Avrupa Bilim ve Teknoloji Dergisi
Özel Sayı 28, S. 1288-1290, Kasım 2021

Araştırma Makalesi

www.ejosat.com ISSN:2148-2683

European Journal of Science and Technology
Special Issue 28, pp. 1288-1290, November 2021

Research Article

The Curvature Property of a Linear Dynamical System

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(1st International Conference on Applied Engineering and Natural Sciences ICAENS 2021, November 1-3, 2021)

(**DOI:** 10.31590/ejosat.1014593)

ATIF/REFERENCE: Bayrakdar, T. & Ok Bayrakdar, Z. (2021). The Curvature Property of a Linear Dynamical System. *European Journal of Science and Technology*, (28), 1288-1290.

Abstract

In this work a two-dimensional smooth autonomous dynamical system is regarded as a three-dimensional Riemannian manifold and it is shown that the scalar curvature of a linear dynamical system $\mathrm{d}x/\mathrm{d}t = ax + by$, $\mathrm{d}y/\mathrm{d}t = cx + dy$ is non-positive. The manifold is scalar-flat iff $b = -c$ and $a = d = 0$.

Keywords: Linear dynamical systems, Riemann curvature tensor, scalar curvature

Bir Lineer Dinamik Sistemin Eğrilik Özelliği

Öz

Bu çalışmada iki-boyutlu, düzgün, otonom bir dinamik sistem üç-boyutlu bir Riemann manifoldu olarak değerlendirilmiş ve bir $\mathrm{d}x/\mathrm{d}t = ax + by$, $\mathrm{d}y/\mathrm{d}t = cx + dy$ lineer dinamik sisteminin skaler eğriliğinin pozitif olmadığı gösterilmiştir. Manifold skaler-düzdür ancak ve ancak $b = -c$ ve $a = d = 0$.

Anahtar Kelimeler: Lineer dinamik sistemler, Riemann eğrilik tensörü, skaler eğrilik.

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1. Introduction

A smooth autonomous dynamical system (SADS) on a two-dimensional manifold $N=(D;x,y)$, where D is a connected open set in $\mathbb{R}^2$ endowed with coordinates (x,y), is given by a system of first order ordinary differential equations

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= f(x,y) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= g(x,y) \end{aligned} \tag{1}$$

such that f and g are smooth functions on N. The system (1) defines a smooth vector field

$$\xi = f(x,y)\partial_x + g(x,y)\partial_y \tag{2}$$

on N which is a smooth section of the tangent bundle TN, i.e. is a mapping $\xi: N \to TN$ defined to be

$$\xi(x,y) = \big(x, y, \dot{x} = f(x,y), \dot{y} = g(x,y)\big).$$

Since the rank of $\mathrm{d}\xi$, the differential of ξ, equals 2 on N, a SADS may be regarded as a surface in TN.

In this work, to capture all the information about the dynamics, we identify a SADS as a three-dimensional manifold $M = I \times N$ with local coordinates (t,x,y). Here t is a coordinate function on an open interval $I \subset \mathbb{R}$ and stands for the time coordinate. This consideration gives rise to define a SADS as a submanifold of the first-order jet bundle $J^1(\mathbb{R},N)$ of maps $\mathbb{R} \to N$ [1, 2].

This paper is devoted to the investigation of the curvature property of a linear system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= ax + by \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= cx + dy, \end{aligned} \tag{3}$$

where $a,b,c,d \in \mathbb{R}$, within the context of Riemannian geometry. By means of the method of moving frames we evaluate the connection 1-form and the curvature 2-form of the Levi-Civita connection in TM compatible with the Riemannian metric which is defined by the sum of squares of the one-forms

$$\begin{aligned} \omega^1 &= \mathrm{d}t \\ \omega^2 &= \mathrm{d}x - (ax+by)\mathrm{d}t \\ \omega^3 &= \mathrm{d}y - (cx+dy)\mathrm{d}t, \end{aligned}$$

and show that the scalar curvature of the connection is non-positive and the curvature vanishes if and only if $b=-c$ and $a=d=0$.

2. Riemannian structure and the curvature

The solutions of (3) are identified with the solutions of the Pfaffian system

$$\omega^2 = 0, \qquad \omega^3 = 0$$

such that on a solution curve we have $\omega^1 \neq 0$. The column vector $\omega = (\omega^1, \omega^2, \omega^3)^t$, where ^{t} denotes the transposition, defines a coframe on M which is dual to the frame of the vector fields

$$e_1 = \frac{\partial}{\partial t} + f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}, \qquad e_2 = \frac{\partial}{\partial x}, \qquad e_3 = \frac{\partial}{\partial y}.$$

If we introduce the Riemannian metric on M

$$ds^2 = \sum_i \omega^i \otimes \omega^i, \tag{4}$$

then M becomes a Riemannian manifold and the frame $e = (e_1, e_2, e_3)$ forms an orthonormal frame. Let ∇ be the connection compatible with the Riemannian metric (4). The structure equations for the coframe ω are given by

$$\begin{aligned} \mathrm{d}\omega^1 &= 0 \\ \mathrm{d}\omega^2 &= a\omega^1 \wedge \omega^2 + b\omega^1 \wedge \omega^3 \\ \mathrm{d}\omega^3 &= c\omega^1 \wedge \omega^2 + d\omega^1 \wedge \omega^3. \end{aligned}$$

As it is considered in [4], the $\mathfrak{so}(3,\mathbb{R})$-valued connection form is obtained by solving the system of equations

$$d\omega^i = -\theta^i_j \wedge \omega^j, \qquad \theta^i_j = -\theta^j_i.$$

The unique 1-form

$$\theta = \begin{pmatrix} 0 & \theta^1_2 & \theta^1_3 \\ -\theta^1_2 & 0 & \theta^2_3 \\ -\theta^1_3 & -\theta^2_3 & 0 \end{pmatrix}$$

satisfying the structure equations is obtained by the Cartan's Lemma, where

$$\theta^1_2 = -a\omega^2 - \frac{1}{2}(b+c)\omega^3$$

$$\theta^1_3 = -\frac{1}{2}(b+c)\omega^2 - d\omega^3$$

$$\theta^2_3 = -\frac{1}{2}(b-c)\omega^1.$$

The 1-form θ is called $\mathfrak{so}(3,\mathbb{R})$-*valued connection form* of the Levi-Civita connection.

The curvature 2-form of the Levi-Civita connection is defined by the anti-symmetric matrix of two-forms:

$$\Omega^i_j = \mathrm{d}\theta^i_j + \sum_k \theta^i_k \wedge \theta^k_j, \qquad \Omega^i_j = -\Omega^j_i.$$

In terms of ω^i's, the components of the Riemannian curvature tensor are determined by

$$\Omega^i_j = \sum_{k<l} R^i_{jkl}\omega^k \wedge \omega^l.$$

For the details on a connection in an arbitrary vector bundle we refer to [3]. By a direct calculation we obtain

$$\Omega^1_2 = -\frac{1}{2}\left(2a^2 + \frac{3}{2}c^2 + bc - \frac{1}{2}b^2\right)\omega^1 \wedge \omega^2$$
$$-(ab+cd)\omega^1 \wedge \omega^3$$

$$\Omega^1_3 = -(ab+cd)\omega^1 \wedge \omega^2$$
$$-\frac{1}{2}\left(\frac{3}{2}b^2 + bc + 2d^2 - \frac{1}{2}c^2\right)\omega^1 \wedge \omega^3$$

$$\Omega_3^2 = \frac{1}{2}\left(\frac{1}{2}(b+c)^2 - 2ad\right)\omega^2 \wedge \omega^3.$$

The independent nonzero components of the Riemann curvature tensor are

$$\boldsymbol{R^1_{212}} = -\frac{1}{2}\left(2a^2 + c(b+c) - \frac{1}{2}(b^2 - c^2)\right),$$

$$\boldsymbol{R^1_{213}} = -(ab + cd),$$

$$\boldsymbol{R^1_{313}} = -\frac{1}{2}\left(b(b+c) + 2d^2 + \frac{1}{2}(b^2 - c^2)\right),$$

$$\boldsymbol{R^2_{323}} = \frac{1}{2}\left(\frac{1}{2}(b+c)^2 - 2ad\right).$$

At a point, the sectional curvatures of the two dimensional subspaces spanned by $(\boldsymbol{e_1}, \boldsymbol{e_2}), (\boldsymbol{e_1}, \boldsymbol{e_3})$ **a**nd $(\boldsymbol{e_2}, \boldsymbol{e_3})$ are given respectively by $\boldsymbol{R^1_{212}}$, $\boldsymbol{R^1_{313}}$, and $\boldsymbol{R^2_{323}}$. The scalar curvature is defined by the trace $\boldsymbol{R} = \sum_{i,j} \boldsymbol{R^i_{jij}}$ and is found to be $\boldsymbol{R} = 2(\boldsymbol{R^1_{212}} + \boldsymbol{R^1_{313}} + \boldsymbol{R^2_{323}})$. Note that the scalar curvature is an invariant, that is, it does not depend on the choice of an orthonormal frame. It follows that

$$\boldsymbol{R} = -\left(2a^2 + \frac{3}{2}c^2 + bc - \frac{1}{2}b^2 + \frac{3}{2}b^2 + bc + 2d^2 - \frac{1}{2}c^2 - \frac{1}{2}b^2 - \frac{1}{2}c^2 - bc + 2ad\right).$$

By arranging the terms we obtain

$$2\boldsymbol{R} = -((b+c)^2 + (2a+d)^2 + 3d^2).$$

As a result we have the following:

Theorem 1 *The Riemannian manifold corresponding to a linear system*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy,$$

has non-positive scalar curvature. The scalar curvature vanishes if and only if $b = -c$ *and* $a = d = 0$.

We say that the Riemannian manifold is flat iff the curvature tensor identically vanishes. Substituting $b = -c$ *and* $a = d = 0$ into the components of the Riemannian curvature tensor yields the following:

Corollary 2 *The Riemannian manifold corresponding to a linear system*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\lambda x,$$

where $\lambda \in \mathbb{R}$, *is flat.*

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