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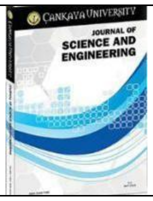

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A Comparative Analysis of Modified Extended Fractional Derivative and Integral Operators Via Modified Extended Beta Function with Applications to Generating Functions

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Keywords

Beta function,
Fractional calculus,
Gamma function,
Mellin transform.

Abstract

This article object is to introduce new extension of the extended beta, Gauss hypergeometric, Appell hypergeometric and Lauricella hypergeometric functions. The new extension of the extended Riemann-Liouville, Caputo and Kober-Erdelyi fractional derivative and integral operators are also examined with their applications to generating functions by considering the extended hypergeometric functions. The Mellin of certain new extension of the extended fractional derivative and integral operators ware obtained.

1. Introduction

In 2020, Saif et al., [1] introduced the following modified Laplace transforms:

$$L_{\sigma}\{f(t)\} = F\{s,\sigma\} = \int_0^{\infty} \sigma^{-st} f(t)dt, \tag{1}$$

where $Re(s) > 0, \sigma \in (0,\infty)\backslash\{1\}$.
Another variety of modified Laplace transform in equation (1) with its applications was also discussed in [2-8].
Kulip et al., [9] proposed the following modified gamma and beta functions:

$$\Gamma_{\lambda_1}(\theta_1;\sigma) = \int_0^{\infty} t^{\theta_1-1}\sigma^{\left(-t-\frac{\lambda_1}{t}\right)}dt, \tag{2}$$

where $Re(\theta_1) > 0, Re(\lambda_1) > 0, \sigma \in (0,\infty)\backslash\{1\}$.
And

$$B_{\lambda_1}(\theta_1,\theta_2;\sigma) = \int_0^1 t^{\theta_1-1}(1-t)^{\theta_2-1}\sigma^{\left(-\frac{\lambda_1}{t(1-t)}\right)}dt,$$

where $Re(\theta_1) > 0, Re(\theta_2) > 0, Re(\lambda_1) > 0, \sigma \in (0,\infty)\backslash\{1\}$.
Barahmah [10] established the following modified extended beta function:

$$B_{\lambda_1,\lambda_2}(\theta_1,\theta_2;\sigma) = \int_0^1 t^{\theta_1-1}(1-t)^{\theta_2-1}\sigma^{\left(-\frac{\lambda_1}{t}-\frac{\lambda_2}{(1-t)}\right)}dt,$$

where $Re(\theta_1) > 0, Re(\theta_2) > 0, Re(\lambda_1) > 0, Re(\lambda_2) > 0, \sigma \in (0,\infty)\backslash\{1\}$.

Inspired by the work of Cetinkaya et al., [11] the modified extended Riemann-Liouville, Caputo and Kober-Erdelyi fractional derivative and integral operators will be introduce and investigate via the following modified extended beta and hypergeometric functions:

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Definition 1: The modified extended beta function is

$$B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1,\theta_2;\sigma)=\int_0^1 t^{\theta_1-1}(1-t)^{\theta_2-1}\sigma^{\left(-\frac{\lambda_1}{t^{\varrho_1}}-\frac{\lambda_2}{(1-t)^{\varrho_2}}\right)}dt, \quad (3)$$

where $Re(\theta_1)>0, Re(\theta_2)>0, Re(\lambda_1)>0, Re(\lambda_2)>0, Re(\varrho_1)>0, Re(\varrho_2)>0, \sigma\in(0,\infty)\backslash\{1\}$.

Throughout this work ρ_1 *and* ρ_2 arbitrary parameters, $\mathbb{N}$ and $\mathbb{C}$ represent sets of natural and complex numbers, respectively.

Definition 2: The modified extended Gauss hypergeometric function is defined by

$$F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1,\theta_2,\theta_3;x;\sigma;\rho_1,\rho_2)=\sum_{\vartheta_1=0}^{\infty}\frac{(\theta_1)_{\vartheta_1}(\theta_2)_{\vartheta_1}}{(\theta_3)_{\vartheta_1}}\frac{B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_2+\vartheta_1-\rho_1,\theta_3-\theta_2+\rho_2;\sigma)}{B(\theta_2+\vartheta_1-\rho_1\theta_3-\theta_2+\rho_2)}\frac{x^{\vartheta_1}}{\vartheta_1}, \quad (4)$$

where $|x|<1$.

Definition 3: The modified extended Appell hypergeometric function are given as

$$F_{1;\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1,\theta_2,\theta_3;\theta_3;x,y;\sigma;\rho_1,\rho_2)$$
$$=\sum_{\vartheta_1,\vartheta_2=0}^{\infty}\frac{(\theta_1)_{\vartheta_1+\vartheta_2}(\theta_2)_{\vartheta_1}(\theta_3)_{\vartheta_2}}{(\theta_3)_{\vartheta_1+\vartheta_2}}\frac{B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1+\vartheta_1+\vartheta_2-\rho_1,\theta_4-\theta_1+\rho_2;\sigma)}{B(\theta_1+\vartheta_1+\vartheta_2-\rho_1,\theta_4-\theta_1+\rho_2)}\frac{x^{\vartheta_1}}{\vartheta_1}\frac{y^{\vartheta_2}}{\vartheta_2}, \quad (5)$$

Where $|x|<1, |y|<1$.
And

$$F_{2;\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1,\theta_2,\theta_3,\theta_4;\theta_5;x,y;\sigma;\rho_1,\rho_2)=\sum_{\vartheta_1,\vartheta_3=0}^{\infty}\frac{(\theta_1)_{\vartheta_1+\vartheta_2}(\theta_2)_{\vartheta_1}(\theta_3)_{\vartheta_2}}{(\theta_4)_{\vartheta_1}(\theta_5)_{\vartheta_2}}\frac{B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_2+\vartheta_1-\rho_1\theta_3-\theta_2+\rho_2;\sigma)}{B(\theta_2+\vartheta_1-\rho_1\theta_3-\theta_2+\rho_2)}$$
$$\times\frac{B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_3+\vartheta_1-\rho_1\theta_5-\theta_3+\rho_2;\sigma)}{B(\theta_3+\vartheta_1+\rho_1\theta_5-\theta_3+\rho_2)}\frac{x^{\vartheta_1}}{\vartheta_1}\frac{y^{\vartheta_2}}{\vartheta_3}, \quad (6)$$

Where $|x|+|y|<1$.

Definition 4: The modified extended Laurecilla hypergeometric function is

$$F_{D;\lambda_1,\lambda_2}^{3;\varrho_1,\varrho_2}(\theta_1,\theta_2,\theta_3,\theta_4;\theta_5;x,y;\sigma;\rho_1,\rho_2)=\sum_{\vartheta_1,\vartheta_2,\vartheta_3=0}^{\infty}\frac{(\theta_1)_{\vartheta_1+\vartheta_2+\vartheta_3}(\theta_2)_{\vartheta_1}(\theta_5)_{\vartheta_2}(\theta_5)_{\vartheta_3}}{(\theta_5)_{\vartheta_1+\vartheta_2+\vartheta_3}}$$
$$\times\frac{B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1+\vartheta_1+\vartheta_2+\vartheta_3-\rho_1,\theta_5-\theta_1+\rho_2)}{B(\theta_1+\vartheta_1+\vartheta_2+\vartheta_3-\rho_1,\theta_4-\theta_1+\rho_2)}\frac{x^{\vartheta_1}}{\vartheta_1}\frac{y^{\vartheta_2}}{\vartheta_2}\frac{z^{\vartheta_3}}{\vartheta_3}, \quad (7)$$

Definition 5: The modified extended Riemann-Liouville fractional derivative is expressed by

$$D_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{f(z);\sigma\}=\begin{cases}\frac{1}{\Gamma(-\varrho)}\int_0^z(z-t)^{-\varrho-1}f(t)\,\sigma^{\left(-\lambda_1\left[\frac{z}{t}\right]^{\varrho_1}-\lambda_2\left[\frac{z}{z-t}\right]^{\varrho_2}\right)}dt, & \text{for } Re(\varrho)<0,\\ \frac{d^{\daleth}}{dx^{\daleth}}\left\{D_{z;\lambda_1,\lambda_2}^{\varrho-\daleth;\varrho_1,\varrho_2}\{f(z);\sigma\}\right\}, & \text{for } \daleth-1<Re(\varrho)<\daleth, \daleth\in N.\end{cases} \quad (8)$$

Definition 6: The modified extended Caputo fractional derivative is expressed by

$$\mathcal{D}_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{f(z);\sigma\}=D_{z;\lambda_1,\lambda_2}^{\varrho-\daleth;\varrho_1,\varrho_2}\{f^{\daleth}(z);\sigma\}$$
$$=\frac{1}{\Gamma(\daleth-\varrho)}\int_0^z(z-t)^{\daleth-\varrho-1}f^{\daleth}(t)\,\sigma^{\left(-\lambda_1\left[\frac{z}{t}\right]^{\varrho_1}-\lambda_2\left[\frac{z}{z-t}\right]^{\varrho_2}\right)}dt, \quad (9)$$

where $\daleth-1<Re(\varrho)<\daleth$.

Definition 7: The modified extended Kober-Erdelyi fractional integral is

$$I_{z;\lambda;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{f(z);\sigma\}=z^{-\varrho-\lambda}D_{z;\lambda_1,\lambda_2}^{-\varrho;\varrho_1,\varrho_2}\{z^{\lambda}f(z);\sigma\}$$
$$=\frac{z^{-\varrho-\lambda}}{\Gamma(\varrho)}\int_0^z(z-t)^{-\varrho-1}t^{\lambda}f(t)\,\sigma^{\left(-\lambda_1\left[\frac{z}{t}\right]^{\varrho_1}-\lambda_2\left[\frac{z}{z-t}\right]^{\varrho_2}\right)}dt, \quad (10)$$

where $Re(\varrho)>0, \lambda\in\mathbb{C}$.

2. Fractional Derivative and Integral Formulas

Theorem 1: Let $Re(\varrho) < 0$ and $Re(\theta) > -1$, then

$$D_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^{\theta};\sigma\} = \frac{\Gamma(\theta+1)}{\Gamma(\theta-\varrho+1)}\frac{B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta+1,-\varrho;\sigma)}{B(\theta+1,-\varrho)} z^{\theta-\varrho}. \quad (11)$$

Proof Using $Re(\varrho) < 0$ and $Re(\theta) > -1$, leads to

$$D_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^{\theta};\sigma\} = \frac{1}{\Gamma(-\varrho)}\int_0^z t^{\theta}(z-t)^{-\varrho-1}\,\sigma^{\left(-\lambda_1\left[\frac{z}{t}\right]^{\varrho_1}-\lambda_2\left[\frac{z}{z-t}\right]^{\varrho_2}\right)}dt.$$

Setting $t = \psi z$, gives

$$D_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^{\theta};\sigma\} = \frac{z^{\theta-\varrho}}{\Gamma(-\varrho)}\int_0^1 \psi^{\theta+1-1}(1-\psi)^{-\varrho-1}\,\sigma^{\left(-\frac{\lambda_1}{\psi^{\varrho_1}}-\frac{\lambda_2}{(1-\psi)^{\varrho_2}}\right)}d\psi.$$

Applying equation (3), yields

$$D_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^{\theta};\sigma\} = \frac{z^{\theta-\varrho}}{\Gamma(-\varrho)} B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta+1,-\varrho;\sigma).$$

On simplifying, one can obtain

$$D_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^{\theta};\sigma\} = \frac{\Gamma(\theta+1)}{\Gamma(\theta-\varrho+1)}\frac{B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta+1,-\varrho;\sigma)}{B(\theta+1,-\varrho)} z^{\theta-\varrho}.$$

Corollary 2: For $\daleth - 1 < Re(\varrho) < \daleth$ and $Re(\theta) > -1$, then

$$D_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^{\theta};\sigma\} = \frac{\Gamma(\theta+1)}{\Gamma(\theta-\varrho+1)}\frac{B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta+1,\daleth-\varrho;\sigma)}{B(\theta+1,\daleth-\varrho)} z^{\theta-\varrho}. \quad (12)$$

Corollary 3: For $\daleth - 1 < Re(\varrho) < \daleth$ and $Re(\theta) > \daleth - 1$, then

$$\mathcal{D}_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^{\theta};\sigma\} = \frac{\Gamma(\theta+1)}{\Gamma(\theta-\varrho+1)}\frac{B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta-\daleth+1,\daleth-\varrho;\sigma)}{B(\theta-\daleth+1,\daleth-\varrho)} z^{\theta-\varrho}. \quad (13)$$

Corollary 4: Let $Re(\varrho) > 0$ and $Re(\theta) > Re(\varrho + \lambda)$, then

$$I_{z;\lambda;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^{\theta};\sigma\} = \frac{\Gamma(\theta+\lambda+1)}{\Gamma(\theta+\lambda+\varrho+1)}\frac{B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta+\lambda+1,\varrho;\sigma)}{B(\theta+\lambda+1,\varrho)} z^{\theta}. \quad (14)$$

Theorem 5: For analytic function $f(z)$ at the origin with its Maclaurin expansion expressed by $f(z) = \sum_{\vartheta_1}^{\infty} a_{\vartheta_1} z^{\vartheta_1}$ with $|z| < \wp$, then

$$D_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{f(z);\sigma\} = \sum_{\vartheta_1}^{\infty} a_{\vartheta_1} D_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^{n};\sigma\}.$$

$$\mathcal{D}_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{f(z);\sigma\} = \sum_{\vartheta_1}^{\infty} a_{\vartheta_1} \mathcal{D}_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^{n};\sigma\}, \qquad \text{for } \daleth - 1 < Re(\varrho) < \daleth.$$

And

$$I_{z;\lambda;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{f(z);\sigma\} = \sum_{\vartheta_1}^{\infty} a_{\vartheta_1} I_{z;\lambda;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^{n};\sigma\}, \qquad \text{for } Re(\varrho) < 0.$$

Proof Considering equations (8), (9), (10), and the fact that term-by-term integration is guaranteed the results follow.

Theorem 6: Suppose $0 < Re(\theta) < Re(\varrho)$, and $|\delta_1 z| < 1$, then

$$D_{z;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\{z^{\theta-1}(1-\delta_1 z)^{-\theta_1};\sigma\} = \frac{\Gamma(\theta)}{\Gamma(\varrho)} F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1,\varrho,\theta;\delta_1 z;\sigma;0,0) z^{\varrho-1}.$$

Proof Using $(1-\delta_1 z)^{-\theta_1} = \sum_{\vartheta_1=0}^{\infty} \frac{(\theta_1)_{\vartheta_1}}{\vartheta_1}(\delta_1 z)^{\vartheta_1}$, gives

$$D_{z;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-vz)^{-\theta_1};\sigma\right\}=\sum_{\vartheta_1=0}^{\infty}\frac{(\theta_1)_{\vartheta_1}}{\vartheta_1}D_{z;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta+\vartheta_1-1};\sigma\right\}.$$

Applying equation (11), yields

$$D_{z;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{-\theta_1};\sigma\right\}=\frac{\Gamma(\theta)}{\Gamma(\varrho)}z^{\varrho-1}\sum_{\vartheta_1=0}^{\infty}\frac{(\theta_1)_{\vartheta_1}(\theta)_{\vartheta_1}}{(\varrho)_{\vartheta_1}}\frac{B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta+\vartheta_1,\theta-\varrho;\sigma)}{B(\theta+\vartheta_1,\theta-\varrho)}\frac{(\delta_1 z)^{\vartheta_1}}{\vartheta_1}.$$

This can be rewriting as

$$D_{z;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{-\theta_1};\sigma\right\}=\frac{\Gamma(\theta)}{\Gamma(\varrho)}F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1,\varrho,\theta;\delta_1 z;\sigma;0,0)z^{\varrho-1}.$$

Corollary 7: For $\daleth-1<Re(\theta-\varrho)<\daleth$ and $Re(\theta)>0$, then

$$D_{z;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{\theta_1};\sigma\right\}=\frac{\Gamma(\theta)}{\Gamma(\varrho)}F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1,\varrho,\theta;\delta_1 z;\sigma;0,\daleth)z^{\varrho-1}.$$

Corollary 8: For $\daleth-1<Re(\theta-\varrho)<\daleth$ and $Re(\theta)>0$, then

$$\mathcal{D}_{z;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{-\theta_1};\sigma\right\}=\frac{\Gamma(\theta)}{\Gamma(\varrho)}F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1,\varrho;\theta;\delta_1 z;\sigma;\daleth,\daleth)z^{\varrho-1}.$$

Corollary 9: For $Re(\theta-\varrho)>0$ and $Re(\theta+\lambda)>0$, then

$$I_{z;\lambda;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{-\theta_1};\sigma\right\}=\frac{\Gamma(\theta+\lambda)}{\Gamma(2\theta+\lambda-\varrho)}F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1,\theta+\lambda;2\theta+\lambda-\varrho;\delta_1 z;\sigma;0,0)z^{\theta-1}.$$

Theorem 10: Suppose $0<Re(\theta)<Re(\varrho)$, $|\delta_1 z|<1$, and $|\delta_2 z|<1$, then

$$D_{z;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{-\theta_1}(1-\delta_2 z)^{-\theta_2};\sigma\right\}=\frac{\Gamma(\theta)}{\Gamma(\varrho)}F_{1;\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta,\theta_1,\theta_2;\varrho;\delta_1 z,\delta_2 z;\sigma;0,0)z^{\varrho-1}.$$

Corollary 11: For $\daleth-1<Re(\theta-\varrho)<\daleth, Re(\theta)>0$, $|\delta_1 z|<1$, and $|\delta_2 z|<1$, then

$$D_{z;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{\theta_1}(1-\delta_2 z)^{-\theta_2};\sigma\right\}=\frac{\Gamma(\theta)}{\Gamma(\varrho)}F_{1;\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta,\theta_1,\theta_2;\varrho;\delta_1 z,\delta_2 z;\sigma;0,\daleth)z^{\varrho-1}.$$

Corollary 12: For $\daleth-1<Re(\theta-\varrho)<\daleth, Re(\theta)>0$, $|\delta_1 z|<1$, and $|\delta_2 z|<1$, then

$$\mathcal{D}_{z;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{-\theta_1}(1-\delta_2 z)^{-\theta_2};\sigma\right\}=\frac{\Gamma(\theta)}{\Gamma(\varrho)}F_{1;\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta,\theta_1,\theta_2;\varrho;\delta_1 z,\delta_2 z;\sigma;\daleth,\daleth)z^{\varrho-1}.$$

Corollary 13: For $Re(\theta-\varrho)>0, Re(\theta+\lambda)>0$, $|\delta_1 z|<1$, and $|\delta_2 z|<1$, then

$$I_{z;\lambda;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{-\theta_1}(1-\delta_2 z)^{-\theta_2};\sigma\right\}$$
$$=\frac{\Gamma(\theta+\lambda)}{\Gamma(2\theta+\lambda-\varrho)}F_{1;\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta+\lambda,\theta_1,\theta_1;2\theta+\lambda-\varrho;\delta_1 z,\delta_2 z;\sigma;0,0)z^{\theta-1}.$$

Theorem 14: Suppose $0<Re(\theta)<Re(\varrho)$, $|\delta_1 z|<1$, $|\delta_1 z|<1$, $|\delta_2 z|<1$, and $|\delta_3 z|<1$, then

$$D_{z;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{-\theta_1}(1-\delta_2 z)^{-\theta_2}(1-\delta_3 z)^{-\theta_3};\sigma\right\}$$
$$=\frac{\Gamma(\theta)}{\Gamma(\varrho)}F_{D;\lambda_1,\lambda_2}^{3;\varrho_1,\varrho_2}(\theta,\theta_1,\theta_2,\theta_2;\varrho;\delta_1 z,\delta_2 z,\delta_2 z;\sigma;0,0)z^{\varrho-1}.$$

Corollary 15: For $\daleth - 1 < Re(\theta - \varrho) < \daleth, Re(\theta) > 0$, $|\delta_1 z| < 1, |\delta_2 z| < 1$, and $|\delta_3 z| < 1$, then

$$D_{z;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{\theta_1}(1-\delta_2 z)^{-\theta_2}(1-\delta_3 z)^{-\theta_3};\sigma\right\}$$
$$= \frac{\Gamma(\theta)}{\Gamma(\varrho)} F_{D;\lambda_1,\lambda_2}^{3;\varrho_1,\varrho_2}(\theta,\theta_1,\theta_2,\theta_2;\varrho;\delta_1 z,\delta_2 z,\delta_2 z;\sigma;0,\daleth)z^{\varrho-1}.$$

Corollary 16: For $\daleth - 1 < Re(\theta - \varrho) < \daleth, Re(\theta) > 0$, $|\delta_1 z| < 1, |\delta_2 z| < 1$, and $|\delta_3 z| < 1$, then

$$\mathcal{D}_{z;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{-\theta_1}(1-\delta_2 z)^{-\theta_2}(1-\delta_3 z)^{-\theta_3};\sigma\right\}$$
$$= \frac{\Gamma(\theta)}{\Gamma(\varrho)} F_{D;\lambda_1,\lambda_2}^{3;\varrho_1,\varrho_2}(\theta,\theta_1,\theta_2,\theta_2;\varrho;\delta_1 z,\delta_2 z,\delta_2 z;\sigma;\daleth,\daleth)z^{\varrho-1}.$$

Corollary 17: For $Re(\theta - \varrho) > 0, Re(\theta + \lambda) > 0$, $|\delta_1 z| < 1, |\delta_2 z| < 1$, and $|\delta_3 z| < 1$, then

$$I_{z;\lambda;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{-\theta_1}(1-\delta_2 z)^{-\theta_2}(1-\delta_3 z)^{-\theta_3};\sigma\right\}$$
$$= \frac{\Gamma(\theta+\lambda)}{\Gamma(2\theta+\lambda-\varrho)} F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta+\lambda,\theta_1,\theta_2,\theta_3;2\theta+\lambda-\varrho;\delta_1 z,\delta_2 z,\delta_3 z;\sigma;0,0)z^{\theta-1}.$$

Theorem 18: Suppose $0 < Re(\theta) < Re(\varrho), |y| + |\delta_1 z| < 1$ and $|\delta_1 z| < 1$, then

$$D_{z;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{-\theta_1}F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}\left(\theta_1,\theta_2,\theta_3;\frac{x}{1-\delta_1 z};\sigma;0,0\right);\sigma\right\}$$
$$= \frac{\Gamma(\theta)}{\Gamma(\varrho)} F_{2;\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1,\theta_2,\theta;\theta_3,\varrho;y,\delta_1 z;\sigma;0,0)z^{\varrho-1}.$$

Corollary 19: For $\daleth - 1 < Re(\theta - \varrho) < \daleth, |y| + |\delta_1 z| < 1$, and $Re(\theta) > 0$, then

$$D_{z;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{\theta_1}F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}\left(\theta_1,\theta_2,\theta_3;\frac{x}{1-\delta_1 z};\sigma;0,\daleth\right);\sigma\right\}$$
$$= \frac{\Gamma(\theta)}{\Gamma(\varrho)} F_{2;\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1,\theta_2,\theta;\theta_3,\varrho;y,\delta_1 z;\sigma;0,\daleth)z^{\varrho-1}.$$

Corollary 20: For $\daleth - 1 < Re(\theta - \varrho) < \daleth, |y| + |\delta_1 z| < 1$, and $Re(\theta) > 0$, then

$$\mathcal{D}_{z;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{-\theta_1}F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}\left(\theta_1,\theta_2,\theta_3;\frac{x}{1-\delta_1 z};\sigma;\daleth,\daleth\right);\sigma\right\}$$
$$= \frac{\Gamma(\theta)}{\Gamma(\varrho)} F_{2;\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1,\theta_2,\theta;\theta_3,\varrho;y,\delta_1 z;\sigma;\daleth,\daleth)z^{\varrho-1}.z^{\varrho-1}.$$

Corollary 21: For $|y| + |\delta_1 z| < 1, Re(\theta - \varrho) > 0$, and $Re(\theta + \lambda) > 0$, then

$$I_{z;\lambda;\lambda_1,\lambda_2}^{\theta-\varrho;\varrho_1,\varrho_2}\left\{z^{\theta-1}(1-\delta_1 z)^{-\theta_1}F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}\left(\theta_1,\theta_2,\theta_3;\frac{x}{1-\delta_1 z};\sigma;0,0\right);\sigma\right\}$$
$$= \frac{\Gamma(\theta+\lambda)}{\Gamma(2\theta+\lambda-\varrho)} F_{2;\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1,\theta_2,\theta+\lambda;\theta_3,2\theta+\lambda-\varrho;y,\delta_1 z;\sigma;0,0)z^{\theta-1}.$$

3. The Mellin Transforms

Theorem 22: Let $Re(s) > 0, Re(r) > 0, Re(\theta + s\varrho_1) > -1, Re(-\varrho + r\varrho_2) > 0, Re(\varrho) < 0$ and $Re(\theta) > -1$, then

$$\mathcal{M}\left\{D_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^\theta;\sigma\};\lambda_1 \to s, \lambda_2 \to r\right\} = \frac{\Gamma(\theta+1)\Gamma(s;\sigma)\Gamma(r;\sigma)}{\Gamma(\theta-\varrho+1)}\frac{B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta+s\varrho_1+1,-\varrho+r\varrho_2;\sigma)}{B(\theta+1,-\varrho)}z^{\theta-\varrho}.$$

Proof In view of equations (3) and (11), one can obtain

$$\mathcal{M}\left\{D_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^\theta;\sigma\};\lambda_1 \to s, \lambda_2 \to r\right\}$$
$$= \frac{\Gamma(\theta+1)z^{\theta-\varrho}}{\Gamma(\theta-\varrho+1)B(\theta+1,-\varrho)}\int_0^1 t^\theta(1-t)^{-\varrho-1}\left\{\int_0^\infty \lambda_1^{s-1}\sigma^{\left(-\frac{\lambda_1}{t^{\varrho_1}}\right)}d\lambda_1\right\}\left\{\int_0^\infty \lambda_2^{r-1}\sigma^{\left(-\frac{\lambda_2}{t^{\varrho_2}}\right)}d\lambda_2\right\}dt.$$

On setting $\psi = \frac{\lambda_1}{t^{\varrho_1}}$ and $\chi = \frac{\lambda_2}{t^{\varrho_2}}$, yields

$$\mathcal{M}\left\{D_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^\theta;\sigma\};\lambda_1 \to s, \lambda_2 \to r\right\}$$
$$= \frac{\Gamma(\theta+1)z^{\theta-\varrho}}{\Gamma(\theta-\varrho+1)B(\theta+1,-\varrho)}\int_0^1 t^{\theta+s\varrho_1}(1-t)^{-\varrho+r\varrho_2-1}\left\{\int_0^\infty \psi^{s-1}\sigma^{(-\psi)}d\psi\right\}\left\{\int_0^\infty \varpi^{r-1}\sigma^{(-\varpi)}d\varpi\right\}dt.$$

Applying equation (2), gives

$$\mathcal{M}\left\{D_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^\theta;\sigma\};\lambda_1 \to s, \lambda_2 \to r\right\} = \frac{\Gamma(\theta+1)\Gamma(s;\sigma)\Gamma(r;\sigma)}{\Gamma(\theta-\varrho+1)}\frac{B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta+s\varrho_1+1,-\varrho+r\varrho_2;\sigma)}{B(\theta+1,-\varrho)}z^{\theta-\varrho}.$$

The following Corollaries also follow from equations (12), (13), and (14).

Corollary 23: For $\daleth - 1 < Re(\varrho) < \daleth$ and $Re(\theta) > -1$, then

$$\mathcal{M}\left\{D_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^\theta;\sigma\};\lambda_1 \to s, \lambda_2 \to r\right\} = \frac{\Gamma(\theta+1)\Gamma(s;\sigma)\Gamma(r;\sigma)}{\Gamma(\theta-\varrho+1)}\frac{B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta+s\varrho_1+1,\daleth-\varrho+r\varrho_2;\sigma)}{B(\theta+1,\daleth-\varrho)}z^{\theta-\varrho}.$$

Corollary 24: For $\daleth - 1 < Re(\varrho) < \daleth$ and $Re(\theta) > \daleth - 1$, then

$$\mathcal{M}\left\{\mathcal{D}_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^\theta;\sigma\};\lambda_1 \to s, \lambda_2 \to r\right\} = \frac{\Gamma(\theta+1)\Gamma(s;\sigma)\Gamma(r;\sigma)}{\Gamma(\theta-\varrho+1)}\frac{B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta+s\varrho_1-\daleth+1,\daleth-\varrho+r\varrho_2;\sigma)}{B(\theta-\daleth+1,\daleth-\varrho)}z^{\theta-\varrho}.$$

Corollary 25: Let $Re(\varrho) > 0$ and $Re(\theta) > Re(\varrho + \lambda)$, then

$$\mathcal{M}\left\{I_{z;\lambda;\lambda;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{z^\theta;\sigma\};\lambda_1 \to s, \lambda_2 \to r\right\} = \frac{\Gamma(\theta+\lambda+1)\Gamma(s;\sigma)\Gamma(r;\sigma)}{\Gamma(\theta+\lambda+\varrho+1)}\frac{B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta+\lambda+s\varrho_1+1,\varrho+r\varrho_2;\sigma)}{B(\theta+\lambda+1,\varrho)}z^{\theta}.$$

Theorem 26: Let $Re(s) > 0, Re(r) > 0$, and $|z| < 1$.

$$\mathcal{M}\left\{D_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{(1-z)^{-\theta_1};\sigma\};\lambda_1 \to s, \lambda_2 \to r\right\}$$
$$= \frac{\Gamma(s;\sigma)\Gamma(r;\sigma)}{\Gamma(-\varrho)}z^{-\varrho}\sum_{\vartheta_1}^{\infty}(\theta_1)_{\vartheta_1}B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\vartheta_1 + s\varrho_1 + 1, -\varrho + r\varrho_2;\sigma)\frac{z^{\vartheta_1}}{\vartheta_1}.$$

Corollary 27: Let $Re(s) > 0, Re(r) > 0$, and $|z| < 1$.

$$\mathcal{M}\left\{D_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\{(1-z)^{-\theta_1};\sigma\};\lambda_1 \to s, \lambda_2 \to r\right\}$$
$$= \frac{\Gamma(s;\sigma)\Gamma(r;\sigma)\Gamma(\daleth-\varrho+1)}{\Gamma(\daleth-\varrho)\Gamma(-\varrho+1)}z^{-\varrho}\sum_{\vartheta_1}^{\infty}\frac{(\theta_1)_{\vartheta_1}(\daleth-\varrho+1)_{\vartheta_1}}{(-\varrho+1)_{\vartheta_1}}B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\vartheta_1 + s\varrho_1 + 1, \daleth - \varrho + r\varrho_2;\sigma)\frac{z^{\vartheta_1}}{\vartheta_1}.$$

Corollary 28: Let $Re(s) > 0, Re(r) > 0$, and $|z| < 1$.

$$\mathcal{M}\left\{\mathcal{D}_{z;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\left\{(1-z)^{-\theta_1};\sigma\right\};\lambda_1 \to s, \lambda_2 \to r\right\}$$
$$= \frac{\Gamma(s;\sigma)\Gamma(r;\sigma)\Gamma(\daleth-\varrho+1)}{\Gamma(\daleth-\varrho)\Gamma(-\daleth+1)} z^{-\varrho} \sum_{\vartheta_1}^{\infty} \frac{(\theta_1)_{\vartheta_1}}{(-\daleth+1)_{\vartheta_1}} B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\vartheta_1 + s\varrho_1 - \daleth + 1, \daleth - \varrho + r\varrho_2; \sigma) z^{\vartheta_1}.$$

Corollary 29: Let $Re(s) > 0, Re(r) > 0$, and $|z| < 1$.

$$\mathcal{M}\left\{I_{z;\lambda;\lambda_1,\lambda_2}^{\varrho;\varrho_1,\varrho_2}\left\{(1-z)^{-\theta_1};\sigma\right\};\lambda_1 \to s, \lambda_2 \to r\right\}$$
$$= \frac{\Gamma(s;\sigma)\Gamma(r;\sigma)}{\Gamma(\varrho)} \sum_{\vartheta_1}^{\infty} (\theta_1)_{\vartheta_1} B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\vartheta_1 + \lambda + s\varrho_1 + 1, \vartheta_1 + r\varrho_2; \sigma) \frac{z^{\vartheta_1}}{\vartheta_1}.$$

4. Generating Functions

This section consists of application of the modified extended fractional derivative and integral to generating functions, the detail proofs is omitted and is similar to those in [11-17].

Theorem 30: Let $|z| < \min\{1, |1-t|\}, |t| < |1-t|,$ and $0 < Re(\theta) < Re(\varrho)$, then

$$\sum_{\vartheta_1}^{\infty} \frac{(\theta_1)_{\vartheta_1}}{\vartheta_1} F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1 + \vartheta_1, \theta; \varrho; z; \sigma; 0,0) t^{\vartheta_1} = (1-t)^{-\theta_1} F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}\left(\theta_1, \theta; \varrho; \frac{z}{1-t}; \sigma; 0,0\right).$$

Corollary 31: Let $|z| < \min\{1, |1-t|\}, |t| < |1-t|, \daleth - 1 < Re(\theta - \varrho) < \daleth, Re(\theta) > 0$, then

$$\sum_{\vartheta_1}^{\infty} \frac{(\theta_1)_{\vartheta_1}}{\vartheta_1} F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1 + \vartheta_1, \theta; \varrho; z; \sigma; 0, \daleth) t^{\vartheta_1} = (1-t)^{-\theta_1} F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}\left(\theta_1, \theta; \varrho; \frac{z}{1-t}; \sigma; 0, \daleth\right).$$

Corollary 32: Let $|z| < \min\{1, |1-t|\}, |t| < |1-t|, \daleth - 1 < Re(\theta - \varrho) < \daleth, Re(\theta) > 0$, then

$$\sum_{\vartheta_1}^{\infty} \frac{(\theta_1)_{\vartheta_1}}{\vartheta_1} F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1 + \vartheta_1, \theta; \varrho; z; \sigma; \daleth, \daleth) t^{\vartheta_1} = (1-t)^{-\theta_1} F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}\left(\theta_1, \theta; \varrho; \frac{z}{1-t}; \sigma; \daleth, \daleth\right).$$

Corollary 33: Let $|z| < \min\{1, |1-t|\}, |t| < |1-t|, Re(\theta - \varrho) > 0,$ and $Re(\theta + \lambda) > 0$, then

$$\sum_{\vartheta_1}^{\infty} \frac{(\theta_1)_{\vartheta_1}}{\vartheta_1} F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1 + \vartheta_1, \theta + \lambda; 2\theta + \lambda - \varrho; z; \sigma; 0,0) t^{\vartheta_1}$$
$$= (1-t)^{-\theta_1} F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}\left(\theta_1, \theta + \lambda; 2\theta + \lambda - \varrho; \frac{z}{1-t}; \sigma; 0,0\right).$$

Theorem 34: Let $|z| < \min\{1, |t^{-1} - 1|\}, |t| < |1-z|^{-1},$ and $0 < Re(\theta) < Re(\varrho)$, then

$$\sum_{\vartheta_1}^{\infty} \frac{(\theta_1)_{\vartheta_1}}{\vartheta_1} F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_3 - \vartheta_1, \theta; \varrho; z; \sigma; 0,0) t^{\vartheta_1} = (1-t)^{-\theta_1} F_{1;\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}\left(\theta, \theta_1, \theta_3; \varrho; \frac{-tz}{1-t}, z; \sigma; 0,0\right).$$

Corollary 35: Let $|z| < \min\{1, |t^{-1} - 1|\}, |t| < |1-z|^{-1}, \daleth - 1 < Re(\theta - \varrho) < \daleth, Re(\theta) > 0$, then

$$\sum_{\vartheta_1}^{\infty} \frac{(\theta_1)_{\vartheta_1}}{\vartheta_1} F_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_3 - \vartheta_1, \theta; \varrho; z; \sigma; 0, \daleth) t^{\vartheta_1} = (1-t)^{-\theta_1} F_{1;\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}\left(\theta, \theta_1, \theta_3; \varrho; \frac{-tz}{1-t}, z; \sigma; 0, \daleth\right).$$

Corollary 36: Let $|z| < \min\{1, |t^{-1} - 1|\}, |t| < |1-z|^{-1}, \daleth - 1 < Re(\theta - \varrho) < \daleth, Re(\theta) > 0$, then

$$\sum_{\vartheta_1}^{\infty}\frac{(\theta_1)_{\vartheta_1}}{\vartheta_1}F_{\gimel_1,\gimel_2}^{\varrho_1,\varrho_2}(\theta_3-\vartheta_1,\theta;\varrho;z;\sigma;\daleth,\daleth)t^{\vartheta_1}=(1-t)^{-\theta_1}F_{1;\gimel_1,\gimel_2}^{\varrho_1,\varrho_2}\left(\theta,\theta_1,\theta_3;\varrho;\frac{-tz}{1-t},z;\sigma;\daleth,\daleth\right).$$

Corollary 37: Let $|z|<\min\{1,|t^{-1}-1|\}, |t|<|1-z|^{-1}, Re(\theta-\varrho)>0,$ and $Re(\theta+\gimel)>0$, then

$$\sum_{\vartheta_1}^{\infty}\frac{(\theta_1)_{\vartheta_1}}{\vartheta_1}F_{\gimel_1,\gimel_2}^{\varrho_1,\varrho_2}(\theta_3-\vartheta_1,\theta+\gimel;2\theta+\gimel-\varrho;z;\sigma;0,0)t^{\vartheta_1}$$
$$=(1-t)^{-\theta_1}F_{\gimel_1,\gimel_2}^{\varrho_1,\varrho_2}\left(\theta+\gimel,\theta_1,\theta_3;2\theta+\gimel-\varrho;\frac{-tz}{1-t},z;\sigma;0,0\right).$$

Theorem 38: Suppose $|z|<1, |(1-y)t|<|1-z|, |z|+|yt|<|1-t|,\ 0<Re(\theta)<Re(\varrho),$ and $0<Re(\theta_3)<Re(\theta_3)$, then

$$\sum_{\vartheta_1}^{\infty}\frac{(\theta_1)_{\vartheta_1}}{\vartheta_1}F_{\gimel_1,\gimel_2}^{\varrho_1,\varrho_2}(\theta_1+\vartheta_1,\theta;\varrho;z;\sigma;0,0)F_{\gimel_1,\gimel_2}^{\varrho_1,\varrho_2}(-\vartheta_1,\theta_3;\theta_4;z;\sigma;0,0)t^{\vartheta_1}$$
$$=(1-t)^{-\theta_1}F_{2;\gimel_1,\gimel_2}^{\varrho_1,\varrho_2}\left(\theta_1,\theta,\theta_3;\varrho,\theta_4;\frac{z}{1-t},\frac{-yt}{1-t};\sigma;0,0\right).$$

Corollary 39: Suppose $|z|<1, |(1-y)t|<|1-z|, |z|+|yt|<|1-t|, \daleth-1<Re(\theta-\varrho)<\daleth, Re(\theta)>0,$ $\daleth-1<Re(\theta_3-\theta_4)<\daleth, Re(\theta_3)>0$, then

$$\sum_{\vartheta_1}^{\infty}\frac{(\theta_1)_{\vartheta_1}}{\vartheta_1}F_{\gimel_1,\gimel_2}^{\varrho_1,\varrho_2}(\theta_1+\vartheta_1,\theta;\varrho;z;\sigma;0,\daleth)F_{\gimel_1,\gimel_2}^{\varrho_1,\varrho_2}(-\vartheta_1,\theta_3;\theta_4;z;\sigma;0,\daleth)t^{\vartheta_1}$$
$$=(1-t)^{-\theta_1}F_{2;\gimel_1,\gimel_2}^{\varrho_1,\varrho_2}\left(\theta_1,\theta,\theta_3;\varrho,\theta_4;\frac{z}{1-t},\frac{-yt}{1-t};\sigma;0,\daleth\right).$$

Corollary 40: Suppose $|z|<1, |(1-y)t|<|1-z|, |z|+|yt|<|1-t|,\ \daleth-1<Re(\theta-\varrho)<\daleth, Re(\theta)>0,$ $\daleth-1<Re(\theta_3-\theta_4)<\daleth, Re(\theta_3)>0$, then

$$\sum_{\vartheta_1}^{\infty}\frac{(\theta_1)_{\vartheta_1}}{\vartheta_1}F_{\gimel_1,\gimel_2}^{\varrho_1,\varrho_2}(\theta_1+\vartheta_1,\theta;\varrho;z;\sigma;\daleth,\daleth)F_{\gimel_1,\gimel_2}^{\varrho_1,\varrho_2}(-\vartheta_1,\theta_3;\theta_4;z;\sigma;\daleth,\daleth)t^{\vartheta_1}$$
$$=(1-t)^{-\theta_1}F_{2;\gimel_1,\gimel_2}^{\varrho_1,\varrho_2}\left(\theta_1,\theta,\theta_3;\varrho,\theta_4;\frac{z}{1-t},\frac{-yt}{1-t};\sigma;\daleth,\daleth\right).$$

Corollary 41: Suppose $|z|<1, |(1-y)t|<|1-z|, |z|+|yt|<|1-t|, Re(\theta-\varrho)>0,$ and $Re(\theta+\gimel)>0,$ $Re(\theta_3-\theta_4)>0,$ and $Re(\theta_3+\aleph)>0$then

$$\sum_{\vartheta_1}^{\infty}\frac{(\theta_1)_{\vartheta_1}}{\vartheta_1}F_{\gimel_1,\gimel_2}^{\varrho_1,\varrho_2}(\theta_1+\vartheta_1,\theta+\gimel;2\theta+\gimel-\varrho;z;\sigma;0,0)F_{\gimel_1,\gimel_2}^{\varrho_1,\varrho_2}(-\vartheta_1,\theta_3+\gimel;2\theta_3+\aleph-\theta_4;y;\sigma;\daleth,\daleth)t^{\vartheta_1}$$
$$=(1-t)^{-\theta_1}F_{2;\gimel_1,\gimel_2}^{\varrho_1,\varrho_2}\left(\theta_1,\theta+\gimel,\theta_4+\aleph;2\theta+\gimel-\varrho,2\theta_4+\aleph-\theta_3;\frac{z}{1-t},\frac{-yt}{1-t};\sigma;0,0\right).$$

Theorem 42: Let $|\delta_1 z|<\min\{1,|1-t|\}, |t|<|1-\delta_1 z|, |\delta_2 z|<1, |\delta_3 z|<1$ and $0<Re(\theta)<Re(\varrho)$, then

$$\sum_{\vartheta_1}^{\infty}\frac{(\theta_1)_{\vartheta_1}}{\vartheta_1}F_{D;\gimel_1,\gimel_2}^{3;\varrho_1,\varrho_2}(\theta,\theta_1+\vartheta_1,\theta_3,\theta_4;\varrho;\delta_1 z,\delta_2 z,\delta_3 z;\sigma;0,0)t^{\vartheta_1}$$
$$=(1-t)^{-\theta_1}F_{D;\gimel_1,\gimel_2}^{3;\varrho_1,\varrho_2}\left(\theta,\theta_1+\vartheta_1,\theta_3,\theta_4;\varrho;\frac{\delta_1 z}{1-t},\delta_2 z,\delta_3 z;\sigma;0,0\right).$$

Corollary 43: Let $|\delta_1 z|<\min\{1,|1-t|\}, |t|<|1-\delta_1 z|, |\delta_2 z|<1, |\delta_3 z|<1, \daleth-1<Re(\theta-\varrho)<$ $\daleth, Re(\theta)>0$, then

$$\sum_{\vartheta_1}^{\infty}\frac{(\theta_1)_{\vartheta_1}}{\vartheta_1}F_{D;\gimel_1,\gimel_2}^{3;\varrho_1,\varrho_2}(\theta,\theta_1+\vartheta_1,\theta_3,\theta_4;\varrho;\delta_1 z,\delta_2 z,\delta_3 z;\sigma;0,\daleth)t^{\vartheta_1}$$

$$= (1-t)^{-\theta_1} F_{D;\lambda_1,\lambda_2}^{3;\varrho_1,\varrho_2}\left(\theta, \theta_1+\vartheta_1, \theta_3, \theta_4; \varrho; \frac{\delta_1 z}{1-t}, \delta_2 z, \delta_3 z; \sigma; 0, \daleth\right).$$

Corollary 44: Let $|\delta_1 z| < \min\{1, |1-t|\}, |t| < |1-\delta_1 z|, |\delta_2 z| < 1, |\delta_3 z| < 1,\ \daleth - 1 < Re(\theta-\varrho) < \daleth, Re(\theta) > 0$, then

$$\sum_{\vartheta_1}^{\infty} \frac{(\theta_1)_{\vartheta_1}}{\vartheta_1} F_{D;\lambda_1,\lambda_2}^{3;\varrho_1,\varrho_2}(\theta, \theta_1+\vartheta_1, \theta_3, \theta_4; \varrho; \delta_1 z, \delta_2 z, \delta_3 z; \sigma; \daleth, \daleth) t^{\vartheta_1}$$
$$= (1-t)^{-\theta_1} F_{D;\lambda_1,\lambda_2}^{3;\varrho_1,\varrho_2}\left(\theta, \theta_1+\vartheta_1, \theta_3, \theta_4; \varrho; \frac{\delta_1 z}{1-t}, \delta_2 z, \delta_3 z; \sigma; \daleth, \daleth\right).$$

Corollary 45: Let $|\delta_1 z| < \min\{1, |1-t|\}, |t| < |1-\delta_1 z|, |\delta_2 z| < 1, |\delta_3 z| < 1,\ Re(\theta-\varrho) > 0,$ and $Re(\theta+\gimel) > 0$, then

$$\sum_{\vartheta_1}^{\infty} \frac{(\theta_1)_{\vartheta_1}}{\vartheta_1} F_{D;\lambda_1,\lambda_2}^{3;\varrho_1,\varrho_2}(\theta+\gimel, \theta_1+\vartheta_1, \theta_3, \theta_4; 2\theta+\gimel-\varrho; \delta_1 z, \delta_2 z, \delta_3 z; \sigma; 0,0) t^{\vartheta_1}$$
$$= (1-t)^{-\theta_1} F_{D;\lambda_1,\lambda_2}^{3;\varrho_1,\varrho_2}\left(\theta+\gimel, \theta_1+\vartheta_1, \theta_3, \theta_4; 2\theta+\gimel-\varrho; \frac{\delta_1 z}{1-t}, \delta_2 z, \delta_3 z; \sigma; 0,0\right).$$

5. Conclusions

New modified extended beta function is introduced, and it's used to defined modified extended Gauss hypergeometric, Appell hypergeometric, and Lauricella hypergeometric functions with their applications to linear and bilinear generating functions by utilizing modified extended Riemann-Liouville, Caputo and Kober-Erdelyi fractional derivative and integral operators. The modified extended beta, hypergeometric functions and operators reduces to some well-known functions and operator in literature if the parameters are replaced appropriately, see for example,

Substituting $\varrho_1 = \varrho_2$ the new modified extended beta function reduces to

$$B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1, \theta_2; \sigma) = \int_0^1 t^{\theta_1-1}(1-t)^{\theta_2-1} \sigma^{\left(-\frac{\lambda_1}{t^{\varrho_1}} - \frac{\lambda_2}{(1-t)^{\varrho_1}}\right)} dt.$$

Putting $\varrho_1 = \varrho_2 = 1$ the new modified extended beta function reduces to modified beta function presented in [10]

$$B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1, \theta_2; \sigma) = \int_0^1 t^{\theta_1-1}(1-t)^{\theta_2-1} \sigma^{\left(-\frac{\lambda_1}{t} - \frac{\lambda_2}{(1-t)}\right)} dt.$$

If $\varrho_1 = \varrho_2 = 1$ and $\lambda_1 = \lambda_2$ the new modified extended beta function reduces to modified beta function in [9]

$$B_{\lambda_1}(\theta_1, \theta_2; \sigma) = \int_0^1 t^{\theta_1-1}(1-t)^{\theta_2-1} \sigma^{\left(-\frac{\lambda_1}{t(1-t)}\right)} dt,$$

Setting $\sigma = e$ the new modified extended beta function reduces to results established by Sahin et al., [18]

$$B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1, \theta_2; \sigma) = \int_0^1 t^{\theta_1-1}(1-t)^{\theta_2-1} e^{\left(-\frac{\lambda_1}{t^{\varrho_1}} - \frac{\lambda_2}{(1-t)^{\varrho_2}}\right)} dt.$$

If $\sigma = e$, and $\varrho_1 = \varrho_2 = 1$ the new modified extended beta function reduces to extended beta function given in [19, 20]

$$B_{\lambda_1,\lambda_2}^{\varrho_1,\varrho_2}(\theta_1, \theta_2; \sigma) = \int_0^1 t^{\theta_1-1}(1-t)^{\theta_2-1} e^{\left(-\frac{\lambda_1}{t} - \frac{\lambda_2}{(1-t)}\right)} dt.$$

If $\sigma = e,\ \varrho_1 = \varrho_2 = 1$, and $\lambda_1 = \lambda_2$ the new modified extended beta function reduces to [21-24]

$$B_{\lambda_1}(\theta_1, \theta_2; \sigma) = \int_0^1 t^{\theta_1-1}(1-t)^{\theta_2-1} e^{\left(-\frac{\lambda_1}{t(1-t)}\right)} dt.$$

If $\sigma = e,\ \varrho_1 = \varrho_2 = 1$, and $\lambda_1 = \lambda_2 = 0$ the new modified extended beta function reduces to classical beta function [25-27]

$$B_{\lambda_1}(\theta_1, \theta_2; \sigma) = \int_0^1 t^{\theta_1-1}(1-t)^{\theta_2-1} dt.$$

The results presented here are hoped to be applicable in the field of probability theory, distribution theory and other areas of science and technology.

Declaration of Competing Interest

No conflict of interest was declared by the authors.

Authorship Contribution Statement

Umar Muhammad Abubakar: Conceptualization, Resources, Methodology, Writing, Reviewing, Visualization and Software

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