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Research Article

THE FORCE OF RUIN

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ABSTRACT

This paper introduces a hazard rate function for the time of ruin to calculate the conditional probability of ruin for very small time intervals. We call this function as the *force of ruin* (FoR). We obtain the expected time of ruin and conditional expected time of ruin from the exact finite time ruin probability with exponential claim amounts. Then, we introduce the FoR which gives the conditional probability of ruin and the condition is that ruin has not occurred at time *t*. We analyse the behaviour of the FoR function for different initial surpluses over a specific time interval. We also obtain FoR under the excess of loss reinsurance arrangement and examine the effect of reinsurance on the FoR.
Keywords: conditional time of ruin, exact finite time ruin probability, the force of ruin, reinsurance.

1. INTRODUCTION

The literature of ruin theory focuses on two particular questions: the time of ruin and the severity of ruin. Over the past two decades, there has been considerable research interest in the analysis of the distributions of the time of ruin. Gerber [12], Delbaen [4] and Picard and Lefevre [20] deal with the moments of the time of ruin. Gerber and Shiu [13], [14] explain the joint distribution of the time of ruin, the surplus before the time of ruin, and the deficit at the time of ruin by considering an expected discounted penalty function. Lin and Willmot [17] improve the idea of Gerber and Shiu [14] about the defective renewal equation. They indicate an explicit solution of a defective renewal equation according to the time of ruin, the surplus immediately before ruin, and the deficit at the time of ruin. Egidio Dos Reis [10] studies the moments of the time of ruin and the duration of the first period of negative surplus under the discrete time compound Poisson process. Dickson and Waters [7] study the distribution of the time of ruin in the classical risk model. They aim to calculate the moments of time of ruin and to investigate the shape of the density of the time of ruin for different approaches. They use the conditional distribution of the time of ruin to obtain the density functions. Drekic and Wilmott [9] investigate the probability density function of the time of ruin in the classical model with exponential claim sizes. In this study, the probability density function of the time of ruin is obtained directly by using the inversion of the associated Laplace transform. They also provide the explicit expression

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for the *kth* moment of the time of ruin. In addition, they obtain the moment-based quantities such as mean, variance, the coefficient of variation, the coefficient of skewness and the coefficient of kurtosis of the time of ruin.

Although the literature mainly discusses the time of ruin, we believe that the probability of the conditional time of ruin is also important for practical purposes. There are few studies Young [18], Moore and Young [19] and Weert et al. [26] which discuss the conditional time of (lifetime) ruin in the literature. Young [18] uses inverse Gaussian distribution and analyses the distribution of the conditional time of lifetime ruin given that ruin occurs. She also introduces the conditional distribution of bequest, given that ruin does not occur. However, Young [18] uses the constant force of mortality which is an unrealistic assumption. Moore and Young [19] improve the idea of Young [18] by considering the lifetime ruin probability and different investment strategies for optimal asset allocation under the general mortality assumptions. Weert et al. [26] discuss the lifetime ruin which is defined as running out of money before death. They particularly focus on investment strategies in order to avoid lifetime ruin and discuss the conditional time of lifetime ruin and the wealth at death. The main difference between Moore and Young [19] and Weert et al. [26] is that the later one works in a discrete-time setting and uses comonotonic approximations for the probability of lifetime ruin.

Weert et al. [26] use the conditional time of lifetime ruin to compare different investment strategies for a retiree who has a specific amount of initial wealth. They show that different investment strategies may lead quite similar ruin probabilities and the retiree will be indecisive between the strategies. However, the conditional time of ruin is significantly different and the retiree will choose the strategy which leads the ruin in later years. Thus, the conditional probability and the expected conditional time of ruin are crucial for a retiree.

Inspired by Weert et al. [26], we propose a formula for the probability of the conditional time of ruin which we call as the force of ruin (FoR). While Weert et al. [26] derive the conditional time of lifetime ruin by multiplying the conditional probability with the survival probability for specific ages, we derive a hazard rate function based on the density of the time of ruin. Therefore, we obtain a general formula for the probability of the conditional time of ruin. FoR enables one to calculate the conditional probability of ruin for very small time intervals. This information might be important for companies which have solvency issues. If the company is close to bankruptcy, the FoR provides an important information about the conditional ruin probability on the condition that the company is solvent until a specific time.

We introduce the FoR step by step. First, we derive the density function of time of ruin. Second, we calculate both the expected and the conditional expected time of ruin. Third, we obtain FoR, the conditional probability of ruin given that ruin has not occurred until a specific time point. Those steps enable us to compare our results with the ones obtained from the formulas for the density of time of ruin in the literature. Finally, we analyse the effect of a reinsurance on the FoR by using the excess of loss reinsurance arrangement.

The paper is organised as follows. In Section 2, we present the finite time ruin probability including the exact formula and the approximations. In Section 3, we derive the density for the time of ruin and obtain the expected and the conditional expected time of ruin. We also conduct a numerical analysis and compare our results with the previous studies. Section 4 introduces the FoR formula and illustrates the numerical results with the graphs. Section 5 presents an analysis of the effect of the excess of loss reinsurance arrangement on the FoR. In Section 6, we conclude with some final remarks.

2. THE FINITE TIME RUIN PROBABILITY

In the classical risk process, it is assumed that the surplus process starts with an initial level u and continues according to two opposing cash flows: the premium income per unit of time, denoted by *c* and the aggregate claim amount up to time *t*, denoted by *S(t)*. The insurer's surplus (or risk) process, $\{U(t)\}_{t\geq 0}$ is defined by

$$U(t) = u + ct - S(t).$$

The aggregate claim amount up to time *t*, *S(t)*, is

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

where *N(t)* denotes the number of claims that occur in the fixed time interval *[0, t]*. The individual claim amounts, modelled as independent and identically distributed (i.i.d.) random variables $\{X_i\}_{i=1}^{\infty}$ with distribution function $F(x) = Pr(X_i \leq x)$ such that $F(0) = 0$ and X_i is the amount of the i^{th} claim. We use the notation f and m_k to represent the density function and k^{th} moment of X_1, respectively, and it is assumed that $c > E[N]m_1$. The finite time ruin probability $\psi(u, t)$ is defined by

$$\psi(u,t) = Pr(U(s) < 0 \quad for \ \ some \ s, \quad 0 < s < t)$$

where $\psi(u, t)$ is the probability that the insurer's surplus falls below zero in the finite time interval *(0, t]*. In this paper, it is assumed that we have a Poisson process for claim frequency with rate λ and thus a compound Poisson process for the aggregate claims. Premiums are assumed as payable with a rate *c* per unit time.

2.1. Exact Finite Time Ruin Probability with Exponential Claim Amounts

There are many approximations and formulas for the calculation of the finite time ruin probability in the literature. Unlike the infinite time case, there is no general finite time ruin probability formula such as Pollaczek-Khinchine formula. A few approximations are obtained by making some adjustments to the infinite time methods. Prabhu [21] gives a formula for the finite time ruin probability in the classical risk process for $u \geq 0$. In this formula, $\psi(0, t)$ is expressed in terms of the distribution function. Seal [22] considers the exponential claims and shows how to apply Prabhu's formula. Seal's approach provides a closed form for $F^{n*}(x)$ and its derivative. De Vylder [5] proposes a simple method that approximates a classical risk process $\{U(t)\}_{t\geq 0}$ by another classical risk process $\{\hat{U}(t)\}_{t\geq 0}$. Segerdahl [23] suggests a formula to obtain the finite time ruin probability. This method extends the Cramer-Lundberg approximation by adding a time factor. This approximation requires the existence of the adjustment coefficient R and the moment generating function M_X. Therefore, it can only be used for the light-tailed distributions. Some diffusion approximations are also developed for the calculation of the finite time ruin probability. In these approximations, it is assumed that the claim amount distributions belong to the domain of attraction of the normal law or have a light tail Iglehart [16] (see also Grandell [15] and Asmussen [2]). Dickson and Waters [6] discuss how to approximate ruin probabilities in the classical risk model by using a Gamma process and a translated Gamma process.

Asmussen [1] presents an exact finite time ruin probability formula when the individual claim amounts are exponentially distributed with parameter $\beta = 1$, the number of claims has a Poisson distribution with the parameter λ and the premium rate per unit time, c, is equal to 1 ($c = 1$). Then the finite time ruin probability is

$$\psi(u, t) = \lambda\, exp\{-(1-\lambda)u\} - \frac{1}{\pi}\int_0^{\pi} \frac{f_1(x,t)\, f_2(x)}{f_3(x)} dx \qquad (1)$$

where

$$f_1(x, t) = \lambda \exp\{2\sqrt{\lambda}\,\mathrm{t}\cos(x) - (1+\lambda)t + u(\sqrt{\lambda}\cos(x) - 1)\},$$

$$f_2(x) = cos(u\sqrt{\lambda}\, sin(x)) - cos(u\sqrt{\lambda}\, sin(x) + 2x),$$

and

$$f_3(x) = 1 + \lambda - 2\sqrt{\lambda}\cos(x).$$

An important implication of this method is that it removes the restriction on the parameter of individual claims distribution and premium rate. When $\beta \neq 1$, the following equation is applied [2]:

$$\psi_{\lambda,\beta}(u,t) = \psi_{\frac{\lambda}{\beta},1}(\beta u, \beta t), \tag{2}$$

and the following equation is valid when $c \neq 1$, [3]

$$\psi_{\lambda,c}(u,t) = \psi_{\frac{\lambda}{c},1}(u, ct). \tag{3}$$

There are two more finite time ruin probability formulas which are suggested by Seal [22] and Takács [24]. Both methods depend on different numerical integrations. However, these methods may not be practical because they produce unstable results for large values of t [2].

3. THE TIME OF RUIN

One of the particular questions of interest in the classical ruin theory is the time of ruin. With the distribution of time of ruin, we obtain the probability of ruin. In this section, we derive the density of the time of ruin for the exact finite time ruin probability with exponential claims and obtain the expected and conditional expected time of ruin. We use the exponential distribution for the claim amounts in order to compare our results with the ones in Dickson [8].

3.1. Density

Dickson [8] defines the density function of the time of ruin, $w_c(u,t)$, by using the Laplace transform and assuming the individual claim amounts have an exponential distribution with parameter β as

$$w_c(u,t) = \frac{1}{\psi(u)}\frac{\partial}{\partial t}\psi(u,t), \tag{4}$$

and the density of the time of ruin is obtained as

$$w_c(u,t) = \frac{\exp\left\{-(\lambda+c\beta)t-\frac{\lambda u}{c}\right\}}{2\lambda t} \times \sum_{j=0}^{\infty}\left(\frac{u}{2c}\right)^j \frac{(j+1)\left(2\sqrt{c\beta\lambda}\right)^{j+1}}{j!}\boldsymbol{I}_{j+1}\left(2t\sqrt{c\beta\lambda}\right), \tag{5}$$

where $\boldsymbol{I}_v(t)$ is called a modified Bessel function of order v and defined as

$$\boldsymbol{I}_v(t) = \sum_{n=0}^{\infty}\frac{(t/2)^{2n+v}}{n!\,(n+v)!}.$$

We use equation (4) to obtain the density for the time of ruin, $\psi_d(u,t)$, based on the exact ruin probability formula with exponential claims given in equation (1) as below:

$$\psi_d(u,t) = \frac{\frac{\partial\psi(u,t)}{\partial t}}{\psi(u)}. \tag{6}$$

The ultimate ruin probability when $F(x) = 1 - exp\,(-\beta x),\ x > 0$ is

$$\psi(u) = \frac{\lambda}{\beta c} exp\left\{-\left(\beta - \frac{\lambda}{c}\right)u\right\}. \tag{7}$$

Thus the following equation is obtained:

$$\psi_d(u,t) = \frac{\frac{d}{dt}\left(\lambda exp\{-(1-\lambda)u\} - \frac{1}{\pi}\int_0^{\pi}\frac{f_1(x,t)f_2(x)}{f_3(x)}dx\right)}{\frac{\lambda}{\beta c}exp\left\{-\left(\beta-\frac{\lambda}{c}\right)u\right\}} \tag{8}$$

Since it is a well-defined and definite integral, under the assumption of *Leibniz rule*, we may interchange of a derivative and an integral. Thus, we derive the formula for $\psi_d(u,t)$ as follow:

$$\psi_d(u,t) = \frac{\left(-\frac{1}{\pi}\int_0^{\pi}\frac{(\partial f_1(x,t)/\partial t) f_2(x)}{f_3(x)}dx\right)}{\frac{\lambda}{\beta c}exp\left\{-\left(\beta-\frac{\lambda}{c}\right)u\right\}}. \quad (9)$$

Then, we take the numerical integral to obtain the probabilities based on the density function.

Table 1 presents the exact values of the density of the time of ruin obtained from Dickson [8] which is given as $w_c(u,t)$ and the values obtained from the density function we derived, $\psi_d(u,t)$ for $u = 40$, $\beta = 1$ and c=1.1.

Table 1. Exact values of the density of time of ruin

t	$w_c(40,t)$	$\psi_d(40,t)$
5	0.00000000	0.00000000
10	0.00000026	0.00000026
20	0.00001227	0.00001227
50	0.00047403	0.00047403
100	0.00185866	0.00185866
200	0.00241480	0.00241480
300	0.00182732	0.00182732
400	0.00125698	0.00125698
500	0.00085022	0.00085022

We obtain the same values for different *t* which proves that our density function is consistent. We observe that the probabilities obtained from two densities differ from the fifth decimal point for $t > 500$. While density $w_c(u,t)$ is only valid for exponential claim amounts, $\psi_d(u,t)$ can be derived for other claim amount distributions. Figure 1 presents the graph of the density function obtained from $\psi_d(40,t)$.

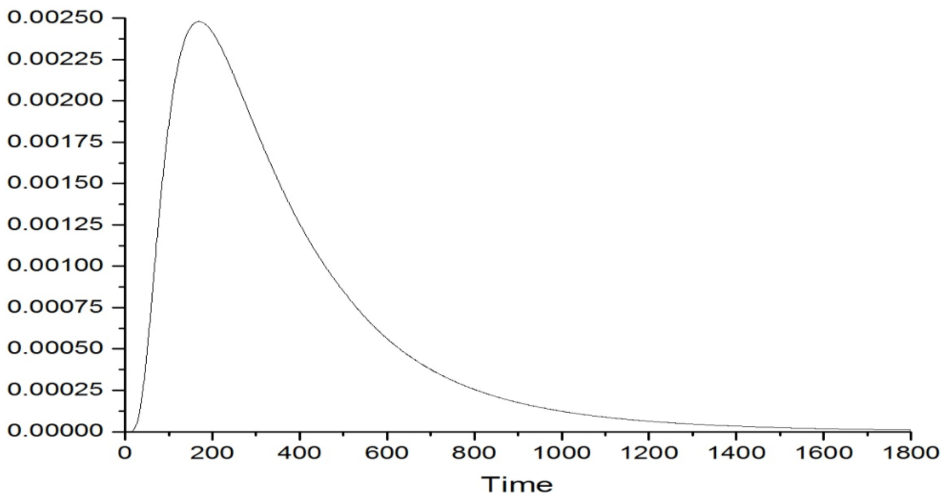

Figure 1. The density of time of ruin

3.2. Expected Time of Ruin

Dickson [8] introduces the random variable T_u, denoting the time of ruin, and calculates the expected time of ruin, given that ruin occurs as:

$$E[T_{u,c}] = \frac{R_0'}{\beta - R_0} + R_0' u,$$
$$= \frac{c + \lambda u}{c(c\beta - \lambda)}.$$

where $T_{u,c} = T_u | T_u < \infty$ and $R_0' = \lambda / c(c\beta - \lambda)$. We derive the expected time of ruin, $E[T_u]$, by using our density function which is introduced in Section 3.1, $\psi_d(u,t)$, as below. $F_u(t)$ is the distribution function of $\psi_d(u,t)$ where:

$$F_u(t) = \int_0^t \psi_d(u,k)\, dk \quad (10)$$

Substituting equation (9) into equation (10), we obtain

$$F_u(t) = \frac{-1}{\pi\psi(u)} \left[\int_0^{\pi} \frac{f_2(x)}{f_3(x)} [f_1(x,t) - f_1(x,0)]\, dx\right]. \quad (11)$$

Then the expected time of ruin, $[T_{u,d}]$ is

$$E[T_{u,d}] = E[T_u | T_u < \infty] = \frac{\int_0^{\infty} (1 - F_u(t))dt}{1 - F_u(t)}, \quad (12)$$

where $F_u(t)$ is obtained as in equation (11) and $E[T_{u,d}]$ is

$$E[T_{u,d}] = \frac{1}{\pi\, \psi(u)} \int_0^{\pi} \frac{f_2(x)}{f_3(x)} \left[\int_0^{\infty} f_1(x,t)dt\right] dx, \quad (13)$$

Equation (13) seems complicated, however taking the numerical integral with respect to *x* is straightforward using programming languages. Table 2 and Figure 2 present the expected time of ruin for different initial surpluses. Both indicate that as initial surplus increases, the expected time of ruin increases. Although it is not obvious from equation (13), there is almost a linear relationship between the initial surplus and the expected time of ruin.

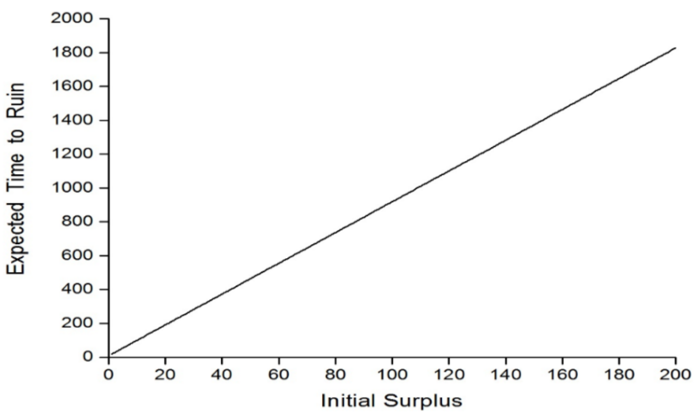

Figure 2. Expected time of Ruin $E[T_{u,d}]$

Table 2. Expected Time of Ruin

u	$E[T_{u,d}]$
1	19.09
5	55.45
10	100.91
15	146.36
20	191.82
25	**237.27**
50	464.55
75	691.82
100	919.09
200	1828.18

3.3. Expected Conditional Time of Ruin

As we explained in Section 1, we propose a general formula (or a function) for the probability of the conditional expected time of ruin. Based on previous studies of Young [18], Moore and Young [19] and Weert et al. [26], conditional time of ruin has an important field of application in pension and investment. We believe that the scope is wider because the conditional time of ruin is a substantial information for any company and financial institution. This is the principal motivation to derive a density function for finite time ruin probability for different claim amount distributions. The expected conditional time of ruin given that ruin does not occur until time *t*, $E[T_{u,t}]$ is:

$$E\left[T_{u,t}\right] = E[T_u - t|T_u > t] = \frac{\int_t^{\infty}(1-F_u(k))dk}{1-F_u(t)}, \tag{14}$$

which leads to

$$E\left[T_{u,t}\right] = \frac{-t+\frac{1}{\pi\psi(u)}\int_0^{\pi}\frac{f_2(x)}{f_3(x)}\left[\int_t^{\infty} f_1(x,t)dt\right]dx}{1+\frac{1}{\pi\psi(u)}\left[\int_0^{\pi}\frac{f_2(x)}{f_3(x)}[f_1(x,t)-f_1(x,0)]dx\right]}. \tag{15}$$

When we take the numerical integral in equation (15), we obtain the expected conditional time of ruin. Table 3 and Figure 3 present the expected conditional time of ruin for initial surplus, $u = 25$. As time *t* increases, the expected conditional time of ruin decreases and there is almost a linear relationship between time and the expected conditional time of ruin. When $t = 0$, the expected conditional time of ruin is equal to the expected time of ruin. Thus, Table 2 and Table 3 give the same result, 237.27 for $u = 25$ and $t = 0$.

Table 3. Expected conditional time of ruin

t	$E[T_{u,d}]$
0	**237.27**
1	235.27
5	227.27
10	217.27
15	207.28
25	187.33
50	138.32
75	91.49
100	47.22

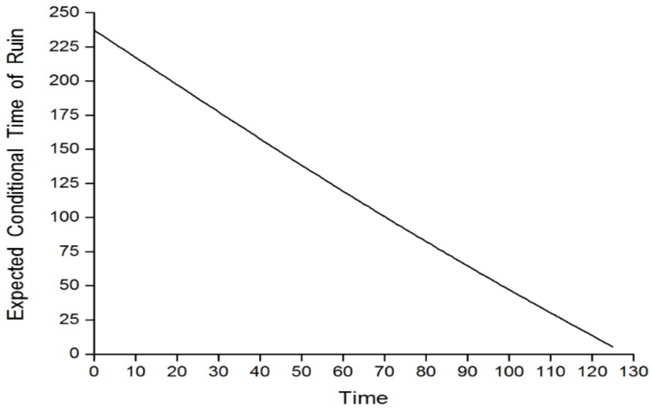

Figure 3. Expected conditional time of ruin $E[T_{u,d}]$

4. FORCE OF RUIN

We take our study one step further and derive a formula to calculate the conditional ruin probability for small time intervals Δt given that the ruin does not occur until a time point, t. We achieve this by defining a *hazard rate function* based on the density, $\psi_d(u,t)$. We call this hazard rate function as *force of ruin* akin to the *force of mortality* in survival analysis. Therefore, we derive a conditional (instantaneous) probability of ruin and the condition is that ruin has not occurred at time *t*. By introducing the force of ruin, we build a bridge between the risk theory and survival analysis. The *force of ruin* is treated as a similar way to the force of mortality.

4.1. Hazard Rate Function

Let T be a continuous lifetime random variable with a cumulative distribution function F and a probability density function *f*. Consider an interval of time $(t, t+\Delta t]$ and we are interested in the probability of failure in this interval given that it did not occur before in $[0,t]$. This probability can be interpreted as the risk of failure in $(t, t+\Delta t]$ given the stated condition, i.e.,

$$\Pr[t < T \leq t+\Delta t \mid T > t] = \frac{\Pr[t < T \leq t+\Delta t]}{\Pr[T > t]} = \frac{F(t+\Delta t) - F(t)}{\overline{F}(t)}$$

and its limit when $\Delta t \longrightarrow 0$ gives the hazard rate function, denoted by $h(t)$ [11].

$$h(t) = \lim_{\Delta t \to 0} \frac{\Pr[t < T \leq t+\Delta t \mid T > t]}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\overline{F}(t)\, \Delta t} = \frac{f(t)}{\overline{F}(t)}. \quad (16)$$

The hazard rate function is also called failure rate function, intensity rate function, force of defaults and force of mortality [25].

4.2. Force of Ruin

We use the exact finite time ruin probability formula to derive the *force of ruin*, $\psi(u,t,\Delta t)$, which is defined as

$$\psi(u,t,\Delta t) = \frac{\psi_d(u,t+\Delta t)}{1 - F_u(t)}, \quad (17)$$

where $\psi_d(u,t)$ is given in equation (9) and we put $(t+\Delta t)$ instead of t. Thus, based on equation (17), we derive the formula for $\psi(u,t,\Delta t)$ by using equation (9) and the denominator of equation (15).

$$\psi(u,t,\Delta t)=\frac{\psi_d(u,t+\Delta t)}{1+\frac{1}{\pi\psi(u)}\left[\int_0^{\pi}\frac{f_2(x)}{f_3(x)}[f_1(x,t)-f_1(x,0)]dx\right]}. \tag{18}$$

Then, we take the numerical integral for various λ, θ and u values. The $\psi(u,t,\Delta t)$ is a hazard rate function which must be non-negative, $\psi(u,t,\Delta t)\geq 0$ and non-decreasing.

Table 4 presents the mean, variance and skewness of $\psi(u,t,\Delta t)s$ for $u = 10$, $u = 25$ and $u = 50$. We divide each time interval into 1000; thus take $\Delta t = 0.001$ and calculate the descriptive statistics for the force of ruin from $t = 1$ to $t = 10$. Mean values indicate that as t increases the force of ruin increases, while as initial surplus u increases the force of ruin decreases. The variances are quite small and very close to zero. The skewness of the force of ruin is relatively high for small t and large u but not significantly different from zero.

Table 4. Force of Ruin

$\psi(u,t,\Delta t)$									
$\lambda = 1, \quad \theta = 0.1$									
	u=10			u=25			u=50		
t	Mean	Var	Skewness	Mean	Var	Skewness	Mean	Var	Skewness
1	2.33E-05	2.86E-11	3.65E-02	8.48E-10	2.41E-19	5.40E-01	3.37E-17	1.31E-34	8.29E-01
2	4.25E-05	2.94E-11	-2.58E-02	4.47E-09	3.11E-18	3.59E-01	2.11E-16	1.59E-32	7.37E-01
3	6.09E-05	2.35E-11	-5.59E-02	1.46E-08	1.83E-17	2.53E-01	1.63E-15	7.72E-31	5.90E-01
4	7.70E-05	1.64E-11	-7.40E-02	3.60E-08	6.84E-17	1.98E-01	9.29E-15	1.77E-29	4.86E-01
5	9.04E-05	1.05E-11	-8.74E-02	7.39E-08	1.90E-16	1.55E-01	4.00E-14	2.37E-28	4.10E-01
6	1.01E-04	6.24E-12	-9.95E-02	1.34E-07	4.26E-16	1.24E-01	1.40E-13	2.16E-27	3.52E-01
7	1.10E-04	3.45E-12	-1.13E-01	2.19E-07	8.20E-16	1.01E-01	4.14E-13	1.46E-26	3.07E-01
8	1.17E-04	1.74E-12	-1.32E-01	3.35E-07	1.40E-15	8.28E-02	1.08E-12	7.78E-26	2.71E-01
9	1.22E-04	7.68E-13	-1.62E-01	4.82E-07	2.19E-15	6.85E-02	2.53E-12	3.43E-25	2.41E-01
10	1.25E-04	2.65E-13	-2.22E-01	6.63E-07	3.17E-15	5.70E-02	5.44E-12	1.29E-24	2.17E-01

Figure 4 illustrates the expected force of ruin from $t = 1$ to $t = 10$ for initial surpluses, $u = 10$, $u = 15$, $u = 25$ and $u = 50$ on four different scales. Those scales enable us to present each graph on their own *y-axes* and observe the shape of the hazard rate functions better. All four functions are increasing. However, while the shape of the force of ruin function for $u = 10$ is concave, the shape for the function $u = 15$ is quite flat and the shape for the functions for $u = 25$ and $u = 50$ are convex. The evolution of the functions proves that as initial surplus increases the increase in the force of ruin decreases.

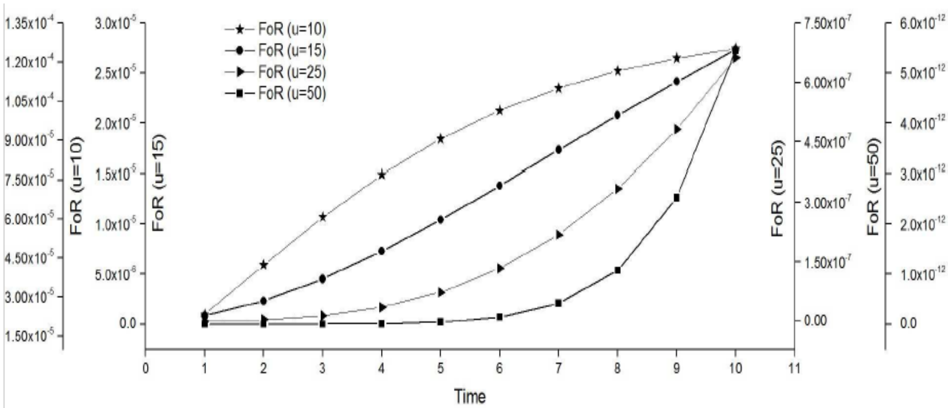

Figure 4. Force of ruin, $\psi(u,t,\Delta t)$

5. REINSURANCE AND THE FORCE OF RUIN

We use the excess of loss reinsurance arrangement to illustrate the effect of the reinsurance on the force of ruin.

5.1. Excess of Loss Reinsurance

The insurer and the reinsurer's expected individual claim amounts are calculated according to a constant retention level M under an excess of loss reinsurance arrangement. When a claim X occurs, the insurer pays $Y = min(X, M)$ and the reinsurer pays $Z = max(0, X - M)$ with $X = Y + Z$. Hence, the distribution function of Y , $F_Y(x)$, is

$$F_Y(x) = \begin{cases} F_X(x) & for \quad x < M, \\ 1 & for \quad x \geq M, \end{cases}$$

and the moments of Y are

$$E[Y^n] = \int_0^M x^n f(x)dx + M^n\big(1 - F(M)\big). \tag{19}$$

Similarly, the moments of Z are

$$E[Z^n] = \int_M^\infty (x - M)^n f(x)\, dx. \tag{20}$$

In the classical risk model, it is assumed that the number of claims has a Poisson distribution with parameter λ. According to the expected value premium principle with the insurance loading factor θ and the reinsurance loading factor ξ, the insurer's premium income per unit time after the reinsurance premium (i.e. net of reinsurance) is defined as

$$c^* = (1+\theta)\,\lambda\, E[X] - (1+\xi)\,\lambda\, E[Z], \tag{21}$$

where we assume that $\xi \geq \theta > 0$ and $c^* > E[Y]$.

The finite time ruin probability for the exponential claim amounts with parameter $\beta = 1$ can be obtained by using equation (2) and equation (3). We use the net of reinsurance premium instead of the premium rate per unit time. The expected individual claim amount of the insurance company will change according to the retention level M. The expected individual claim amount for the insurance company $E[X_I]$ and the reinsurance company $E[X_R]$ are obtained as follows

$$E[X_I] = \frac{1 - \exp(-\beta M)}{\beta}$$

and

$$E[X_R] = \frac{\exp(-\beta M)}{\beta}$$

where $E[X_I] + E[X_R] = 1/\beta$.

Figure 5 shows the values of the force of ruin from $t = 1$ to $t = 10$ for a fix initial surplus $u = 25$ but different retention levels, $M = 0.5$, $M = 2.5$, $M = 7.5$. As the retention level increases, the force of ruin increases. When we compare Figure 4 and Figure 5 for $u = 25$, we see that having an excess of loss reinsurance arrangement does not always reduce the force of ruin for the insurance company. The probabilities are lower for $M = 0.5$ but higher for $M = 2.5$, $M = 7.5$ under the reinsurance arrangement. This result indicates that the force of ruin might be considered as a criterion to decide the optimal retention level for reinsurance arrangements.

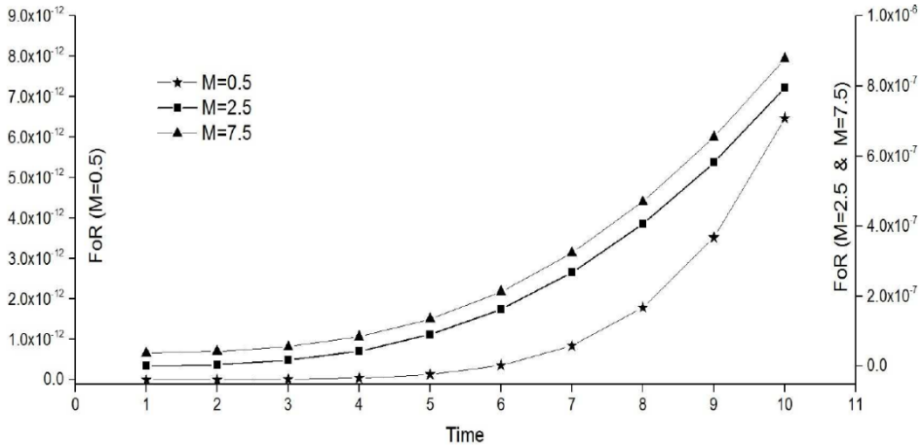

Figure 5. Force of ruin for different retention levels, $\psi(u, t, \Delta t)$

Figure 6 illustrates the force of ruin from $t = 1$ to $t = 10$ for different initial surpluses, $u = 10$; $u = 15$; $u = 25$, $u = 50$ and different retention levels, $M = 1$, $M = 1.5$, $M = 2.5$ and $M = 5$ on four different $y-axes$. The retention level has been chosen as 10% of the initial surplus.

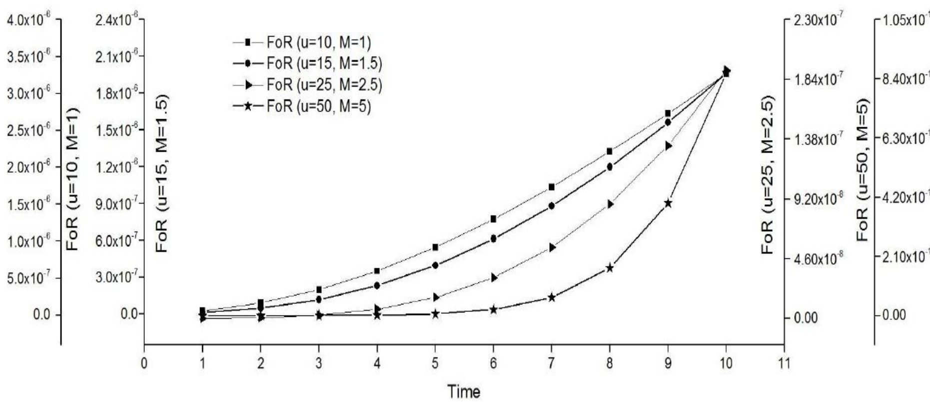

Figure 6. Force of ruin for different initial surplus and retention levels, $\psi(u, t, \Delta t)$

Figure 6 shows that the increase in the initial surplus leads a decrease in the force of ruin as in Figure 4. Thus, the existence of reinsurance does not make a difference considering the initial surplus and the force of ruin relation. As time increases, the force of ruin increases. However, the reinsurance arrangement affects the shape of the hazard rate functions by changing the slope of the curves. The slopes of the force of ruin functions decrease as the initial surpluses increase under the excess of loss reinsurance arrangement. When we compare Figure 4 and Figure 6, we see that for $u = 10$, $u = 15$, $u = 25$ the values of the force of ruin are lower, for $u = 50$ the values of the force of ruin are higher under reinsurance arrangement. This result indicates that the effect of the reinsurance on the force of ruin is determined by both the initial surplus and retention level. The existence of excess of loss reinsurance decreases the force of ruin as initial surplus increases assuming that the retention level is some proportion of the initial surplus. However, we see that the force of ruin function for $u = 50$ and $M = 5$ does not follow this conclusion. This means that not only the proportional relation between the initial surplus and the retention level, but also the value of the retention level determines the force of ruin. This finding confirms that the force of ruin might be considered as a criterion to decide the optimal retention levels for the excess of loss reinsurance arrangements.

6. CONCLUSIONS

In this paper, we derived a hazard rate function based on the exact finite time ruin probability formula proposed by Asmussen [1] to obtain the probabilities for conditional time of ruin. First, we derived the density for the time of ruin based on exponential claim amounts. Besides being consistent with the other density functions for the time of ruin in the literature, our approach is applicable to the other claim amount distributions. Then we calculated the expected time of ruin and conditional expected time of ruin. Following the idea of Weert et al. [26], we obtained the probabilities for the conditional time of ruin but we proposed a different method by deriving the hazard rate function which we call the force of ruin. Proposing a hazard rate function to find the instantaneous ruin probabilities is the main contribution of our study. We analysed the behaviour of the force of ruin function based on numerical results. The force of ruin increases as time increases and decreases as the initial surplus increases. Although it is non-decreasing, the shape of the force of ruin function differs significantly based on initial surpluses. We also investigated the effect of an excess of loss reinsurance arrangement on the force of ruin. Our findings suggest that the force of ruin might be considered as a criterion to find the optimal retention level but this is left as a further research.

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