Almost complex structures on coframe bundle with Cheeger-Gromoll metric

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Abstract
In this paper we introduce several almost complex structures compatible with Cheeger-Gromoll metric on the coframe bundle and investigate their integrability conditions.

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1. Introduction
The geometric structures of the fiber bundles over Riemannian manifold \((M,g)\) is one of the essential topics in the differential geometry. First Sasaki [13] constructed a Riemannian metric \(Sg\) on the tangent bundle \(T(M)\) which depend only on the base manifold. Kowalski [8] proved that if the Sasaki metric \(Sg\) is locally symmetric, then the base metric \(g\) is flat and hence \(Sg\) is also flat. Musso and Tricerri [10] obtained an explicit expression of the Cheeger-Gromoll metric \(CGg\) introduced by Cheeger and Gromoll in [3] (see also [6]). Sekizawa [14] defined some geometric objects related \(CGg\). Tahara, Vanhecke and Watanabe [15] constructed several almost complex structures compatible with some natural defined Riemannian metrics on the tangent bundle of an almost Hermitian manifold. Bejan and Drută [2] defined harmonic almost complex structures with respect to general natural metrics in the tangent bundle. In [9] Munteanu introduced Cheeger-Gromool type metrics and showed the conditions for which the tangent bundle is almost Kahlerian or Kahlerian (see also [7]). To construct an almost Hermitian structure on the cotangent bundle \(T^*(M)\) of a Riemannian manifold \((M,g)\) Oproiu and Poroşniuc used some natural lifts of geometric objects [11]. (see also [4]).

In this paper, we construct an almost Hermitian structures on the bundle of linear coframes \(F^*(M)\) over a Riemannian manifold \((M,g)\) with the Cheeger-Gromoll metric \(CGg\). In 2 we briefly describe the definitions and results that are needed later, after which the adapted frame on coframe bundle \(F^*(M)\) introduced in 3. The Cheeger-Gromoll metric \(CGg\) on \(F^*(M)\) and its Levi-Civita connection \(CG\nabla\) are determined in 4. In 5 we define an almost Hermitian structures \((CG, J_\beta), \beta = 1, 2, \ldots, n\) on the linear coframe bundle \(F^*(M)\). The integrability conditions for almost complex structures \(J_\beta, \beta = 1, 2, \ldots, n\), are studied in 6.

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2. Preliminaries

In this section we shall summarize briefly the main definitions and results which be used later. Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold. Then the linear coframe bundle \(F^*(M)\) over \(M\) consists of all pairs \((x, u^*)\), where \(x\) is a point of \(M\) and \(u^*\) is a basis (coframe) for the cotangent space \(T_x^* M\) of \(M\) at \(x\) [5]. We denote by \(\pi\) the natural projection of \(F^*(M)\) to \(M\) defined by \(\pi(x, u^*) = x\). If \((U; x^1, x^2, \ldots, x^n)\) is a system of local coordinates in \(M\), then a coframe \(u^* = (X^\alpha) = (X^1, X^2, \ldots, X^n)\) for \(T_x^* M\) can be expressed uniquely in the form \(X^\alpha = X^\alpha (dx^i)_x\). From mentioned above it follows that

\[
\left( \pi^{-1}(U); x^1, x^2, \ldots, x^n, X_1^1, X_1^2, \ldots, X_n^n \right)
\]

is a system of local coordinates in \(F^*(M)\) (see, [5]), that is \(F^*(M)\) is a \(C^\infty\) manifold of dimension \(n + n^2\). We note that indices \(i, j, k, \ldots, \alpha, \beta, \gamma, \ldots\) have range in \(\{1, 2, \ldots, n\}\), while indices \(A, B, C, \ldots\) have range in \(\{1, 2, \ldots, n + 1, \ldots, n + n^2\}\). We put \(i_\alpha = \alpha \cdot n + i\). Obviously that indices \(i_\alpha, j_\beta, k_\gamma, \ldots\) have range in \(\{n + 1, n + 2, \ldots, n + n^2\}\). Summation over repeated indices is always implied. Let \(\nabla\) be a symmetric linear connection on \(M\) with components \(\Gamma^k_{ij}\). Then the tangent space \(T_{(x,u^*)}(F^*(M))\) of \(F^*(M)\) at \((x, u^*) \in F^*(M)\) splits into the horizontal and vertical subspaces with respect to \(\nabla\):

\[
T_{(x,u^*)}(F^*(M)) = H_{(x,u^*)}(F^*(M)) \oplus V_{(x,u^*)}(F^*(M)).
\] (2.1)

We denote by \(\mathfrak{S}^r_s(M)\) the set of all differentiable tensor fields of type \((r, s)\) on \(M\). From (2.1) it follows that for every \(X \in \mathfrak{S}^0_0(F^*(M))\) is obtained unique decomposing \(X = hX + vX\), where \(hX \in H(F^*(M))\), \(vX \in V(F^*(M))\). \(H(F^*(M))\) and \(V(F^*(M))\) the horizontal and vertical distributions for \(F^*(M)\), respectively. Now we define naturally \(n\) different vertical lifts of \(1\)-form \(\omega \in \mathfrak{S}^0_1(M)\). If \(Y\) be a vector field on \(M\), i.e. \(Y \in \mathfrak{S}^1_0(M)\), then \(i^\mu Y\) are functions on \(F^*(M)\) defined by \((i^\mu Y)(x, u^*) = X^\mu (Y)\) for all \((x, u^*) = (x, X^1, X^2, \ldots, X^n) \in F^*(M)\), where \(\mu = 1, 2, \ldots, n\). The vertical lifts \(V^\lambda \omega\) of \(\omega\) to \(F^*(M)\) are the \(n\) vector fields such that

\[
V^\lambda \omega(i^\mu Y) = \omega(Y) \delta^\lambda_\mu
\]

hold for all vector fields \(Y\) on \(M\), where \(\lambda, \mu = 1, 2, \ldots, n\) and \(\delta^\lambda_\mu\) denote the Kronecker’s delta. The vertical lifts \(V^\lambda \omega\) of \(\omega\) to \(F^*(M)\) have the components

\[
\begin{pmatrix}
V^\lambda \omega_k \\
V^\lambda \omega^k_\mu
\end{pmatrix} = \begin{pmatrix}
0 \\
\omega^k_\mu \delta^\lambda_\mu
\end{pmatrix}
\] (2.2)

with respect to the induced coordinates \((x^i, X^\alpha_i)\) in \(F^*(M)\) (see, [12]).

Let \(V \in \mathfrak{S}^1_0(M)\). The complete lift \(C^V \in \mathfrak{S}^1_0(F^*(M))\) of \(V\) to the linear coframe bundle \(F^*(M)\) is defined by

\[
C^V(i^\mu Y) = i^\mu (LV_Y) = X^\mu_m (LV_Y)^m
\]

for all vector fields \(Y \in \mathfrak{S}^1_0(M)\), where \(LV\) be the Lie derivation with respect to \(V\). The complete lift \(C^V\) has the components

\[
C^V = \begin{pmatrix}
C^V_k \\
C^V^k_\mu
\end{pmatrix} = \begin{pmatrix}
V^k \\
-X^\mu_m \partial_k V^m
\end{pmatrix}
\]

with respect to the induced coordinates \((x^i, X^\alpha_i)\) in \(F^*(M)\).

The horizontal lift \(H^V \in \mathfrak{S}^1_0(F^*(M))\) of \(V\) to the linear coframe bundle \(F^*(M)\) is defined by

\[
H^V(i^\mu Y) = i^\mu (\nabla_V Y) = X^\mu_m (\nabla_V Y)^m
\]
for all vector fields $Y \in \mathfrak{X}_0^1(M)$, where $\nabla_Y$ be the covariant derivative with respect to $V$. The horizontal lift $HV$ has the components

$$HV = \left( \frac{HV^k}{HV^{\kappa \mu}} \right) = \left( \frac{V^k}{X^\mu_{\kappa \mu} \Gamma^\mu_{\kappa \mu}} \right)$$

(2.3)

with respect to the induced coordinates $(x^i, X^\mu_i)$ in $F^*(M)$, where $\Gamma^k_{ij}$ are the components of Levi-Civita connection on $M$.

The bracket operation of vertical and horizontal vector fields is given by the formulas

$$[V^\beta \omega, V^\gamma \theta] = 0,$$

$$[H X, V^i \theta] = V^i (\nabla_X \theta),$$

$$[H X, H Y] = H [X, Y] + \sum_{\sigma=1}^n V^\sigma (X^\sigma \circ R(X, Y))$$

(2.4)

for all $X, Y \in \mathfrak{X}_0^1(M)$ and $\omega, \theta \in \mathfrak{X}_0^1(M)$, where $R$ is the Riemannian curvature of $g$. If $f$ is a differentiable function on $M$, $Vf = f \circ \pi$ denotes its canonical vertical lift to the $F^*(M)$.

3. **Adapted frames on** $F^*(M)$

Suppose $(U, x^i)$ be a local coordinate system in $M$. In $U \subset M$, we put

$$X_{(i)} = \partial f/\partial x^i, \quad \theta_{(i)} = dx^i, i = 1, 2, ..., n.$$

Taking into account of (2.2) and (2.3), we see that

$$HV_{(i)} = D_i = \left( \frac{\delta^j_i}{X^\mu_{\kappa \mu} \Gamma^\mu_{\kappa \mu}} \right),$$

(3.1)

$$V^\nu_\alpha \theta_{(i)} = D_{\nu a} = \left( \frac{0}{\delta^0_\beta \delta^0_j} \right)$$

(3.2)

with respect to the natural frame $\{\partial_j, \partial_{\beta j}\}$. It follows that this $n + n^2$ vector fields are linearly independent and generate, respectively the horizontal distribution of linear connection $\nabla$ and the vertical distribution of linear coframe bundle $F^*(M)$. The set $\{D_I\} = \{D_i, D_{\nu a}\}$ is called the frame adapted to linear connection $\nabla$ on $\pi^{-1}(U) \subset F^*(M)$. From (2.2), (2.3), (3.1) and (3.2), we deduce that the horizontal lift $HV$ of $V \in \mathfrak{X}_0^1(M)$ and vertical lift $V^\nu_\alpha \omega$ for each $\alpha = 1, 2, ..., n$, of $\omega \in \mathfrak{X}_0^1(M)$ have respectively, components:

$$HV = V^i D_i = \left( \frac{V^i}{0} \right),$$

(3.3)

$$V^\beta_\omega = \sum_i \omega_i \delta^\beta_\omega D_{\nu a} = \left( \frac{0}{\delta^0_\beta \omega_i} \right)$$

(3.4)

with respect to the adapted frame $\{D_I\}$. The non-holonomic objects $\Omega_{IJ}^K$ of the adapted frame $\{D_I\}$ are defined by

$$[D_I, D_J] = \Omega_{IJ}^K D_K$$

and have the following non-zero components:

$$\Omega_{j\beta}^k \gamma = -\Omega_{j\beta}^k \gamma = -\delta_\beta^j \Gamma^k_{\beta \gamma},$$

$$\Omega_{ij}^k \gamma = X^\mu_m R_{ijkm},$$

where $R_{ijkm}$ local components of the Riemannian curvature $R$. 

1262

A. Salimov, H. Fattayev
4. The Cheeger-Gromoll metric on the linear coframe bundle

**Definition 4.1.** Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold. A Riemannian metric \(\tilde{g}\) on the linear coframe bundle \(F^*(M)\) is said to be natural with respect to \(g\) on \(M\) if

\[
\tilde{g}(H X, H Y) = g(X, Y),
\]

\[
\tilde{g}(H X, V_\alpha \omega) = 0
\]

for all \(X, Y \in \mathcal{I}_\mathbb{R}(M)\) and \(\omega \in \mathcal{I}_\mathbb{R}(M)\).

For any \(x \in M\) the scalar product on the cotangent space \(T^*_x M\) is defined by

\[
g^{-1}(\omega, \theta) = g^{ij} \omega_i \theta_j
\]

for all \(\omega, \theta \in \mathcal{I}_\mathbb{R}(M)\).

The Cheeger-Gromoll metric \(CG\) is a positive definite metric on linear coframe bundle \(F^*(M)\) which is described in terms of lifted vector fields as follows.

**Definition 4.2.** Let \(g\) be a Riemannian metric on a manifold \(M\). Then the Cheeger-Gromoll metric is a Riemannian metric \(CG\) on the linear coframe bundle \(F^*(M)\) such that

\[
CG g(H X, H Y) = V(g(X, Y)) = g(X, Y) \circ \pi,
\]

\[
CG g(V_\alpha \omega, H Y) = 0,
\]

\[
CG g(V_\alpha \omega, V_\beta \theta) = 0, \quad \alpha \neq \beta,
\]

\[
CG g(V_\alpha \omega, V_\alpha \theta) = \frac{1}{1 + r_\alpha^2}(g^{-1}(\omega, \theta) + g^{-1}(\omega, X^\alpha)g^{-1}(\theta, X^\alpha))
\]

for all \(X, Y \in \mathcal{I}_\mathbb{R}(M)\) and \(\omega, \theta \in \mathcal{I}_\mathbb{R}(M)\), where \(r_\alpha^2 = ||X^\alpha||^2 = g^{-1}(X^\alpha, X^\alpha)\).

We note that the Cheeger-Gromoll metric on the cotangent bundle of Riemannian manifold introduced by Salimov and Ağa and studied in [1].

From (4.1) we determine that metric \(CG\) has components

\[
CG g_{ij} = CG g(D_i, D_j) = V(g(\partial_i, \partial_j)) = g_{ij},
\]

\[
CG g_{i\alpha} = CG g(D_{i\alpha}, D_j) = 0,
\]

\[
CG g_{i\alpha j\beta} = CG g(D_{i\alpha}, D_{j\beta}) = 0, \quad \alpha \neq \beta,
\]

\[
CG g_{i\alpha j\alpha} = CG g(D_{i\alpha}, D_{j\alpha}) = \frac{1}{1 + r_\alpha^2}(g^{-1}(dx^i, dx^j) + g^{-1}(dx^i, X^\alpha)g^{-1}(dx^j, X^\alpha))
\]

with respect to the adapted frame \(\{D_i\}\) of linear coframe bundle \(F^*(M)\).

The Levi-Civita connection \(CG \nabla\) satisfies the following relations

i) \(CG \nabla H X, H Y = H(\nabla X Y) + \frac{1}{2} \sum_{\sigma=1}^n V_\sigma(X^\sigma \circ R(X, Y))\),

ii) \(CG \nabla H X, V_\beta \theta = V_\beta(\nabla X \theta) + \frac{1}{2r_\beta^2} H(X^\beta (g^{-1} \circ R(\ , X) \theta))\),

iii) \(CG \nabla V_\alpha \omega, H Y = \frac{1}{2r_\alpha^2} H(X^\alpha (g^{-1} \circ R(\ , Y) \omega))\),

iv) \(CG \nabla V_\alpha \omega, V_\beta \theta = 0\) for \(\alpha \neq \beta\),

\[
CG \nabla V_\alpha \omega, V_\alpha \omega \theta = -\frac{1}{r_\alpha}(CG g(V_\alpha \omega, \gamma \delta)V_\alpha \theta + CG g(V_\alpha \theta, \gamma \delta)V_\alpha \omega) + \frac{1}{r_\alpha}(CG g(V_\alpha \theta, \gamma \delta)CG g(V_\alpha \omega, \gamma \delta)\gamma \delta
\]
for all $X, Y \in \mathfrak{X}_0^1(M)$, $\omega, \theta \in \mathfrak{X}_1^0(M)$, where $\tilde{\omega} = g^{-1} \circ \omega, R(\cdot, X)\tilde{\omega} \in \mathfrak{X}_1^1(M), h_\alpha = 1 + r_\alpha^2$, $R$ and $\gamma \delta$ denotes respectively the Riemannian curvature of $g$ and the canonical vertical vector field on $F^*(M)$ with local expression $\gamma \delta = X^\alpha D_{\alpha}$.

5. Almost complex structures on $(F^*(M), CG\ g)$

First of all, let us introduce the almost complex structures $J_\beta, \beta = 1, 2, \ldots, n$, which are compatible with $CG\ g$ on the linear coframe bundle $F^*(M)$. Suppose that for each $\beta = 1, 2, \ldots, n$, $J_\beta$ is defined to be the following form

\begin{align*}
J_\beta^H X &= a_1 V_\beta \tilde{X} + b_1 X^{\beta}(X)V_\beta X^\beta, \\
J_\beta^V \omega &= 0, \quad \beta \neq \gamma, \\
J_\beta^{\tilde{\omega}} &= a_2^H \tilde{\omega} + b_2 g^{-1}(X^\beta, \omega)H \tilde{X}^\beta,
\end{align*}

where $X \in \mathfrak{X}_0^1(M), \omega \in \mathfrak{X}_1^0(M), \tilde{X} = g \circ X \in \mathfrak{X}_0^1(M), \tilde{\omega} = g^{-1} \circ \omega \in \mathfrak{X}_1^0(M)$ and $a_1, a_2, b_1$ and $b_2$ are functions on colinear frame bundle $F^*(M)$ determined by conditions

\begin{align*}
J_\beta^2 &= -I, \\
CG\ g(J_\beta^H X, J_\beta^H X) &= CG\ g(H X, H X) = g(X, Y).
\end{align*}

Substituting (5.1) into (5.2), we obtain:

\begin{align*}
J_\beta^{2H} X &= J_\beta(J_\beta^H X) = J_\beta(a_1 V_\beta \tilde{X} + b_1 X^{\beta}(X)V_\beta X^\beta) \\
&= a_1 (J_\beta^{V_\beta} \tilde{X}) + b_1 X^{\beta}(X)(J_\beta^{V_\beta} X^\beta) = a_1 (a_2^H X + b_2 g^{-1}(X^\beta, \tilde{X})H \tilde{X}^\beta) \\
&+ b_1 X^{\beta}(X)(a_2^H \tilde{X} + b_2 g^{-1}(X^\beta, X^\beta)H \tilde{X}^\beta) = a_1 a_2^H X \\
&+ a_1 b_2 g^{-1}(X^\beta, \tilde{X})H \tilde{X}^\beta + b_1 a_2 X^{\beta}(X)H \tilde{X}^\beta \\
&+ b_2 b_1 X^{\beta}(X)(h_\beta - 1)H \tilde{X}^\beta = a_1 a_2^H X + (b_1 a_2 + b_2 b_1) \\
&+ b_2 b_1 (h_\beta - 1)X^{\beta}(X)H \tilde{X}^\beta = -H X,
\end{align*}

from which it follows that

\begin{align*}
a_1 a_2 &= -1, \\
a_1 b_2 + b_1 a_2 + b_2 b_1 (h_\beta - 1) &= 0.
\end{align*}

Direct calculations using (5.1) and (5.3) give

\begin{align*}
CG\ g(J_\beta^H X, J_\beta^H X) &= CG\ g(a_1 V_\beta \tilde{X} + b_1 X^{\beta}(X)V_\beta X^\beta, a_1 V_\beta \tilde{X} \\
&+ b_1 X^{\beta}(X)V_\beta X^\beta) = a_1^2 CG\ g(V_\beta \tilde{X}, V_\beta \tilde{X}) + a_1 b_1 X^{\beta}(X)CG\ g(V_\beta \tilde{X}, V_\beta X^\beta) \\
&+ b_1 a_1 X^{\beta}(X)CG\ g(V_\beta X^\beta, V_\beta \tilde{X}) + b_1^2 X^{\beta}(X)CG\ g(V_\beta X^\beta, V_\beta X^\beta) \\
&= \frac{a_1^2}{h_\beta} (g^{-1}(\tilde{X}, \tilde{X}) + g^{-1}(X^\beta, X^\beta)g^{-1}(X^\beta, X^\beta)) \\
&+ \frac{a_1 b_1 X^{\beta}(X)}{h_\beta} (g^{-1}(\tilde{X}, X^\beta) + g^{-1}(\tilde{X}, X^\beta)g^{-1}(X^\beta, X^\beta)) \\
&+ \frac{b_1 a_1 X^{\beta}(X)}{h_\beta} (g^{-1}(X^\beta, \tilde{X}) + g^{-1}(X^\beta, X^\beta)g^{-1}(\tilde{X}, X^\beta)) \\
&+ \frac{b_1^2 X^{\beta}(X)X^{\beta}(X)}{h_\beta} (g^{-1}(X^\beta, X^\beta) + g^{-1}(X^\beta, X^\beta)g^{-1}(X^\beta, X^\beta)) \\
&= \frac{a_1^2}{h_\beta} g(X, X) + \left(\frac{a_1^2}{h_\beta} + 2a_1 b_1 + b_1^2 (h_\beta - 1)\right)(X^\beta(X))^2 = g(X, X).
\end{align*}
Almost complex structures on coframe bundle

From the last relation we obtain:

\[ \frac{a_1^2}{h_{\beta}} = 1, \]  
\[ (5.6) \]

\[ \frac{a_1^2}{h_{\beta}} + 2a_1b_1 + b_2^2(h_{\beta} - 1) = 0. \]  
\[ (5.7) \]

Using (5.6) and (5.4), we get first

\[ a_1 = \pm \sqrt{h_{\beta}} \text{ and } a_2 = \mp \frac{1}{\sqrt{h_{\beta}}}. \]

Without lost of the generality we can take

\[ a_1 = \sqrt{h_{\beta}} \text{ and } a_2 = -\frac{1}{\sqrt{h_{\beta}}}. \]

Then for these values from (5.7) we get

\[ b_2^2(h_{\beta} - 1) + 2\sqrt{h_{\beta}b_1} + 1 = 0, \]

from which it follows

\[ b_1 = \frac{-\sqrt{h_{\beta}} \pm 1}{h_{\beta} - 1}. \]

We can take

\[ b_1 = \frac{-\sqrt{h_{\beta}+1}}{h_{\beta}-1} = -\frac{1}{\sqrt{h_{\beta}+1}}. \]

Then by using of (5.5) we obtain:

\[ \sqrt{h_{\beta}}b_2 + \frac{1}{\sqrt{h_{\beta}(\sqrt{h_{\beta}} + 1)}} - b_2\frac{1}{\sqrt{h_{\beta}}}(h_{\beta} - 1) = 0, \]

or

\[ b_2 = \frac{-1}{\sqrt{h_{\beta}(\sqrt{h_{\beta}} + 1)}}. \]

Therefore, we have the almost complex structures

\[ J_\beta, \beta = 1, 2, \ldots, n, \] on linear coframe bundle \( F^\ast (M) \)

\[ \left\{ \begin{array}{l}
J_\beta H X = \sqrt{h_{\beta}} \tilde{X} - \frac{1}{\sqrt{h_{\beta}+1}} X^\beta (X)^V_\beta X^\beta, \\
J_\beta V^\gamma \omega = 0, \quad \beta \neq \gamma, \\
J_\beta V^\gamma \omega = -\frac{1}{\sqrt{h_{\beta}}} \left( H \omega + \frac{1}{(\sqrt{h_{\beta}+1})} g^{-1}(X^\beta, \omega)^H \tilde{X} \right),
\end{array} \right. \]  
\[ (5.8) \]

which are satisfies the compability conditions (5.3) with the Cheeger-Gromoll metric \( CG \).

**Remark 5.1.** Taking into account that equality \( J_\beta V^\gamma \omega = 0 \) holds for \( \gamma \neq \beta \), of interest is the case when \( \gamma = \beta \).

Now it follows by a direct computations that

\[ CG (J_\beta H X, J_\beta V^\rho \omega) = CG (H X, V^\rho \omega), \]
\[ CG (J_\beta V^\rho \omega, J_\beta V^\theta \omega) = CG (V^\rho \omega, V^\theta \omega), \]

whenever

\[ CG (J_\beta H X, J_\beta H X) = CG (H X, H X). \]
Indeed, using (4.1) and (5.8), we have

\[ CG(g(J_\beta^H X, J_\beta^V \omega)) = CG(g(\sqrt{h_\beta} V_\beta \tilde{X}) \]
\[ - \frac{1}{\sqrt{h_{\beta+1}}} X_\beta^\beta (X) V_\beta X_\beta^\beta, \quad \frac{1}{\sqrt{h_\beta}} (H_\tilde{\omega} + \frac{1}{\sqrt{h_{\beta+1}}} g^{-1}(X_\beta^\beta, \omega) H_\tilde{X}_\beta^\beta)) \]
\[ = -\delta_\beta^CG g(V_\beta \tilde{X}, H_\tilde{\omega}) - \frac{1}{\sqrt{h_{\beta+1}}} g^{-1}(X_\beta^\beta, \omega) CG g(V_\beta \tilde{X}, H_\tilde{X}_\beta^\beta) \]
\[ + \frac{1}{\sqrt{h_\beta(\sqrt{h_{\beta+1}})^2}} X_\beta^\beta(X) g^{-1}(X_\beta^\beta, \omega) CG g(V_\beta X_\beta^\beta, H_\tilde{X}_\beta^\beta) \]
\[ = 0 = CG g(H X, V_\beta \omega). \]

Similarly we get

\[ CG(g(J_\beta^V \omega, J_\beta^V \theta)) = CG(g(-\frac{1}{\sqrt{h_\beta}} (H_\tilde{\omega} \]
\[ + \frac{1}{\sqrt{h_{\beta+1}}} g^{-1}(X_\beta^\beta, \omega) H_\tilde{X}_\beta^\beta), \quad \frac{1}{\sqrt{h_\beta}} (H_\tilde{\theta} + \frac{1}{\sqrt{h_{\beta+1}}} g^{-1}(X_\beta^\beta, \theta) H_\tilde{X}_\beta^\beta)) \]
\[ = \frac{1}{h_\beta} CG g(H_\tilde{\omega}, H_\tilde{\theta}) + \frac{1}{h_\beta(\sqrt{h_{\beta+1}})^2} g^{-1}(X_\beta^\beta, \omega) \]
\[ + \frac{1}{h_\beta(\sqrt{h_{\beta+1}})^2} g^{-1}(X_\beta^\beta, \omega) g^{-1}(X_\beta^\beta, \theta) CG g(H_\tilde{X}_\beta^\beta, H_\tilde{X}_\beta^\beta) \]
\[ = \frac{1}{h_\beta} g^{-1}(\omega, \theta) + \frac{2}{h_\beta(\sqrt{h_{\beta+1}})^2} g^{-1}(X_\beta^\beta, \omega) g^{-1}(X_\beta^\beta, \theta) \]
\[ + \frac{1}{h_\beta(\sqrt{h_{\beta+1}})^2} g^{-1}(X_\beta^\beta, \omega) g^{-1}(X_\beta^\beta, \theta)(h_\beta - 1) \]
\[ = (\sqrt{h_\beta} + 1) g^{-1}(\omega, \theta) + (\sqrt{h_\beta} + 1) g^{-1}(X_\beta^\beta, \omega) g^{-1}(X_\beta^\beta, \theta) \]
\[ h_\beta(\sqrt{h_{\beta+1}})^2 \]
\[ = \frac{1}{h_\beta} g^{-1}(\omega, \theta) + g^{-1}(X_\beta^\beta, \omega) g^{-1}(X_\beta^\beta, \theta)) = CG g(V_\beta \omega, V_\beta \theta). \]

Thus the following theorem holds.

**Theorem 5.2.** The triple \((F^\ast(M), CG, J_\beta)\) is an almost Hermitian manifold for any \(\beta = 1, 2, ..., n\).

6. The integrability of \(J_\beta, \beta = 1, 2, ..., n\)

It is known that the almost complex structure \(J\) of a Riemannian manifold \((M, g)\) is integrable if and only if its Nijenhuis tensor


for all \(X, Y \in \mathfrak{X}_0(M)\) ([16, p. 118]).

The Nijenhuis tensor of an almost complex structure \(J_\beta\) on \(F^\ast(M)\) for any \(\beta = 1, 2, ..., n\), is given by

\[ N_{J_\beta}(\tilde{X}, \tilde{Y}) = [\tilde{X}, \tilde{Y}] + J_\beta[J_\beta \tilde{X}, \tilde{Y}] + J_\beta[\tilde{X}, J_\beta \tilde{Y}] - [J_\beta \tilde{X}, J_\beta \tilde{Y}], \quad (6.1) \]
where \( \tilde{X}, \tilde{Y} \in \mathfrak{S}_1^0(F^*(M)) \). It is easy to check that the values \( N_{J_\beta}(H X, V_\theta) \) and \( N_{J_\beta}(V_\alpha, V_\theta) \) of the Nijenhuis tensor \( N_{J_\beta} \) can be expressed in terms of the values \( N_{J_\beta}(H X, H Y) \) of this tensor, where \( X, Y \in \mathfrak{S}_1^0(M) \), \( \omega, \theta \in \mathfrak{S}_1^0(M) \). Indeed, using (5.2) and (6.1), we have

\[
N_{J_\beta}(H X, V_\theta) = [H X, V_\theta] + J_\beta[J_\beta(H X, V_\theta)] + J_\beta[H X, J_\beta V_\theta]
\]

\[
-[J_\beta H X, J_\beta V_\theta] = [H X, \delta_\beta J_\beta H W] + J_\beta[J_\beta H X, \delta_\beta J_\beta H W]
\]

\[
+J_\beta[H X, J_\beta(\delta_\beta J_\beta H W)] - [J_\beta H X, J_\beta(\delta_\beta J_\beta H W)] = \delta_\beta[H X, J_\beta H W]
\]

\[
+\delta_\beta J_\beta[H X, J_\beta H W] - \delta_\beta J_\beta[H X, H W] + \delta_\beta[J_\beta H X, H W]
\]

\[
= -\delta_\beta J_\beta N_{J_\beta}(H X, H W),
\]

where

\[
V_\theta = \delta_\beta J_\beta H W = \delta_\beta(\sqrt{\Omega^\beta} \tilde{W} - \frac{1}{\sqrt{h_{\beta+1}}} X^\beta(W)V_\beta X^\beta)
\]

\[
= \delta_\beta(\sqrt{\Omega^\beta} \tilde{W} - \frac{1}{\sqrt{h_{\beta+1}}} X^\beta(W)X^\beta), W \in \mathfrak{S}_1^0(M).
\]

Similarly, we have

\[
N_{J_\beta}(V_\alpha, V_\theta) = [V_\alpha, V_\theta] + J_\beta[J_\beta(V_\alpha, V_\theta)] + J_\beta[V_\alpha, J_\beta V_\theta]
\]

\[
\]

\[
+J_\beta[\delta_\beta J_\beta H Z, J_\beta(\delta_\beta J_\beta H W)] - [J_\beta(\delta_\beta J_\beta H Z), J_\beta(\delta_\beta J_\beta H W)]
\]

\[
= \delta_\beta[\delta_\beta J_\beta H Z, J_\beta H W] - \delta_\beta J_\beta[H Z, J_\beta H W] - \delta_\beta J_\beta[H Z, H W]
\]

\[
-\delta_\beta J_\beta N_{J_\beta}(H Z, H W),
\]

where \( V_\alpha = \delta_\beta J_\beta H Z, Z \in \mathfrak{S}_1^0(M) \). Therefore, we have

**Lemma 6.1.** An almost complex structure \( J_\beta \) on \((F^*(M), CG g)\) for each \( \beta = 1, 2, ..., n \), is integrable if and only if \( N_{J_\beta}(H X, H Y) = 0 \) for any \( X, Y \in \mathfrak{S}_1^0(M) \).

Let us calculate

\[
\]

\[
-[J_\beta H X, J_\beta H Y].
\]

Before calculating \( N_{J_\beta}(H X, H Y) \) it is necessary to prove the following.

**Lemma 6.2.** Let \((M, g)\) be a Riemannian manifold and \( f : R \to R \) a smooth function. Then for all \( X \in \mathfrak{S}_1^0(M) \) and \( \omega, \theta \in \mathfrak{S}_1^0(M) \), we have

\[
1. V_\beta(\omega(f(r_\alpha^2))) = 2\delta_\beta f'(r_\alpha^2)g^{-1}(\omega, X^\alpha),
\]

\[
2. H X(g^{-1}(X^\alpha, \theta)) = g(X^\alpha, \nabla X^\alpha),
\]

where \( r_\alpha^2 = g^{-1}(X^\alpha, X^\alpha) \).

**Proof.** Direct calculations using (3.3) and (3.4) give

\[
1. V_\beta(\omega(f(r_\alpha^2))) = \omega_\beta f'(r_\alpha^2)\partial_\alpha(g^{rs}X^r_\alpha X^\alpha)
\]
\[ 2. \quad H(X(g^{-1}(X^\alpha, \theta)) = (X^i D_i)(g^{-1}(X^\alpha, \theta)) = X^i(\partial_i), \]
\[ + X^i \Gamma^i_{ip} \partial_{ip} (g^{-1}(X^\alpha, \theta)) = X^i(\partial_i g^r s) X^\alpha r \theta_s \]
\[ + \Gamma^r g^r m X^\alpha \theta_s + X^i X^r \Gamma^r_{ip} g^r s \delta^s \theta_s = X^i (-\Gamma^r g^m) \]
\[ = -X^i \Gamma^r g^m X^\alpha \theta_s - X^i \Gamma^r g^m X^\alpha \theta_s + X^i g^r s X^\alpha \partial_s \theta_s \]
\[ + X^i X^r \Gamma^r_{ir} g^s \theta_s = X^i g^r s X^\alpha \theta_s - X^i \Gamma^r g^r m X^\alpha \theta_s \]
\[ = X^r X^i (\partial_s \theta_s - \Gamma^r_{ip} \theta_m) g^r s = X^r (\nabla \chi \theta)_s g^s = g^{-1}(X^\alpha, \nabla \chi \theta). \]

This completes the proof of the lemma.

Direct calculations using (2.4), (3.3), (3.4), (5.8), (6.2) and (6.3) give
\[
[H X, H Y] = H[X, Y] + \sum_{\sigma=1} V_\sigma (X^\sigma \circ R(X, Y)),
\]
\[
J_\beta J_\beta^H [H X, H Y] = J_\beta [\sqrt{h_\beta} V_\beta \hat{X}, \frac{1}{\sqrt{h_\beta}} X^\beta(X)V_\beta X^\beta, H Y]
\]
\[
= J_\beta (\sqrt{h_\beta} V_\beta \hat{X}, \frac{1}{\sqrt{h_\beta}} g(\hat{X}^\beta, X)[V_\beta X^\beta, H Y]
\]
\[
+ \frac{1}{\sqrt{h_\beta}} H Y(g(\hat{X}^\beta, X)) V_\beta X^\beta = J_\beta (-\sqrt{h_\beta} V_\beta (\nabla \chi \hat{X})
\]
\[
+ \frac{1}{\sqrt{h_\beta}} (g^{-1}(X^\beta, \hat{X}^\beta) V_\beta \nabla \chi \hat{X}^\beta + H Y(g^{-1}(X^\beta, \hat{X}^\beta) V_\beta X^\beta)
\]
\[
= J_\beta (-\sqrt{h_\beta} V_\beta (\nabla \chi \hat{X}) + \frac{1}{\sqrt{h_\beta}} g^{-1}(\nabla \chi \hat{X}, X^\beta) V_\beta X^\beta
\]
\[
= J_\beta (-J_\beta H (\nabla \chi X)) = -J_\beta^2 H (\nabla \chi X) = H (\nabla \chi X),
\]
\[
\]
\[
[J_\beta^H X, J_\beta^H Y] = [\sqrt{h_\beta} V_\beta \hat{X} - \frac{1}{\sqrt{h_\beta}} g(\hat{X}^\beta, X) V_\beta X^\beta, \sqrt{h_\beta} V_\beta \hat{Y}
\]
\[- \frac{1}{\sqrt{h_\beta}} g(\hat{X}^\beta, Y) V_\beta X^\beta] = [\sqrt{h_\beta} V_\beta \hat{X}, \sqrt{h_\beta} V_\beta \hat{Y}] \]
Almost complex structures on coframe bundle

\[ + [\sqrt{h_\beta} \tilde{X}, - \frac{1}{\sqrt{\hbar_\beta + 1}} g(\tilde{X}_\beta, Y) V_\beta X_\beta] \]

\[ + [- \frac{1}{\sqrt{h_\beta + 1}} g(\tilde{X}_\beta, X) V_\alpha X_\alpha, \sqrt{h_\beta} V_\beta Y] \]

\[ + [- \frac{1}{\sqrt{h_\beta + 1}} g(\tilde{X}_\beta, X) V_\alpha X_\alpha, - \frac{1}{\sqrt{h_\beta + 1}} g(\tilde{X}_\beta, Y) V_\alpha X_\alpha] \]

\[ = \sqrt{h_\beta} V_\beta \tilde{X}(\sqrt{h_\beta}) V_\beta \tilde{Y} - \sqrt{h_\beta} V_\beta \tilde{Y}(\sqrt{h_\beta}) V_\beta \tilde{X} \]

\[ + \frac{1}{\sqrt{h_\beta + 1}} g(\tilde{X}_\beta, Y) V_\beta X_\beta (\sqrt{h_\beta}) V_\beta \tilde{X} + \frac{\sqrt{h_\beta}}{\sqrt{h_\beta + 1}} g(\tilde{X}_\beta, Y)[V_\beta X_\beta, V_\beta \tilde{X}] - \]

\[ - \frac{1}{\sqrt{h_\beta + 1}} g(\tilde{X}_\beta, X) V_\beta X_\beta (\sqrt{h_\beta}) V_\beta \tilde{Y} - \frac{\sqrt{h_\beta}}{\sqrt{h_\beta + 1}} g(\tilde{X}_\beta, X)[V_\beta X_\beta, V_\beta \tilde{Y}] \]

\[ = g^{-1}(X_\beta, \tilde{X}) V_\beta \tilde{Y} - g^{-1}(X_\beta, \tilde{Y}) V_\beta \tilde{X} \]

\[ + \frac{1}{\sqrt{h_\beta (h_\beta + 1)}} g^{-1}(X_\beta, \tilde{Y}) g^{-1}(X_\beta, X_\beta) V_\beta \tilde{X} - \frac{\sqrt{h_\beta}}{\sqrt{h_\beta + 1}} g^{-1}(X_\beta, \tilde{Y}) V_\beta \tilde{X} \]

\[ - \frac{1}{\sqrt{h_\beta (h_\beta + 1)}} g^{-1}(X_\beta, \tilde{X}) g^{-1}(X_\beta, X_\beta) V_\beta \tilde{Y} + \frac{\sqrt{h_\beta}}{\sqrt{h_\beta + 1}} g^{-1}(X_\beta, \tilde{X}) V_\beta \tilde{Y} \]

\[ = V_\beta \left( g^{-1}(X_\beta, \tilde{X}) \tilde{Y} - g^{-1}(X_\beta, \tilde{Y}) \tilde{X} \right) \left( 1 - \frac{r_\beta^2}{\sqrt{h_\beta (h_\beta + 1)}} + \frac{\sqrt{h_\beta}}{\sqrt{h_\beta + 1}} \right). \]

Therefore,

\[ N_{J_\beta} (H(X, H Y)) = H[X, Y] + \sum_{\sigma=1}^{n} (X_\sigma o R(X, Y)) + H((\nabla Y)X) - (\nabla X Y)) \]

\[ - V_\beta \left( g^{-1}(X_\beta, \tilde{X}) \tilde{Y} - g^{-1}(X_\beta, Y) \tilde{X} \right) \left( 1 - \frac{r_\beta^2}{\sqrt{h_\beta (h_\beta + 1)}} + \frac{\sqrt{h_\beta}}{\sqrt{h_\beta + 1}} \right) \]

\[ = \sum_{\sigma=1}^{n} (X_\sigma o R(X, Y)) - V_\beta \left( g^{-1}(X_\beta, \tilde{X}) \tilde{Y} \right. \]

\[ - g^{-1}(X_\beta, \tilde{Y}) \tilde{X} \left. \right) \left( 1 - \frac{r_\beta^2}{\sqrt{h_\beta (h_\beta + 1)}} + \frac{\sqrt{h_\beta}}{\sqrt{h_\beta + 1}} \right) \]

\[ = \sum_{\sigma=1}^{n} (X_\sigma o R(X, Y)) - \frac{1 + \sqrt{h_\beta + h_\beta}}{\sqrt{h_\beta (h_\beta + 1)}} V_\beta \left( g^{-1}(X_\beta, \tilde{X}) \tilde{Y} - g^{-1}(X_\beta, \tilde{Y}) \tilde{X} \right). \]

Thus, the following theorem holds.

**Theorem 6.3.** An almost complex structure \( J_\beta \) on \( (F^*(M), Cg_{g}) \) for each \( \beta = 1, 2, ..., n \), is integrable if and only if

\[ \gamma R(X, Y) = \sum_{\sigma=1}^{n} (X_\sigma o R(X, Y)) \]

\[ = \frac{1 + \sqrt{h_\beta + h_\beta}}{\sqrt{h_\beta (h_\beta + 1)}} V_\beta \left( g^{-1}(X_\beta, \tilde{X}) \tilde{Y} - g^{-1}(X_\beta, \tilde{Y}) \tilde{X} \right). \]
References