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Araştırma Makalesi / Research Article

Zero-Divisor Graphs of Order-Decreasing Full Transformation Semigroups

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Abstract

Let $n \in \mathbb{Z}^+$ and $X_n = \{1,2,\dots,n\}$ be a finite set. Let D_n be the order-decreasing full transformation semigroup on X_n. In this paper, we find the left zero-divisors, the right zero-divisors and two sided zero-divisors of D_n. Moreover, for $n \geq 4$ we define an undirected graph $\Gamma(D_n)$ whose vertices are two-sided zero divisors of D_n excluding the zero element θ of D_n. In the graph, distinct two vertices α and β are adjacent if and only if $\alpha\beta = \theta = \beta\alpha$. In this paper, we prove that $\Gamma(D_n)$ is a connected graph, and we find diameter, girth, the degrees of all vertices, the maximum degree and the minimum degree in $\Gamma(D_n)$. Moreover, we give lower bounds for clique number and choromatic number of $\Gamma(D_n)$.

Sıra Azaltan Dönüşüm Yarıgruplarının Sıfır-Bölen Çizgesi

Anahtar kelimeler
Sıfır-bölen çizge; Sıra azaltan dönüşümler; Çap; Klik sayısı

Öz

$n \in \mathbb{Z}^+$ olmak üzere $X_n = \{1,2,\dots,n\}$ sonlu bir küme olsun. X_n üzerindeki tüm sıra azaltan dönüşümlerin yarıgrubu D_n olsun. Bu çalışmada D_n yarıgrubunun sol sıfır bölenleri, sağ sıfır bölenleri ve iki-yönlü sıfır bölenleri bulunmuştur. Ayrıca, $n \geq 4$ için köşeleri D_n yarıgrubunun sıfır elemanı θ dışındaki iki-yönlü sıfır bölenleri olmak üzere $\Gamma(D_n)$ yönsüz çizgesi tanımlanmıştır. Bu çizgede α ve β farklı köşeler olmak üzere bu iki köşenin çizgede bir kenar oluşturması için gerek ve yeter koşul $\alpha\beta = \theta = \beta\alpha$ olmasıdır. Bu çalışmada $\Gamma(D_n)$ çizgesinin bağlantılı olduğu ispatlanmış olup, çizgenin çapı, çizgedeki en kısa devir uzunluğu, tüm köşelerin dereceleri, en büyük derece ve en küçük derece bulunmuştur. Ayrıca, $\Gamma(D_n)$ çizgesinde klik ve kromatik sayıları için bir alt sınır bulunmuştur.

1. Introduction and Definitions

The zero-divisor graphs were first defined on commutiative rings by Beck (Beck 1988). The zero element of ring is a vertex in the zero-divisor graph within Beck's definition, then the standart zero-divisor graphs on commutative rings were defined by Anderson and Livingston (Anderson and Livingston 1999). Let R be a commutative ring and $Z(R)$ be the set of zero-divisor elements of R. The zero-divisor graph of R is defined by the vertex set $Z(R)\backslash\{0\}$ and distinct two vertices α and β are adjacent if and only if $\alpha\beta = 0$. The zero-divisor graph of R is denoted by $\Gamma(R)$. DeMeyer *et al*. have considered this definition on commutative semigroups, they found some basic properties of zero-divisor graphs of commutative semigroups (DeMeyer *et al*. 2002, DeMeyer *et al*. 2005). There are some papers about zero-divisor graphs of some special classes of commutative semigroups (Das *et al*. 2013, Toker 2016). Redmond defined some zero-divisor graphs for the noncommutative rings (Redmond 2002). Let R be a noncommutative ring and $Z_T(R)$ be the set of two-sided zero-divisor elements of R. Then zero-divisor graph of R is defined by the vertex set $Z_T(R)\backslash\{0\}$ and distinct two vertices α and β are adjacent if and only if $\alpha\beta =$

$0=\beta\alpha$. The zero-divisor graph of R is denoted by $\Gamma(R)$. If R is a noncommutative ring, then $\Gamma(R)$ does not need to be connected graph. Moreover, these definitions can be considered on noncommutative semigroups. Let S be a semigroup with 0 (zero), $S^*=S\backslash\{0\}$ and

$$T(S)=\{z\in S|zx=0=yz \text{ for some } x,y\in S^*\}.$$

If $T(S)\backslash\{0\}\neq\emptyset$, then we similarly define the (undirected) zero-divisor graph $\Gamma(S)$ whose the set of vertices is $T(S)\backslash\{0\}$ and distinct two vertices x and y are adjacent by an edge if and only if $xy=0=yx$ for some $x,y\in T(S)\backslash\{0\}$.

Recently, some properties of zero-divisor graphs of Catalan monoid and zero-divisor graphs of partial transformation semigroups researched (Toker 2021, Toker 2021). In this paper, our aim is research of zero-divisor graphs of order-decreasing transformation semigroups. Let $n\in\mathbb{Z}^+$ and $X_n=\{1,2,\dots,n\}$ be a finite set. Let T_n and D_n be the full transformation semigroup on X_n, order-decreasing full transformation semigroup on X_n, respectively. Then,

$$D_n=\{\alpha\in T_n|(\forall x\in X_n)\ x\alpha\leq x\}.$$

D_n is a noncommutative semigroup for $n\geq 3$ and it is also a monoid. Let 1_{D_n} be the identity element of D_n. Then $x1_{D_n}=x$ for all $x\in X_n$. Umar studied some algebraic properties of $D_n\backslash\{1_{D_n}\}$ (Umar 1992).

It is clear that $|D_n|=n!$ and $1\alpha=1$ for all $\alpha\in D_n$. Let $\theta\in D_n$ such that $x\theta=1$ for all $x\in X_n$. Then we have $\alpha\theta=\theta\alpha=\theta$ for all $\alpha\in D_n$, so θ is the zero element of D_n. Throughout the paper, the zero element of D_n is denoted by θ. Let ${D_n}^*=D_n\backslash\{\theta\}$ for $n\geq 2$. We define the following sets

$$L=L(D_n)=\{\alpha\in D_n|\alpha\beta=\theta \text{ for some } \beta\in {D_n}^*\},$$

$$R=R(D_n)=\{\alpha\in D_n|\beta\alpha=\theta \text{ for some } \beta\in {D_n}^*\},$$

$$T=T(D_n)$$
$$=\{\alpha\in D_n|\alpha\beta=\gamma\alpha=\theta \text{ for some } \gamma,\beta\in {D_n}^*\}$$

which are called the set of left zero-divisors, right zero-divisors and two-sided zero-divisors of D_n. Then it is clear that $T=L\cap R$.

For semigroup terminology see (Howie 1995) and graph theory terminology see (Thulasiraman *et al.* 2015).

2. Preliminaries

In this section, we find the set of left zero-divisors, right zero-divisors and two sided zero-divisors of D_n, and then we find their numbers.

Lemma 2.1 Let $n\geq 2$. If $\alpha,\beta\in D_n$, then $\alpha\beta=\theta$ if and only if $\mathrm{Im}(\alpha)\subseteq 1\beta^{-1}$. In particular, $\alpha^2=\theta$ if and only if $\mathrm{Im}(\alpha)\subseteq 1\alpha^{-1}$.

Proof: Let $\alpha,\beta\in D_n$. If $\alpha\beta=\theta$, then we have $x(\alpha\beta)=(x\alpha)\beta=x\theta=1$ for all $x\in X_n$. So we have $y\beta=1$ for all $y\in\mathrm{Im}(\alpha)$. It follows that $\mathrm{Im}(\alpha)\subseteq 1\beta^{-1}$. If $\mathrm{Im}(\alpha)\subseteq 1\beta^{-1}$, then we have $x(\alpha\beta)=(x\alpha)\beta=1$ for all $x\in X_n$, it follows that $\alpha\beta=\theta$.

Lemma 2.2 For $n\geq 2$, let L be the set of left zero-divisors of D_n and R be the set of right zero-divisors of D_n. Then, $L=D_n\backslash\{1_{D_n}\}$, $R=\{\alpha\in D_n|\ |1\alpha^{-1}|\geq 2\}$. Moreover, $|L|=n!-1$ and $|R|=n!-(n-1)!$.

Proof: Let $n\geq 2$. Let L be the set of left zero-divisors of D_n and $\alpha\in D_n\backslash\{1_{D_n}\}$. Then we have $\mathrm{Im}(\alpha)\neq X_n$ from the definition of D_n. Let $\beta\in T_n$ such that $x\beta=1$ for all $x\in\mathrm{Im}(\alpha)$ and $y\beta=2$ for all $y\in X_n\backslash\mathrm{Im}(\alpha)$. Then we have $\beta\in {D_n}^*$ and $\alpha\beta=\theta$. Thus, α is a left zero-divisor of D_n. If $\alpha=1_{D_n}$ and $\alpha\beta=\theta$ for any $\beta\in D_n$, then $\beta=\theta$ since $\mathrm{Im}(\alpha)=X_n$ and by Lemma2.1. Thus, 1_{D_n} is not a left zero-divisor of D_n. So $L=D_n\backslash\{1_{D_n}\}$ and it is clear that

$|L| = n! - 1$. Let R be the set of right zero-divisors of D_n and $\alpha \in \{\alpha \in D_n | \, |1\alpha^{-1}| \geq 2\}$. Then we have $t\alpha = 1$ for some $t \in X_n \backslash \{1\}$. Let $\beta \in T_n$ such that $x\beta = 1$ for all $x < t$ and $x\beta = t$ for all $x \geq t$. So $\beta \in D_n{}^*$ and $\beta\alpha = \theta$. Thus, α is a right zero-divisor of D_n. If $\alpha \in D_n$ and $\alpha \notin \{\alpha \in D_n | \, |1\alpha^{-1}| \geq 2\}$, then we have $x\alpha \neq 1$ for all $x \geq 2$ and $1\alpha^{-1} = \{1\}$. Let $\beta\alpha = \theta$ for any $\beta \in D_n$. Then we have $\mathrm{Im}(\beta) = \{1\}$ by Lemma 2.1 and so $\beta = \theta$. If $\alpha \in D_n$ and $\alpha \notin \{\alpha \in D_n | \, |1\alpha^{-1}| \geq 2\}$, then α is not a right zero-divisor of D_n. So $R = \{\alpha \in D_n | \, |1\alpha^{-1}| \geq 2\}$. Let

$$B = \{\alpha \in D_n | \, |1\alpha^{-1}| = 1\}$$
$$= \{\alpha \in D_n | 1\alpha^{-1} = \{1\}\}.$$

It is clear that $|B| = (n-1)!$. Moreover, $R \cup B = D_n$ and $R \cap B = \emptyset$. So we have $|R| = |D_n| - |B| = n! - (n-1)!$.

We have the following corollary since $T = L \cap R$ and $R \subseteq L$.

Corollary 2.3 For $n \geq 2$, let T be the set of (two-sided) zero-divisors of D_n. Then $T = L \cap R = R$. So $|T| = n! - (n-1)!$.

3. Results and Discussions

Let $G = (V(G), E(G))$ be an undirected graph where $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G. A graph whose edge set is empty set is called as a null graph. If G does not have any loops and multiple edges, then G is called a simple graph. We consider simple graphs for the following definitions. If $u, v \in V(G)$ and there is a path from u to v, then it is said u and v are connected vertices in G. If all vertices are connected in G, then G is called a connected graph, otherwise G is called a disconnected graph. A simple graph is called complete graph if every pair of distinct vertices is connected by an edge. The complete graph on n vertices is denoted by K_n. Now we give some examples about those definitons.

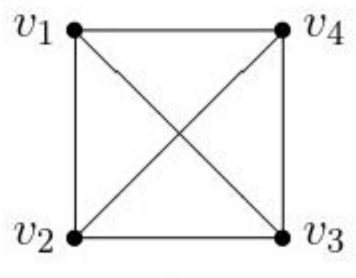

Figure 1. (Complete graph with 4 vertices) K_4.

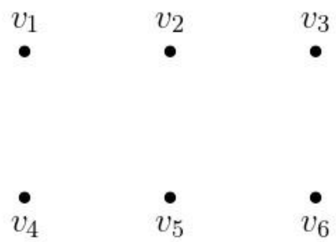

Figure 2. Null graph with 6 vertices.

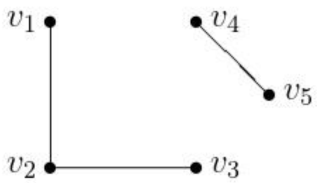

Figure 3. (Disconnected graph) G.

v_1 and v_2 are adjacent and connected vertices in G, v_1 and v_3 are not adjacent vertices but they are connected vertices since there is a path from v_1 to v_3. There is not any path from v_1 to v_4, so G is a disconnected graph.

Let $u, v \in V(G)$, the length of the shortest path between u and v is denoted by $d_G(u, v)$. The diameter of G is denoted by $\mathrm{diam}(G)$ and defined by

$$\mathrm{diam}(G) = \max\{d_G(u, v) | u, v \in V(G)\}.$$

The degree of a vertex $v \in V(G)$ is denoted by $\deg_G(v)$ and defined as the number of adjacent vertices to v in G. Among all the vertex degrees in G, the maximum degree in G is denoted by $\Delta(G)$ and the minimum degree in G is denoted by $\delta(G)$.

The length of the shortest cycle in G is called girth of G and it is denoted by $\mathrm{gr}(G)$. If G does not have any cycles, then its girth is defined to be infinity. Let C be the nonempty subset of $V(G)$. If u and v are adjacent vertices for all $u, v \in C$ in G, then C is called a clique. The number of vertices in any maximal clique in G is called clique number of G, it is denoted by $\omega(G)$. The chromatic number of G is defined by the number of the minimum number of colours required to colour all the vertices of G with the rule no two adjacent vertices have the same colour, and it is denoted by $\chi(G)$.

Let $I \subseteq V(G)$. If G' be a subgraph of G which has vertex set I and edge set consists of all of the edges in $E(G)$ that have both endpoints in I, then G' is called (vertex) induced subgraph of G.

In this section, we prove that $\Gamma(D_n)$ is a connected graph for $n \geq 4$. We find diameter, girth, the vertex degrees, the maximum degree, the minimum degree and we give lower bounds for clique number and choromatic number of $\Gamma(D_n)$ for $n \geq 4$. In this paper, we use Γ instead of $\Gamma(D_n)$. Let $T^* = T\backslash\{\theta\}$. Then we have $T^* = V(\Gamma)$ and

$$|T^*| = [n! - (n-1)!] - 1.$$

Let $\alpha, \beta \in V(\Gamma)$. α and β are adjacent vertices if and only if $\mathrm{Im}(\alpha) \subseteq 1\beta^{-1}$ and $\mathrm{Im}(\beta) \subseteq 1\alpha^{-1}$ by Lemma 2.1.

Lemma 3.1 Γ is a connected graph for $n \geq 4$.

Proof: Let $n \geq 4$. Let $\bar{\alpha} \in T_n$ such that $x\bar{\alpha} = 1$ for $1 \leq x \leq n-1$ and $n\bar{\alpha} = 2$. So

$$\bar{\alpha} = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ 1 & 1 & \ldots & 1 & 2 \end{pmatrix}.$$

Then $\bar{\alpha} \in V(\Gamma)$. Let $\beta \in V(\Gamma)\backslash\{\bar{\alpha}\}$, we will show that there is a path from β to $\bar{\alpha}$ in Γ.

Case 1: If $\beta \in V(\Gamma)\backslash\{\bar{\alpha}\}$ such that $2\beta = 1$ and $n\beta \neq n$, then $\bar{\alpha}$ and β are adjacent vertices in Γ by Lemma 2.1.

Case 2: Let $\beta \in V(\Gamma)\backslash\{\bar{\alpha}\}$ such that $2\beta = 2$ and $n\beta \neq n$. If there exists $t \in X_n$ such that $2 < t < n$ and $t\beta = 1$, then there exists $\gamma \in T_n$ such that $x\gamma = 1$ for $1 \leq x \leq n-1$ and $n\gamma = t$. So $\gamma \in V(\Gamma)$ and there is a path in Γ such that $\beta - \gamma - \bar{\alpha}$ by Lemma 2.1. Othercase, we have $2\beta = 2$, $n\beta = 1$ and $t\beta \neq 1$ for $2 \leq t \leq n-1$ since $|1\beta^{-1}| \geq 2$. There exists $\rho \in T_n$ such that $x\rho = 1$ for $1 \leq x \leq n-1$ and $n\rho = n$, $\eta \in T_n$ such that $3\eta = 2$ and $x\eta = 1$ for $x \in X_n\backslash\{3\}$. Then there is a path in Γ such that $\beta - \rho - \eta - \bar{\alpha}$ by Lemma 2.1.

Case 3: Let $\beta \in V(\Gamma)\backslash\{\bar{\alpha}\}$ such that $2\beta = 1$ and $n\beta = n$. Then there exists $k \in X_n$ such that $1 < k < n$ and $k \notin \mathrm{Im}(\beta)$. Let $\lambda \in T_n$ such that $k\lambda = 2$ and $x\lambda = 1$ for all $x \in X_n\backslash\{k\}$, $\tau \in T_n$ such that $3\tau = 3$ and $x\tau = 1$ for all $x \in X_n\backslash\{3\}$. Then we have $\lambda, \tau \in V(\Gamma)$. If $k \neq 2$, then there is a path in Γ such that $\beta - \lambda - \bar{\alpha}$ by Lemma 2.1. If $k = 2$, then there is a path in Γ such that $\beta - \lambda - \tau - \bar{\alpha}$ by Lemma 2.1.

Case 4: Let $\beta \in V(\Gamma)\backslash\{\bar{\alpha}\}$ such that $2\beta = 2$ and $n\beta = n$. Let $A = \{x \in X_n\backslash\{1\} | x\beta = 1\}$ and $B = \{x \in X_n | x \notin \mathrm{Im}(\beta)\}$. Then it is clear that $A \neq \emptyset$, $B \neq \emptyset$, $2 \notin A$ and $n \notin B$. Let $k = \min A$ and $t = \max B$. We have $k \neq n$ and $t \neq 2$. Moreover, we have $t \geq k$ from definition of D_n. Let $\mu \in T_n$ such that $t\mu = k$ and $x\mu = 1$ for all $x \in X_n\backslash\{t\}$. Then $\mu \in V(\Gamma)$ and there is a path in Γ such that $\beta - \mu - \bar{\alpha}$ by Lemma 2.1.

Thus, Γ is a connected graph for $n \geq 4$.

Lemma 3.2 $\mathrm{diam}(\Gamma) = 4$ for $n \geq 4$.

Proof: For $n \geq 4$, let $\alpha, \beta \in V(\Gamma)$ and α, β be different vertices. First of all, we will show that $d_\Gamma(\alpha, \beta) \leq 4$. If α and β are adjacent vertices in Γ, then $d_\Gamma(\alpha, \beta) = 1$. Suppose that α and β are not adjacent vertices in Γ. Let

$$A = \{x \in X_n\backslash\{1\} | x\alpha = 1\},$$

$$B = \{x \in X_n | x \notin \mathrm{Im}(\alpha)\},$$

$$C = \{x \in X_n\backslash\{1\} | x\beta = 1\},$$

$$D = \{x \in X_n | x \notin \mathrm{Im}(\beta)\}.$$

Let $k_1 = \min A$, $t_1 = \max B$, $k_2 = \min C$ and $t_2 = \max D$. Let $\gamma \in T_n$ such that $t_1\gamma = k_1$ and $x\gamma = 1$ for all $x \in X_n \backslash \{t_1\}$. Let $\rho \in T_n$ such that $t_2\rho = k_2$ and $x\rho = 1$ for all $x \in X_n \backslash \{t_2\}$. We have $t_1 \geq k_1$, $t_2 \geq k_2$ and it is clear that $\gamma, \rho \in V(\Gamma)$. Moreover, α and γ are adjacent vertices in Γ, similarly β and ρ are adjacent vertices in Γ by Lemma 2.1. If $k_1 \neq t_2$ and $k_2 \neq t_1$ then γ and ρ are adjacent vertices in Γ by Lemma 2.1 and so $d_\Gamma(\alpha,\beta) \leq 3$. Let $k_1 = t_2$. Then we have $k_1 \geq k_2$ since $t_2 \geq k_2$. If $t_1 = k_1 = k_2$, then $\rho = \gamma$ and so $d_\Gamma(\alpha,\beta) \leq 2$. If $t_1 > k_1 = k_2$, then there exists $r \in X_n \backslash \{1, t_1, k_1\}$ since $n \geq 4$. Let $\lambda \in T_n$ such that $r\lambda = r$ and $x\lambda = 1$ for all $x \in X_n \backslash \{r\}$. Then $\lambda \in V(\Gamma)$ and there is a path in Γ such that $\alpha - \gamma - \lambda - \rho - \beta$ by Lemma 2.1 and so $d_\Gamma(\alpha,\beta) \leq 4$. If $t_1 = k_1 > k_2$, then there exists $r \in X_n \backslash \{1, k_1, k_2\}$ since $n \geq 4$. Let $\mu \in T_n$ such that $r\mu = r$ and $x\mu = 1$ for all $x \in X_n \backslash \{r\}$. Then $\mu \in V(\Gamma)$ and there is a path in Γ such that $\alpha - \gamma - \mu - \rho - \beta$ by Lemma 2.1 and so $d_\Gamma(\alpha,\beta) \leq 4$. Let $t_1 > k_1 > k_2$ and $\eta \in T_n$ such that $t_1\eta = k_2$ and $x\eta = 1$ for all $x \in X_n \backslash \{t_1\}$. Then $\eta \in V(\Gamma)$ and there is a path in Γ such that $\alpha - \gamma - \eta - \rho - \beta$ by Lemma 2.1 and so $d_\Gamma(\alpha,\beta) \leq 4$. If $t_1 = k_2$, then we have similar case. So if $\alpha, \beta \in V(\Gamma)$, then $d_\Gamma(\alpha,\beta) \leq 4$. Let $\alpha_1 \in T_n$ such that $3\alpha_1 = 1$ and $x\alpha_1 = x$ for all $x \in X_n \backslash \{3\}$, $\alpha_2 \in T_n$ such that $1\alpha_2 = 2\alpha_2 = 1$, $3\alpha_2 = 2$ and $x\alpha_2 = x$ for all $x \geq 4$. Then $\alpha_1, \alpha_2 \in V(\Gamma)$. Moreover, α_1 and α_2 are different vertices and they are not adjacent vertices in Γ. α_1 has only one adjacent vertex which is $\mu_1 \in V(\Gamma)$, $3\mu_1 = 3$ and $x\mu_1 = 1$ for all $x \in X_n \backslash \{3\}$, similarly α_2 has only one adjacent vertex which is $\mu_2 \in V(\Gamma)$, $3\mu_2 = 2$ and $x\mu_2 = 1$ for all $x \in X_n \backslash \{3\}$. Furthermore, μ_1 and μ_2 are not adjacent vertices and so $d_\Gamma(\alpha_1,\alpha_2) = 4$. Thus, $\text{diam}(\Gamma) = 4$ for $n \geq 4$.

Notice that if S is a commutative semigroup with zero, then $\Gamma(S)$ is a connected graph and $\text{diam}(\Gamma(S)) \leq 3$ (Demeyer *et al*. 2002). However, these results may not be correct in noncommutative semigroups. So we have showed that $\Gamma(D_n)$ is a connected graph for $n \geq 4$. Moreover, we have proved that $\text{diam}(\Gamma(D_n)) = 4$ for $n \geq 4$.

Lemma 3.3 $\text{gr}(\Gamma) = 3$ for $n \geq 4$.

Proof: It is clear that $\text{gr}(\Gamma) \geq 3$ since Γ is a simple graph for $n \geq 4$. Let $n \geq 4$ and $\alpha, \beta, \gamma \in V(\Gamma)$ such that $2\alpha = 2$, $x\alpha = 1$ for all $x \in X_n \backslash \{2\}$, $3\beta = 3$, $y\beta = 1$ for all $y \in X_n \backslash \{3\}$, $4\gamma = 4$, $z\gamma = 1$ for all $z \in X_n \backslash \{4\}$. Then there exists a cycle in Γ which is $\alpha - \beta - \gamma - \alpha$. So $\text{gr}(\Gamma) = 3$ for $n \geq 4$.

To find vertex degree of any vertex in Γ, we will define functions associate with vertices. Let $\alpha \in V(\Gamma)$, $A = X_n \backslash \text{Im}(\alpha) = \{a_1, a_2, \dots, a_k\}$, $1\alpha^{-1} = \{1 = b_1, b_2, \dots, b_r\}$ with $1 = b_1 < b_2 < \cdots < b_r$. If $a_i \in A$, then $a_i \geq b_r$ or there exists $j \in \{1,2,\dots,n-1\}$ and $b_j \leq a_i < b_{j+1}$. Let $f_\alpha : X_n \to X_n$ such that

$$(x)f_\alpha = \begin{cases} 1, & \text{if } x \in \text{Im}(\alpha) \\ j, & \text{if } x \notin \text{Im}(\alpha) \text{ and } b_j \leq x = a_i < b_{j+1} \\ r, & \text{if } x \notin \text{Im}(\alpha) \text{ and } x = a_i \geq b_r. \end{cases}$$

Theorem 3.4 Let $n \geq 4$ and $\alpha \in V(\Gamma)$, $A = X_n \backslash \text{Im}(\alpha) = \{a_1, a_2, \dots, a_k\}$, $1\alpha^{-1} = \{1 = b_1, b_2, \dots, b_r\}$ with $1 = b_1 < b_2 < \cdots < b_r$. Then

$$deg_\Gamma(\alpha) = \begin{cases} \left(\prod_{i=1}^{n} if_\alpha\right) - 1, & \text{if } \text{Im}(\alpha) \nsubseteq 1\alpha^{-1} \\ \left(\prod_{i=1}^{n} if_\alpha\right) - 2, & \text{if } \text{Im}(\alpha) \subseteq 1\alpha^{-1}. \end{cases}$$

Proof: Let $n \geq 4$ and $\alpha \in V(\Gamma)$, $A = X_n \backslash \text{Im}(\alpha) = \{a_1, a_2, \dots, a_k\}$, $1\alpha^{-1} = \{1 = b_1, b_2, \dots, b_r\}$ with $1 = b_1 < b_2 < \cdots < b_r$. Let $\beta \in V(\Gamma)$, α and β be the adjacent vertices in Γ. If $\text{Im}(\alpha) \nsubseteq 1\alpha^{-1}$, then $\alpha^2 \neq \theta$ by Lemma 2.1. We have $x\beta = 1$ for all $x \in \text{Im}(\alpha)$, thus $|1\beta^{-1}| \geq 2$. If $x \notin \text{Im}(\alpha)$, then $x\beta \in 1\alpha^{-1}$ and $x\beta \leq x$. If $x \notin \text{Im}(\alpha)$, then $x = a_i$ for $1 \leq i \leq k$. So, it is clear that we have $(a_i)f_\alpha$ different choices for $x\beta$ where $x \notin \text{Im}(\alpha)$. However, we have $\beta \in T$ with those choices. If we take $x\beta = 1$ for all $x \notin \text{Im}(\alpha)$, then $\beta = \theta$. So, if $\text{Im}(\alpha) \nsubseteq 1\alpha^{-1}$, then $deg_\Gamma(\alpha) = (\prod_{i=1}^{n} if_\alpha) - 1$. If $\text{Im}(\alpha) \subseteq 1\alpha^{-1}$, then we have $\alpha^2 = \theta$ by Lemma 2.1. Moreover, we have similar proof for this case. So, if $\mathrm{Im}(\alpha) \subseteq 1\alpha^{-1}$, then $deg_\Gamma(\alpha) = (\prod_{i=1}^{n} if_\alpha) - 2$ since $\alpha^2 = \theta$.

Let $n \geq 4$ and $\alpha \in V(\Gamma)$, $A = X_n \backslash \mathrm{Im}(\alpha) = \{a_1, a_2, \ldots, a_k\}$, $1\alpha^{-1} = \{1 = b_1, b_2, \ldots, b_r\}$ with $1 = b_1 < b_2 < \cdots < b_r$. We have $|A| \leq n-2$ since $|\mathrm{Im}(\alpha)| \geq 2$, moreover we have $2 \leq r \leq n-1$ since $\alpha \in T^*$. So, $if_\alpha \leq i$ for $1 \leq i \leq n-1$ and $nf_\alpha \leq n-1$ since $r \leq n-1$ and the definiton of f. It can be $if_\alpha \neq 1$ at most $n-2$ different elements in X_n since $|\mathrm{Im}(\alpha)| \geq 2$. Thus, for the maximum degree we take $1f_\alpha = 1$, $2f_\alpha = 1$, $if_\alpha = i$ for $3 \leq i \leq n-1$ and $nf_\alpha = n-1$. In this case, we have

$$\alpha = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ 1 & 1 & \ldots & 1 & 2 \end{pmatrix},$$

$\alpha^2 = \theta$ and so $deg_\Gamma(\alpha) = \left(\frac{(n-1)!}{2}.(n-1)\right) - 2$. Moreover, it is clear that

$$\left(\frac{(n-1)!}{2}.(n-1)\right) - 2 > \left(\frac{(n-1)!}{k}.(n-1)\right) - 1$$

for $n \geq 4$ and $k \geq 3$. Thus, α is the unique vertex which has maximum degree in Γ. Let

$$\beta = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ 1 & 2 & \ldots & n-1 & 1 \end{pmatrix},$$

then $deg_\Gamma(\beta) = 1$. So we have the following corollary.

Corollary 3.5 If $n \geq 4$, then

$$\Delta(\Gamma) = \left(\frac{(n-1)!}{2}.(n-1)\right) - 2$$

and $\delta(\Gamma) = 1$.

Theorem 3.6 For $n \geq 4$, $\omega(\Gamma) \geq r^{n-r} - 1$ for $2 \leq r \leq n-1$.

Proof: Let $n \geq 4$ and $X_r = \{1,2,\ldots,r\}$ for $2 \leq r \leq n-1$. Let

$$A = \{\alpha \in V(\Gamma) | 1\alpha^{-1} \supseteq X_r \text{ and } \mathrm{Im}(\alpha) \subseteq X_r\}.$$

If $\alpha \in A$, then we have $\mathrm{Im}(\alpha) \subseteq X_r \subseteq 1\alpha^{-1}$. Let $\alpha, \beta \in A$ and $\alpha \neq \beta$. Then we have $\mathrm{Im}(\alpha) \subseteq X_r \subseteq 1\beta^{-1}$ and $\mathrm{Im}(\beta) \subseteq X_r \subseteq 1\alpha^{-1}$ and so α and β are adjacent vertices in Γ. If G be an induced subgraph of Γ induced by the vertex set A, then G is a complete graph. Moreover, it is clear that $|A| = r^{n-r} - 1$. Thus, we have $\omega(\Gamma) \geq r^{n-r} - 1$ for $2 \leq r \leq n-1$.

For any graph G, it is known that $\chi(G) \geq \omega(G)$ (Chartrand *et al*. 2009). So we have the following corollary.

Corollary 3.7 For $n \geq 4$, $\chi(\Gamma) \geq r^{n-r} - 1$ for $2 \leq r \leq n-1$.

Example 3.8 Let $\Gamma = \Gamma(D_4)$. Then Γ is a connected graph, $V(\Gamma) = 17$, $\mathrm{diam}(\Gamma) = 4$, $\mathrm{gr}(\Gamma) = 3$, $\Delta(\Gamma) = 7$, $\delta(\Gamma) = 1$, $\omega(\Gamma) \geq 3$ and $\chi(\Gamma) \geq 3$. Moreover, Γ is isomorphic to following graph.

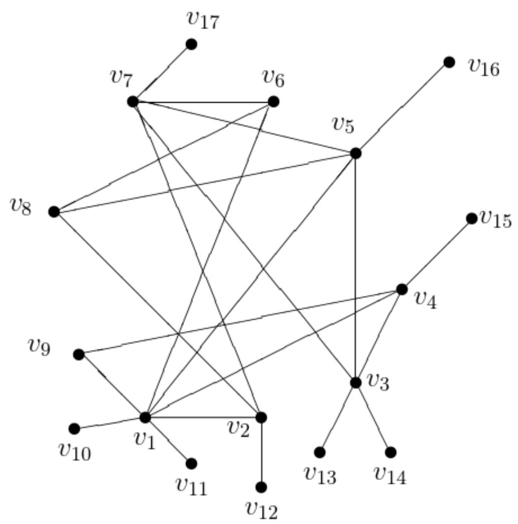

Figure 4. $\Gamma(D_4)$.

4. Conclusion

In this study, we find the set of left zero-divisors, right zero-divisors and two sided zero divisors of D_n for $n \geq 2$. It is well known that D_n is a noncommutative semigroup for $n \geq 3$. We define a graph associated with D_n which is called zero-divisor graph of D_n and it is denoted by $\Gamma(D_n)$. One can see that $\Gamma(D_2)$ is a null graph and $\Gamma(D_3)$ is not a connected graph. We have introduced $\Gamma(D_n)$ for $n \geq 4$.

5. References

Anderson, D.F., Livingston, P.S., 1999. The zero-divisor graph of a commutative ring. *Journal of Algebra,* **217**(2), 434-447.

Beck, I., 1988. Coloring of commutative rings. *Journal of Algebra,* **116**(1), 208-226.

Chartrand, G., Zhang, P., 2009. Chromatic Graph Theory. Boca Raton, FL, USA: CRC Press, 1-483.

Das, K.C., Akgüneş, N., Çevik, A.S., 2013. On a graph of monogenic semigroup. *Journal of Inequalities and Applications,* **44**, 1-13.

DeMeyer, F., DeMeyer, L., 2005. Zero divisor graphs of semigroups. *Journal of Algebra,* **283**(1), 190-198.

DeMeyer, F., McKenzie, T., Schneider, K., 2002. The zero-divisor graph of a commutative semigroup. *Semigroup Forum,* **65**(2), 206-214.

Howie, J.M., 1995.Fundamentals of semigroup theory. New York, NY, USA: Oxford University Press, 1-351.

Redmond S.P., 2002. The zero-divisor graph of a non-commutative ring. *International Journal of Commutative Rings,* **1**(4), 203-211.

Thulasiraman, K., Arumugam, S., Brandstädt, A., Nishizeki, T., 2015. Handbook of Graph Theory, Combinatorial Optimization, and Algorithms. Boca Raton, CRC Press, 1-1195.

Toker, K., 2016. On the zero-divisor graphs of finite free semilattices. *Turkish Journal of Mathematics,* **40**(4), 824-831.

Toker, K., 2021. Zero-divisor graphs of Catalan monoid. *Hacettepe Journal of Mathematics and Statistics,* **50**(2), 387-396.

Toker, K., 2021. Zero-divisor graphs of partial transformation semigroups. *Turkish Journal of Mathematics,* **45**(5), 2323-2330.

Umar, A., 1992. On the semigroup of order-decreasing full transformation. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics,* **120**(A), 129-142.