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A COMPOUND POSITIVELY DEPENDENT FARLIE-GUMBEL-MORGENSTERN BIVARIATE COPULA

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Abstract: In this study, we propose a two parameter Farlie-Gumbel-Morgenstern (FGM) copula that maintains membership in the family in a way while adding the extra dependence parameter to the model by using the compound method. Also, we assess the performance of the new compound FGM copula among all the most used two-parameter families of FGM copulas. The new copula performs best for the real data having moderate dependence structure.

Key words: Copula, Compound disribution, FGM copula.

1. Introduction

Let X and Y be random variables having joint cumulative distribution function H and margins F, G, respectively. Sklar [15] defined copula representation of H as given by $H(x,y) = C(F(x),G(y))$, where C is a unique cumulative distribution function having uniform margins on unit interval. Copula must satisfy the following properties:

Definition 1. A bivariate copula is a function with following properties:

1. C is 2-increasing function for all $x_1 \le x_2$, $y_1 \le y_2 \in [0,1]$ such that

$$C(x_2,y_2) - C(x_2,y_1) - C(x_1,y_2) + C(x_1,y_1) \ge 0,$$

2. C is grounded such that $C(x,0) = C(0,y) = 0$ for all $x,y \in [0,1]$,
3. C has uniform margins such that $C(x,1) = x$ and $C(1,y) = y$ for all $x,y \in [0,1]$,
4. $\dfrac{\partial^2 C(u,v)}{\partial u \, \partial v} \ge 0.$

We note that first condition is an equivalent condition for fourth condition in Definition 1 if $C(u,v)$ is twice differentiable. See Lu and Ghosh [10].

Many nonparametric measures of dependence can be viewed as functions of copula C. For bivarate case, Kendall's tau τ and Spearman's rho ρ can be defined, respectively, as

$$\tau = 4\int_{[0,1]^2} C(u,v)dC(u,v) - 1,$$

$$\rho = 12\int_{[0,1]^2} uvdC(u,v) - 3.$$

One of the most used bivariate family of the copula, because of its simple form, is the FGM family defined as

$$C(u,v) = uv + \theta u(1-u)v(1-v); \; \theta \in [-1,1],$$

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and studied in Farlie [4], Gumbel [5] and Morgenstern [11]. Kendall's tau and Spearman's rho for FGM copula can given as

$$\tau = 2\frac{\theta}{9},\ \rho = \frac{\theta}{3}.$$

It is clear that, FGM copula has the positive dependence struructure for the dependence parameter $\theta \in [0,1]$. Essepecially, in this paper, we mainly interested for the FGM copula having positive dependence structures.

FGM copula have been stuied in different fields, such as finance (Cossette et al. [3]), economics (Patton [14]), and reliability engineering (Navarro et al. [13]; Navarro and Durante [12]). Since the FGM copula has only one parameter, many FGM generated copulas have been introduced with the aim of adding extra parameters to the model. Huang and Kotz [6] proposed the following copula:

$$C_1(u,v) = uv\Big(1 + \theta(1-u)(1-v) + \beta uv(1-u)(1-v)\Big);$$

where

$$\theta \in [-1,1],\ \beta \leq \frac{3 - \theta + \sqrt{9 - 6\theta - 3\theta^2}}{2}.$$

Huang and Kotz [7] proposed the following two copulas:

$$C_2(u,v) = uv\Big(1 + \theta(1-u^\beta)(1-v^\beta)\Big),\ \beta > 0, -\min(1, \frac{1}{\beta^2}) < \theta < \frac{1}{\beta},$$

$$C_3(u,v) = uv\Big(1 + \theta(1-u)^\beta(1-v)^\beta\Big),\ \beta > 1, -1 < \theta < (\frac{\beta+1}{\beta-1})^{\beta-1}.$$

In addition, Lai and Xie [9], Bairamov and Bairamov [1] gave a generalization of the FGM copula family in their works.

In this study we prefer to use a different approach for enriching the FGM copula inspired by the Kelner et al. [8]. We attempt to replace θ, the generator parameter θ by new two parameters α and β, by using the compound tecnique in order to create new FGM copula while preserving its membership in FGM type copula. The paper is organized as follows: In Section 2, the new compound FGM copula is proposed and also, we investigate the its dependence structure. In Section 3, the performance of the proposed copula is investigated for the real data examples according to its goodness of fit results. Finally, last section is devoted for the conclusion.

2. Compound FGM copula

In this section, we present a tool for generating new compound positively dependent FGM copula having two dependence parameter α and β. This is achieved by using a compound of an existing positively dependent FGM copula cdf with respect to $f_{\alpha,\beta}(\theta)$ which is probability density of the dependence parameter θ,

$$C_T(u,v) = \int_0^1 C_\theta(u,v) f_{\alpha,\beta}(\theta) d\theta, \tag{2.1}$$

where $f_{\alpha,\beta}$ is the probabilty distribution function of beta distribution defined as

$$f_{\alpha,\beta}(\theta) = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{Beta(\alpha,\beta)};\ \alpha,\beta > 0.$$

Then the compuond FGM copula can be derived as following:

$$\begin{aligned} C_T(u,v) &= \int_0^1 \Big(uv + \theta u(1-u)v(1-v)\Big)\frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{Beta(\alpha,\beta)} d\theta \\ &= uv + \frac{\alpha}{\alpha+\beta} u(1-u)v(1-v);\ \alpha,\beta > 0, \end{aligned}$$

where $Beta$ is the Beta function defined as as $Beta(x_1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1}dt$ for x_1, x_2 positive integers. We now show that the new compound FGM copula satisfies the all properties of copula function.

LEMMA 1. *Let $C_T(u,v) = \int_0^1 C_\theta(u,v) f_{\alpha,\beta}(\theta) d\theta$ be a compound FGM copula based on density function $f_{\alpha,\beta}(\theta)$ of $\theta \in [0,1]$. Then for any $C_\theta(u,v)$ the compound FGM copula C_T is also an valid copula function.*

For the proof of Lemma 1, using the fact that for existing FGM copula complies properties defined in Definition 1 we get

$$C_T(0,v) = C_T(u,0) = \int_0^1 C_\theta(0,v) f_{\alpha,\beta}(\theta) d\theta = \int_0^1 C_\theta(u,0) f_{\alpha,\beta}(\theta) d\theta = 0,$$

$$C_T(1,v) = \int_0^1 C_\theta(1,v) f_{\alpha,\beta}(\theta) d\theta = v \int_0^1 f_{\alpha,\beta}(\theta) d\theta = v,$$

$$C_T(u,1) = \int_0^1 C_\theta(u,1) f_{\alpha,\beta}(\theta) d\theta = u \int_0^1 f_{\alpha,\beta}(\theta) d\theta = u.$$

Also we know that $\frac{\delta^2 C_\theta(u,v)}{\delta\, u\, \delta\, v} \geq 0$ then

$$\frac{\partial^2 C_T(u,v)}{\partial\, u\, \partial\, v} = \int_0^1 \frac{\delta^2 C_\theta(u,v)}{\delta\, u\, \delta\, v} f_{\alpha,\beta}(\theta) d\theta \geq 0,$$

then proof is completed.

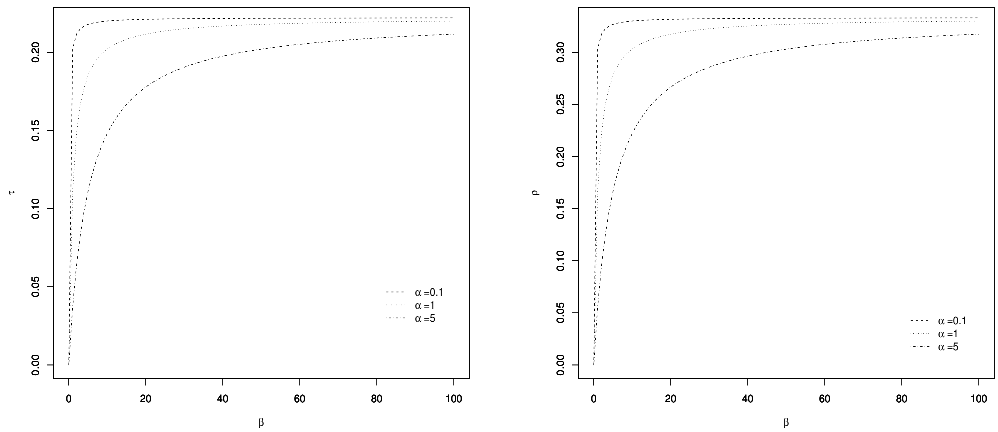
Proposed compound FGM copula has the following Kendall's tau, Spearman's rho, lower tail dependence and upper tail dependence coefficients given by respectively

$$\tau_T = \frac{11\,\alpha + 9\,\beta}{9(\alpha+\beta)} - 1,$$

$$\rho_T = \frac{10\,\alpha + 9\,\beta}{3(\alpha+\beta)} - 3,$$

$$\lambda_{T,L} = 0,\ \lambda_{T,U} = 0.$$

Figure 1 represents the values for the Kendall's tau and Spearman's rho of the compound FGM copula with different fixed values of α and varying values of β. From the this figure, it can be conclude that the compound FGM copula exhibits a varying dependence structures depending on the dependence parameters. Also, we can state that Kendall's tau and Spearman's rho are, respectively, limited to $(0, 0.22)$ and $(0, 0.33)$ as in FGM copula.



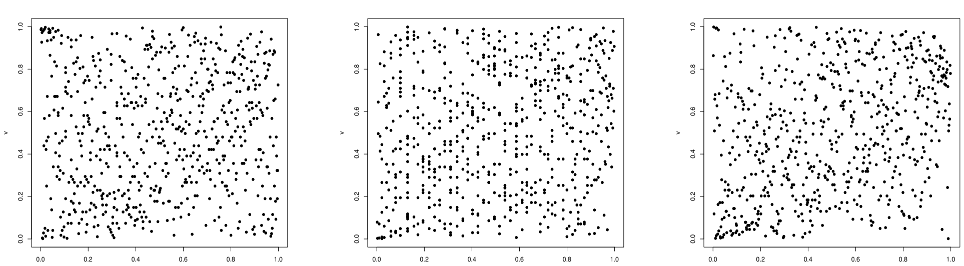
(a) τ values (b) ρ values

FIGURE 1. τ and ρ values for compound FGM copula

3. Case study

In this section, we put the new compound copula into the most used double parameter families of FGM copula as discussed in Section 1. Especially, in this case study, it is aimed to investigating the goodness of fit performance of proposed compund FGM copula under the different dependence structures. We use uranium dataset available in R package "copula". According the this package "These data consist of log concentrations of 7 chemical elements in 655 water samples collected near Grand Junction, CO (from the Montrose quad-rangle of Western Colorado). Concentrations were measured for the following elements: Uranium (U), Lithium (Li), Cobalt (Co), Potassium (K), Cesium (Cs), Scandium (Sc), And Titanium (Ti)." We prefer to modelling the pairs of variables U-Co, Li-Sc and K-Cs.

To avoid decision about marginal distributions, the observations were transformed to pseudo-observation (normalized ranked data) by their corresponding empirical distribution functions. Figure 2(a), 2(b) and 2(c) show the scatter plots of pseudo-observation for the pairs U-Co, Li-Sc and K-Cs, respectively. Looking at the graphs, strong positive dependence structure with $\tau = 0.2074$, mild positive dependence structure with $\tau = 0.0595$ and moderate dependence structure with $\tau = 0.1021$ can be observed for the pairs of K-Cs, U-Co and Li-Sc, respectively.



(a) U-Co ($\tau = 0.0595$) (b) Li-Sc ($\tau = 0.1021$) (c) K-Cs ($\tau = 0.2074$)

FIGURE 2. Scatter plots of real data sets

TABLE 1. Goodness-of-Fit results for real data sets

Copula	Data set	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	CvM	P-Val
C_θ	U-Co	0.2681	—	—	0.0612	0.0241
C_1		0.3812	—	-0.4700	0.0565	0.501
C_2		0.7100	—	0.5142	0.0583	0.0487
C_3		0.4488	—	1.3734	0.0521	0.0547
C_T		—	0.0676	0.1920	0.0611	0.0251
C_θ	Li-Sc	0.3762	—	—	0.0618	0.0231
C_1		0.3081	—	-0.1990	0.0308	0.0905
C_2		0.5650	—	0.5873	0.0307	0.0912
C_3		0.3347	—	1.1980	0.0293	0.0847
C_T		—	0.0742	0.2121	0.0218	0.1027
C_θ	K-Cs	0.9333	—	—	0.0397	0.0784
C_1		0.9543	—	-0.3472	0.0340	0.0817
C_2		1.2087	—	0.7888	0.0326	0.0907
C_3		0.4255	—	0.6905	0.0991	0.0034
C_T		—	0.2682	0.0401	0.0367	0.0797

In order to asses goodness of fit we use Cramér-von Mises distance which measure the distance beetwen empirical copula and null hypothesis copula distribution functions are given by

$$CvM = \int_0^1 \int_0^1 n\Big(C_n(u,v) - C_\theta(u,v)\Big)^2 dC_n(u,v), \tag{3.1}$$

where empirical copula is defined by

$$C_n(u,v) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{I}(U_i \le u, V_i \le v).$$

Thus test statistic defined in Eq. (3.1) allows us to compare the distances among copulas (smaller is the better). Also, the parameters of the copulas are estimated by minimizing the Eq. (3.1) under the consideration of constraints (α, $\beta > 0$). Goodness of fit results and estimated parameters for the pairs of K-Cs, U-Co and Li-Sc are shown in Table 1. P-values of the test statistic is computed according to Berg [2]. According to the this table, C_3 is the best performing copula model for the pair U-Co since it possesses the greatest p-value (0.0547) and lowest CvM (0.0521) values. Similarly, from Table 1, the best fit among all possible copulas for the pairs of Li-Sc and K-Cs are C_T (P-val:0.1027) and C_2 (P-val:0.0907), respectively.

For the performance of the new compound FGM copula in real data examples, it has smaller CvM diastance and greates P-Value for the pair Li-Sc which has a modarate dependence coefficient. On the contrary, for the pairs of U-Co and K-Cs which have mild and high dependence respectively, there is no difference with the classical FGM copula in terms of real data performance.

4. Conclusions

We have introduced compound FGM copula, describing its Kendall's tau and Spearman's rho with closed-form. We create new copula using a compound distribution method with a Beta probability density function of its dependence parameter θ. The proposed compound FGM copula make us possible to work with powerful models that can provide a much better goodness-of-fit results for the data set which have moderate dependence structure.

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