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Some inclusion results for the new Tribonacci-Lucas matrix

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ABSTRACT

The main purpose of this paper is first to establish a new regular matrix by using one of the important sequences of integer number called Tribonacci-Lucas. Also, we class this new Tribonacci-Lucas matrix with some well-known summability methods such as Riesz means, Nörlund means and Cesaro means. To do this, we show that the Tribonacci-Lucas matrix is a regular summability method and in addition to this, we give some inclusion results and finally prove that Cesaro matrix is stronger than the Tribonacci-Lucas matrix.

1. Introduction

In 1963, Tribonacci concept was introduced by Feinberg in [1]. Later, Tribonacci and Tribonacci-Lucas numbers were investigated by Catalani in [2]. These numbers must be regarded as a generalization of the well-known Fibonacci numbers. Also, the Tribonacci-Lucas numbers are members of the following general Tribonacci recurrence

$$U_{n+1} = U_n + U_{n-1} + U_{n-2}, \quad U_0 = 0, U_1 = U_2 = 1 .$$

The Tribonacci-Lucas sequence is

$$(v_n) = (3,1,3,7,11,21,39,71,131,241,\ldots)$$

and it can be easily seen from the elements of the sequence (v_n) that $v_0 = 3, v_1 = 1, v_2 = 3$ and

$$v_{n+1} = v_n + v_{n-1} + v_{n-2} .$$

The following expressions for the sums of the Tribonacci and Tribonacci-Lucas numbers can be found in [3-4]:

$$\sum_{k=1}^{n} U_k = \frac{U_{n+2} + U_n - 1}{2} ,$$

$$\sum_{k=1}^{n} (-1)^{k-1} U_k = \frac{(-1)^{n+1}(U_{n+1} - U_{n-1}) + 1}{2} ,$$

$$\sum_{k=1}^{n} v_k = \frac{v_{n+2} + v_n - 6}{2} ,$$

$$\sum_{k=1}^{n} (-1)^{k-1} v_k = \frac{(-1)^{n+1}(v_{n+1} - v_{n-1}) + 2}{2} .$$

The sequences of integer number defined by recurrence relations have been studied by many authors in [5-10]. In these studies, authors have given Fibonacci, Lucas, Padovan and Catalan numbers and their various properties.

Let us denote the space of all real valued sequences by w and each vector subspace of w is named sequence space. We indicate the spaces of null, convergent, bounded sequences and

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$p-$ absolutely convergent series by c_0, c, ℓ_∞ and $\ell_p (1 \le p < \infty)$.

Let $s=(s_n)$ be a sequence of non-negative real numbers with $s_0 > 0$ and take $S_n = \sum_{k=0}^{n} s_k$ for all $n \in \mathbb{N}$. Now, we give the following some well-known examples of particular summability matrices which satisfy the Toeplitz conditions.

Definition 1.1. The Riesz means according to the sequence $s=(s_n)$ is defined by the following matrix for all $n,k \in \mathbb{N}$:

$$a_{nk} = \begin{cases} \frac{s_k}{S_n}, & 0 \le k \le n \\ 0, & k > n \end{cases}.$$

Riesz mean (R,s) is also stated for a sequence (x_n) as follows:

$$s_n = \frac{s_1 x_1 + s_2 x_2 + \ldots + s_n x_n}{S_n} \text{ [11].}$$

Definition 1.2. The Nörlund means according to the sequence $s=(s_n)$ is defined by the following matrix for all $n,k \in \mathbb{N}$:

$$\tilde{a}_{nk} = \begin{cases} \frac{s_{n-k+1}}{S_n}, & k \le n \\ 0, & k > n \end{cases}.$$

Nörlund mean (N,s) is also stated for a sequence (x_n) as follows:

$$\tilde{s}_n = \frac{s_n x_1 + s_{n-1} x_2 + \ldots + s_1 x_n}{S_n} \text{ [11].}$$

The transformation (R,s) is regular if $S_n \to \infty (n \to \infty)$ and (N,s) is regular if $s_n / S_n \to \infty (n \to \infty)$ [12]. Also, both the Riesz and the Nörlund means are reduced to the following Cesaro mean $(C,1)$ in the case $s_n = 1$ for all n:

$$C_{nk} = \begin{cases} \frac{1}{n}, & k \le n \\ 0, & k > n \end{cases}.$$

Definition 1.3. Let (λ_n) be a strictly increasing sequence of positive integers. For a sequence (x_n), $C_\lambda -$ transformation is defined as follows:

$$s_n = \frac{x_1 + x_2 + \ldots + x_{\lambda_n}}{\lambda_n} \text{ [13].}$$

Definition 1.4. The matrix $B=(B_{m,n})$ is a (M) matrix if B is triangular and

$$\left| \sum_{k=1}^{n} b_{m,k} x_k \right| \le T \left| \sum_{k=1}^{n'} b_{n',k} x_k \right|$$

for some $n', n' = n'(n) (0 \le n' \le n), (n = 1,2,3,\ldots)$ and for all $m (m \ge n)$ [11]. Herein, n' is interdepend n and $\{x_n\}$ but it is independent of m. Also, the class (M) isn't confined to the regular matrices.

If $k < n+1$ for the matrix $(C,1)$, then we have $\frac{1}{n+1} \sum_{m=0}^{k} t_m \le \frac{1}{k+1} \sum_{m=0}^{k} t_m$ and so the matrix $(C,1)$ is a (M) matrix [11].

Theorem 1.5. Let $A=(a_{m,n})$ and $B=(b_{m,n})$ be regular triangular matrices and A be a (M) matrix. Therefore, if

$$\sum_{n=1}^{m} \left| \frac{b_{m,n}}{a_{m,n}} - \frac{b_{m,n+1}}{a_{m,n+1}} \right| < K,$$

from which it is concluded that B is stronger than A [11].

Theorem 1.6. The matrix $A=(a_{m,n})$ is (M) matrix if it is triangular and holds the following conditions:

$$a_{m,k} = 0, 0 \le \frac{a_{m,k}}{a_{n,k}} \le T (0 \le k \le n \le m) \tag{1}$$

and

$$\frac{a_{m,k}}{a_{n,k}} \ge \frac{a_{m,k+1}}{a_{n,k+1}} (0 \le k \le n \le m) \tag{2}$$

2. Inclusion results for the Tribonacci-Lucas matrix

In this part of the paper, we are first going to introduce a new Tribonacci-Lucas matrix. Then, we give some relations and inclusion results between the matrix $V=(v_{nk})$ and some well-known summability matrices by comparing them.

Now, let us define our new Tribonacci-Lucas matrix as follows:

$$V = (v_{nk}) = \begin{cases} \frac{2v_k}{v_{n+2} + v_n - 6}, & 1 \le k \le n \\ 0, & k > n \end{cases} \tag{3}$$

If we write the terms of this matrix, then we have

$$V = \begin{bmatrix} \frac{2v_1}{v_3 + v_1 - 6} & 0 & 0 & 0 & \cdots \\ \frac{2v_1}{v_4 + v_2 - 6} & \frac{2v_2}{v_4 + v_2 - 6} & 0 & 0 & \cdots \\ \frac{2v_1}{v_5 + v_3 - 6} & \frac{2v_2}{v_5 + v_3 - 6} & \frac{2v_3}{v_5 + v_3 - 6} & 0 & \cdots \\ \frac{2v_1}{v_6 + v_4 - 6} & \frac{2v_2}{v_6 + v_4 - 6} & \frac{2v_3}{v_6 + v_4 - 6} & \frac{2v_4}{v_6 + v_4 - 6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and so,

$$V=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & \cdots \\ \frac{1}{11} & \frac{3}{11} & \frac{7}{11} & 0 & \cdots \\ \frac{1}{22} & \frac{3}{22} & \frac{7}{22} & \frac{11}{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It can be clearly seen from above that the Tribonacci-Lucas matrix is triangular.

Now, let us define the following real valued sequence $y=(y_n)$ which is named $V-$ transform of a sequence $x=(x_n)$ for all $n\in\mathbb{N}$:

$$y_n=V(x_n)=\frac{2}{v_{n+2}+v_n-6}\sum_{k=1}^{n}v_k \tag{4}$$

First, we are going to give the definition of $V-$ convergence in defiance of $F-$convergence in [14].

Definition 2.1. If $(V(x_n-l))\to 0$ for $n\in\mathbb{N}$ and $l\in\mathbb{R}$, then a real valued sequence $x=(x_n)$ is named $V-$ convergent to l.

Theorem 2.2. The Tribonacci-Lucas matrix $V=(v_{nk})$ is a regular summability method $\Leftrightarrow$ $v_{n+2}+v_n-6\to\infty$ as $n\to\infty$.

Proof. Let $V=(v_{nk})$ be a regular summability method. Then, $\lim\limits_{n\to\infty}v_{nk}=\lim\limits_{n\to\infty}\frac{2v_k}{v_{n+2}+v_n-6}=0$ from Silverman-Toeplitz theorem in [8]. Thus, $v_{n+2}+v_n-6\to\infty$, $n\to\infty$. Now contrarily, assume that $v_{n+2}+v_n-6\to\infty$ as $n\to\infty$. Therefore, $\sum\limits_{k=1}^{\infty}\frac{2v_k}{v_{n+2}+v_n-6}=\sum\limits_{k=1}^{n}\frac{2v_k}{v_{n+2}+v_n-6}=1$ and also,

$$\lim_{n\to\infty}v_{nk}=\lim_{n\to\infty}\frac{2v_k}{v_{n+2}+v_n-6}=0$$

and

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}v_{nk}=\lim_{n\to\infty}\sum_{k=1}^{n}v_{nk}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2v_k}{v_{n+2}+v_n-6}=1.$$

In that case, the Tribonacci-Lucas matrix V is a regular summability method.

Theorem 2.3. The Tribonacci-Lucas matrix $V=(v_{nk})$ is a (M) matrix.

Proof. Since the inequalities

$$0\le\frac{2v_k}{v_{m+2}+v_m-6}\cdot\frac{v_{n+2}+v_n-6}{2v_k}=\frac{v_{n+2}+v_n-6}{v_{m+2}+v_m-6}\le\frac{v_{n+2}}{v_{m+2}}\le 1$$

and

$$\frac{\dfrac{2v_{k+1}}{v_{m+2}+v_m-6}}{\dfrac{2v_{k+1}}{v_{n+2}+v_n-6}}=\frac{v_{n+2}+v_n-6}{v_{m+2}+v_m-6}\le\frac{v_{n+2}+v_n-6}{v_{m+2}+v_m-6}\cdot\frac{2v_k}{2v_k}$$

$$=\frac{2v_k}{v_{m+2}+v_m-6}\cdot\frac{v_{n+2}+v_n-6}{2v_k}$$

hold, the Tribonacci-Lucas matrix V is (M) matrix.

Definition 2.4. Let $x=(x_n)$ and $y=(y_n)$ be two real valued sequences. Then, if there are two positive real numbers t and T such that the inequality $t.x_n\le y_n\le T.x_n$ holds for all $n\in\mathbb{N}$, they are named equivalent.

Now, let us give the relation between V and (R,s) :

Theorem 2.5. Let $V=(v_{nk})$ be a Tribonacci-Lucas matrix and $x=(x_n)$ be a real valued sequence. Then, $x_n\to l(V)\Leftrightarrow x_n\to l(R,s)$ for any sequence (s_n) such that (s_n) and (v_n) are equivalent for all $n\in\mathbb{N}$.

Proof. Assume that $x_n\to l(V)$. In this case, we get

$$\lim_{n\to\infty}\frac{2}{v_{n+2}+v_n-6}\sum_{k=1}^{n}v_k(x_k-l)=0.$$

From here, under the supposition on (s_n), the following inequality holds:

$$\begin{aligned}\frac{1}{S_n}\sum_{k=1}^{n}s_k(x_k-l)&\le\frac{1}{S_n}\sum_{k=1}^{n}T.v_k(x_k-l)\\&\le\frac{T}{t}\frac{2}{v_{n+2}+v_n-6}\sum_{k=1}^{n}v_k(x_k-l).\end{aligned}\tag{5}$$

Also, by using the similar technique, we find

$$\frac{t}{T}\frac{2}{v_{n+2}+v_n-6}\sum_{k=1}^{n}v_k(x_k-l)\le\frac{1}{S_n}\sum_{k=1}^{n}s_k(x_k-l).\tag{6}$$

Resulting from $x_k\to l(V)$, the inequalities (5) and (6) give us that $\lim\limits_{n\to\infty}\frac{1}{S_n}\sum\limits_{k=1}^{n}s_k(x_k-l)=0$. The sufficient condition of this theorem can be easily shown by use of the same method. So, the proof is completed.

Theorem 2.6. Let $B=(b_{nk})$ be a regular matrix and suppose that $\sum\limits_{k=1}^{n}|b_{nk}-v_{nk}|\to 0$ as $n\to\infty$. Then for any bounded sequence, $x_n\to l(B)\Leftrightarrow x_n\to l(V)$.

Proof. For any n and bounded sequence $x=(x_n)$, we have

$$\left|(Bx)_n-(Vx)_n\right|=\left|\sum_{k=1}^{n}b_{nk}x_k-\sum_{k=1}^{n}v_{nk}x_k\right|$$
$$\leq\sum_{k=1}^{n}\left|b_{nk}-v_{nk}\right|\left|x_k\right|\leq\|x\|\sum_{k=1}^{n}\left|b_{nk}-v_{nk}\right|.$$

Hence, if $x_n\to l(B)$ and $x_n\to l(V)$, then we get

$$\left|(Vx)_n-l\right|\leq\left|(Vx)_n-(Bx)_n\right|+\left|(Bx)_n-l\right|\to 0, n\to\infty \qquad (7)$$

and in a similar way

$$\left|(Bx)_n-l\right|\leq\left|(Bx)_n-(Vx)_n\right|+\left|(Vx)_n-l\right|\to 0, n\to\infty. \qquad (8)$$

Consequently, the inequalities (7) and (8) complete the proof.

Now, we are going to establish the following associate matrix $\tilde{V}=(\tilde{v}_{nk})$:

$$\tilde{v}_{nk}=\begin{cases}\dfrac{2v_{n-k}}{v_{n+2}+v_n-6}, & k\leq n\\ 0, & k>n\end{cases} \qquad (9)$$

We can state the associate matrix $\tilde{V}=(\tilde{v}_{nk})$ as a Nörlund type Tribonacci-Lucas matrix when the matrix $\tilde{V}=(\tilde{v}_{nk})$ can be written as a Riesz type Tribonacci-Lucas matrix. Accordingly, we first give the following lemma which will be used in the next theorems.

Lemma 2.7. The series $\sum_{n=1}^{\infty}s_nx^{n-1}$ and $\sum_{n=1}^{\infty}S_nx^{n-1}$ are convergent for all x where $|x|<1$, if (N,s_n) is a regular Nörlund matrix.

This lemma is also suitable for the matrix $\tilde{V}=(\tilde{v}_{nk})$ just because the matrix $\tilde{V}=(\tilde{v}_{nk})$ is a Nörlund type matrix. In the continuation of this study, we can use V_n in place of $\tilde{V}_n$ having regard to the definition of $\tilde{V}=(\tilde{v}_{nk})$.

Remark 2.8. Due to the fact that the series $v(x)=\sum_{n=1}^{\infty}v_nx^{n-1}$ and $V(x)=\sum_{n=1}^{\infty}V_nx^{n-1}$ are convergent for all $|x|<1$, the series below are also convergent:

$$q(x)=\frac{s(x)}{v(x)}=\frac{S(x)}{V(x)},\quad q(x)=\sum_{n=1}^{\infty}q_nx^{n-1},$$

$$r(x)=\frac{v(x)}{s(x)}=\frac{V(x)}{S(x)},\quad r(x)=\sum_{n=1}^{\infty}r_nx^{n-1}.$$

Theorem 2.9. $(N,s_n)\subseteq(\tilde{V})$ if and only if there is $T>0$ such that for every n $\quad |q_1|S_n+|q_2|S_{n-1}+\ldots+|q_n|S_1\leq T.V_n$ and $\lim_{n\to\infty}\frac{q_n}{V_n}=0$.

Proof. Let (k_n) and (h_n) be the $(N,s)-$ transformation and $(\tilde{V})-$ transformation of a real valued sequence (p_n). Then, we get

$$\sum_{n=1}^{\infty}V_nh_nx^{n-1}=\sum_{n=1}^{\infty}V_n\frac{(v_np_1+v_{n-1}p_2+\ldots+v_1p_n)}{V_n}x^{n-1}$$
$$=(v_1p_1)x^0+(v_2p_1+v_1p_2)x^1+(v_3p_1+v_2p_2+v_1p_3)x^2+\ldots$$
$$+(v_np_1+v_{n-1}p_2+\ldots+v_1p_n)x^{n-1}+\ldots$$
$$=p_1\left(v_1x^0+v_2x^1+v_3x^2+\ldots\right)+p_2\left(v_1x+v_2x^2+v_3x^3+\ldots\right)$$
$$+p_3\left(v_1x^2+v_2x^3+v_3x^4+\ldots\right)+\ldots+p_n\left(v_1x^{n-1}\right)+\ldots$$
$$=p_1x^0\left(v_1x^0+v_2x^1+v_3x^2+\ldots+v_nx^{n-1}\right)+p_2x^1\left(v_1x^0+v_2x^1+v_3x^2+\ldots+v_{n-1}x^{n-2}\right)$$
$$+\ldots+p_nx^{n-1}\left(v_1x^0\right)+\ldots$$

$$=\left(\sum_{n=1}^{\infty}p_nx^{n-1}\right)\left(\sum_{n=1}^{\infty}v_nx^{n-1}\right)=p(x)v(x). \qquad (10)$$

In a similar way, we also have

$$\sum_{n=1}^{\infty}S_nk_nx^{n-1}=p(x)s(x). \qquad (11)$$

Now, from the hypothesis, we know that $v(x)=q(x)s(x)$ and $v(x)p(x)=q(x)s(x)p(x)$.

If we consider (17), (18) and the Cauchy product of series, then we find $\sum_{n=1}^{\infty}V_nh_nx^{n-1}=\sum_{n=1}^{\infty}\sum_{m=1}^{n}q_{n-m+1}S_mk_mx^{n-1}$ and so for all $n\in\mathbb{N}$,

$V_nh_n=q_nS_1k_1+q_{n-1}S_2k_2+\ldots+q_1S_nk_n$. Thus, $h_n=\sum_{n=1}^{\infty}b_{nm}k_m$ and

$b_{nm}=\begin{cases}\dfrac{q_{n-m+1}S_m}{V_n}, & m\leq n\\ 0, & m>n\end{cases}$. The matrix b_{nm} is regular, in truth

$$\lim_{n\to\infty}b_{nm}=\lim_{n\to\infty}\frac{q_{n-m+1}S_m}{V_n}=\lim_{n\to\infty}\frac{q_{n-m+1}S_m}{V_{n-m+1}}=0,$$
$$\sum_{m=1}^{\infty}|b_{nm}|=\frac{|q_1|S_n+\ldots+|q_n|S_1}{V_n}\leq T,$$
$$\lim_{n\to\infty}\sum_{n=1}^{m}b_{nm}=\frac{q_1S_n+\ldots+q_nS_1}{V_n}=\frac{V_n}{V_n}=1.$$

Therefore, the proof of sufficient condition is completed. The proof of necessary condition can be done by taking advantage of the specifications in the expression of theorem.

Definition 2.10. Let $\beta=(\beta_n)$ be a strictly increasing sequence of positive integers. Let us define the $V_\beta-$ transformation of a sequence $x=(x_n)$ as follows:

$$z_n=\frac{2}{v_{\beta(n)+2}+v_{\beta(n)}-6}\sum_{k=1}^{\beta(n)}v_kx_k.$$

Litlle o notation, also called Landau's symbol is usually used in mathematics. Informally, $f(t)=o(g(t))$ is supposed to mean that f grows much slower than g and it is insignificant in comparison. Formally, we write $f(t)=o(g(t))$ if and only if for every $T>0$ there exists a real number N such that for all $t>N$ we get $|f(t)|<T|g(t)|$ and if $g(t)\neq 0$, this is equivalent to $\lim_{t\to\infty}\frac{f(t)}{g(t)}=0$.

Theorem 2.11. Let $\beta=\{\beta(n)\}$ and $\gamma=\{\gamma(n)\}$ be a strictly increasing sequences of natural number. Then, V_β is equivalent to V_γ on ℓ_∞ if $\lim_{n\to\infty}\frac{v_{\beta(n)+2}+v_{\beta(n)}-6}{v_{\gamma(n)+2}+v_{\gamma(n)}-6}=1$.

Proof. Let $x=x(n)$ be a bounded sequence and $T(n)=\max\{\beta(n),\gamma(n)\}, t(n)=\min\{\beta(n),\gamma(n)\}$. Then, we have for any n

$$\begin{aligned}
\left|(V_\beta x)_n-(V_\gamma x)_n\right| &= \left|\frac{2}{v_{\beta(n)+2}+v_{\beta(n)}-6}\sum_{k=1}^{\beta(n)}v_kx_k-\frac{2}{v_{\gamma(n)+2}+v_{\gamma(n)}-6}\sum_{k=1}^{\gamma(n)}v_kx_k\right| \\
&= \left|\frac{2}{v_{T(n)+2}+v_{T(n)}-6}\sum_{k=1}^{T(n)}v_kx_k-\frac{2}{v_{t(n)+2}+v_{t(n)}-6}\sum_{k=1}^{t(n)}v_kx_k\right| \\
&= \left|\frac{2}{v_{T(n)+2}+v_{T(n)}-6}\sum_{k=1}^{t(n)}v_kx_k+\frac{2}{v_{T(n)+2}+v_{T(n)}-6}\sum_{k=t(n)+1}^{T(n)}v_kx_k-\frac{2}{v_{t(n)+2}+v_{t(n)}-6}\sum_{k=1}^{t(n)}v_kx_k\right| \\
&= \left|\sum_{k=1}^{t(n)}v_kx_k\left(\frac{2}{v_{T(n)+2}+v_{T(n)}-6}-\frac{2}{v_{t(n)+2}+v_{t(n)}-6}\right)+\frac{2}{v_{T(n)+2}+v_{T(n)}-6}\sum_{k=t(n)+1}^{T(n)}v_kx_k\right| \\
&\le \|x\|_\infty\left(\sum_{k=1}^{t(n)}v_k\left|\frac{2\left(v_{t(n)+2}+v_{t(n)}-v_{T(n)+2}-v_{T(n)}\right)}{\left(v_{T(n)+2}+v_{T(n)}-6\right)\left(v_{t(n)+2}+v_{t(n)}-6\right)}\right|+\sum_{k=t(n)+1}^{T(n)}v_k\left|\frac{2}{v_{T(n)+2}+v_{T(n)}-6}\right|\right) \\
&\le \|x\|_\infty\left(\frac{\left(v_{t(n)+2}+v_{t(n)}-v_{T(n)+2}-v_{T(n)}\right)\left(v_{t(n)+2}+v_{t(n)}-6\right)}{\left(v_{T(n)+2}+v_{T(n)}-6\right)\left(v_{t(n)+2}+v_{t(n)}-6\right)}+\frac{v_{T(n)+2}+v_{T(n)}-v_{t(n)+2}-v_{t(n)}}{v_{T(n)+2}+v_{T(n)}-6}\right) \\
&\le 2\|x\|_\infty\left(\frac{\left(v_{T(n)+2}+v_{T(n)}-6\right)-\left(v_{t(n)+2}+v_{t(n)}-6\right)}{v_{T(n)+2}+v_{T(n)}-6}\right) \\
&\le 2\|x\|_\infty\left(1-\frac{v_{t(n)+2}+v_{t(n)}-6}{v_{T(n)+2}+v_{T(n)}-6}\right)=o(1)
\end{aligned}$$

since $\lim_{n\to\infty}\frac{v_{\beta(n)+2}+v_{\beta(n)}-6}{v_{\gamma(n)+2}+v_{\gamma(n)}-6}=1$ and $\lim_{n\to\infty}\frac{v_{t(n)+2}+v_{t(n)}-6}{v_{T(n)+2}+v_{T(n)}-6}=1$.

Therefore, if x is $V_\beta-$ summable to L, then we obtain

$$0\le\left|(V_\gamma x)_n-L\right|\le\left|(V_\gamma x)_n-(V_\beta x)_n\right|+\left|(V_\beta x)_n-L\right|=o(1)+o(1)=o(1)$$

and in a similar way, ,if x is $V_\gamma-$ summable to L, then we obtain

$$0\le\left|(V_\beta x)_n-L\right|\le\left|(V_\beta x)_n-(V_\gamma x)_n\right|+\left|(V_\gamma x)_n-L\right|=o(1)+o(1)=o(1).$$

Theorem 2.12. Cesaro matrix C_{nk} is stronger than Tribonacci-Lucas matrix $V=(v_{nk})$.

Proof. From Theorem 1.5, if we take $B=V$ (Tribonacci-Lucas matrix) and $A=C$ (Cesaro matrix), then we find

$$\begin{aligned}
\sum_{k=1}^{n}\left|\frac{2nv_k}{v_{n+2}+v_n-6}-\frac{2nv_{k+1}}{v_{n+2}+v_n-6}\right| &= \sum_{k=1}^{n}\frac{2n}{v_{n+2}+v_n-6}\left(v_k-v_{k+1}\right) \\
&= \frac{2n}{v_{n+2}+v_n-6}\sum_{k=1}^{n}\left(v_k-v_{k+1}\right).
\end{aligned}$$

Since the inequality $2n\le v_{n+2}+v_n-6$ holds for all $n\in\mathbb{N}$, we have

$$\begin{aligned}
&\frac{2n}{v_{n+2}+v_n-6}\sum_{k=1}^{n}\left(v_k-v_{k+1}\right)\le\frac{v_{n+2}+v_n-6}{v_{n+2}+v_n-6}\sum_{k=1}^{n}\left(v_k-v_{k+1}\right) \\
&\le v_1-v_2+v_2-v_3+v_3-v_4+\ldots+v_n-v_{n+1}\le 1-v_{n+1}<1.
\end{aligned}$$

Consequently, we obtain $V\subset C$ and the proof is completed.

In general, the converse of this theorem is not true. Indeed, for the sequence $x_n=\frac{(-1)^n}{n}$, $(C_nx)=\frac{1}{n}\sum_{k=1}^{n}\frac{(-1)^k}{k}$ is convergent but the $V-$ transformation of (x_n), that is $(V_nx)=\frac{2}{v_{n+2}+v_n-6}\sum_{k=1}^{n}v_k\frac{(-1)^k}{k}$ is not convergent.

3. Summary and Conclusions

In our study, a new regular matrix was first defined by using the well-known sequence of integer number called Tribonacci-Lucas. Then, we compared the Tribonacci-Lucas matrix with the other summability matrices such as Nörlund mean, Riesz mean and Cesaro mean and also investigated the relation between these matrices. Since the matrix $V=(v_{nk})$ is regular, the sequence (V_nx) is convergent for a sequence (x_n). So, for the matrix $V=(v_{nk})$, both statistical convergence and the studies with regular matrices can be investigated.

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