \geq 0.
\end{aligned}$$

Consequently

$$\int_{-1}^{\xi_1-} \frac{W^2(f, g; x)}{f^2(x)}dx + \int_{\xi_1+}^{\xi_2-} \frac{W^2(f, g; x)}{f^2(x)}dx + \int_{\xi_2+}^{1} \frac{W^2(f, g; x)}{f^2(x)}dx \leq 0.$$

The right hand side is identically zero if $f(x)$ and $g(x)$ are linearly dependent in which case the result is trivially true. So if $f(x)$ and $g(x)$ are linearly independent, the left hand side is positive and we get a contradiction. Hence $f(x)$ must vanish in $\cup_{i=1}^{3}\Omega_i$. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The authors have read and agreed to the published version of the manuscript.

References

[1] Akbarfam, I., Jodayree, A., *Resolvent operator and self-adjointness of Sturm-Liouville operators with a finite number of transmission conditions*, Mediterranean Journal Of Mathematics, **11**(2)(2014), 447–462.

[2] Allahverdiev, B.P., Bairamov, E., Ugurlu, E., *Eigenparameter dependent Sturm-Liouville problems in boundary conditions with transmission conditions*, Journal of Mathematical Analysis and Applications, **401**(1)(2013), 388–396.

[3] Allahverdiev, B.P., Tuna, H., *Titchmarsh-weyl theory for Dirac systems with transmission conditions*, Mediterranean Journal of Mathematics, **15**(4)(2018), 1–12.

[4] Allahverdiev, B. P., Tuna, H., *Eigenfunction Expansion for singular Sturm-Liouville problems with transmission conditions*, Electronic Journal of Differential Equations, **3**(2019), 4286–4302.

[5] Aydemir, K., Mukhtarov, O.Sh., *Qualitative analysis of eigenvalues and eigenfunctions of one boundary value-transmission problem*, Boundary Value Problems, **2016**(1)(2016), 1–16.

[6] Aydemir, K., Olğar, H., Mukhtarov, O.Sh., *Differential operator equations with interface conditions in modified direct sum spaces*, Filomat, **32**(3)(2018), 921–931.

[7] Allegretto, W., *Sturm theorems for degenerate elliptic equations*, Proc. Am. Math. Soc., **129**(2001), 3031–3035.

[8] Dunninger,Dr., *A Sturm comparison theorem for some degenerate quasilinear elliptic operators*, Boll. Unione Mat. Ital., **A7**(9)(1995), 117–121.

[9] Jaroš, J., Takaŝi, K., Yoshida, N.,*Picone-type inequalities for nonlinear elliptic equations with first-order terms and their applications*, J. Inequal. Appl., (2006), 1–17.

[10] Jaro, J., Kusano, T.,*Second-order semilinear differential equations with external forcing terms*, RIMS Kokyuroku, **984**(1997), 191–197.

[11] Jaro, J., Kusano, T., *A Picone type identity for second order half-linear differential equations*, Acta Math. Univ. Comen., **68**(1999), 117–121.

[12] Karlin, S., Lee, J.W., *Periodic boundary-value problems and cyclic totally positive Green's functions with applications to periodic spline theory*, J. Differmtial Equations, **8**(1970), 374–396.

[13] Kreith, K, Oscillation Theory, Lecture Notes in Mathematics, vol. 324. Springer, Berlin, 1963.

[14] Kreith, K., *Picone's identity and generalizations*, Rend. Mat., **8**(1975), 251–261.

[15] Olğar, H., Mukhtarov, O.Sh., *Weak eigenfunctions of two-interval Sturm-Liouville problems together with interaction conditions*, Journal of Mathematical Physics, **58**(2017), 042201.

[16] Mukhtarov, O.Sh., Olğar, H., Aydemir, K., Jabbarov, I.Sh., *The operator-pencil realization of one Sturm-Liouville problem with transmission conditions*, Applied and Computational Mathematics, **17**(3)(2018), 284–294.

[17] Pham Huy, H., Sanchez-Palencia, E., *Ph´enom'enes des transmission 'a travers des couches minces de conductivit´e ´elev´ee*, J. Math. Anal. Appl., **47**(1974), 284–309.

[18] Picone, M., *Sui valori eccezionali di un parametro da cui dipende un'equazione differenziale lineare ordinaria del second'ordine*, Ann. Scuola Norm. Sup. Pisa., **11**(1909), 1–141.

[19] Swanson, C.A., Comparison and Oscillation Theory of Linear Differential Equations, Vol. 48, Academic Prees, New York and London, 1968.

[20] Sturm, C., *Sur les équations différentielles linéaires du second ordre*, J. Math. Pures Appl., **1**(1836), 106–186.

[21] Şen, E., *Computation of eigenvalues and eigenfunctions of a Schrodinger-type boundary-value-transmission problem with retarded argument*, Mathematical Methods in the Applied Sciences, **41**(2018), 6604–6610.

[22] Şen, E., *Spectral properties of boundary-value-transmission problems with a constant retarded argument*, Turkish J Math., **43**(2)(2019), 612–619.

[23] Uğurlu, E., Bairamov, E., *O Spectral analysis of eigenparameter dependent boundary value transmission problems,* Journal Of Mathematical Analysis And Applications, **443**(1)(2014), 482–494.

[24] Yoshida, N, *Oscillation criteria for half-linear partial differential equations via Picone's identity*, In: Proceedings of Equadiff, **11**(2005), 589–598

[25] Yoshida, N., Oscillation Theory of Partial Differential Equations, World Scientific, 2008.