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Research Article

On Certain Power Horadam Sequences

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ABSTRACT

In this study, firstly, we analyzed power Fibonacci sequences defined by Ide and Renault in [13]. Then, we described two power Horadam sequences modulo s for $u = 1, v = 3$ and $u = 3, v = 1$, respectively. We determined those modulus s for which the two power Horadam sequences exist and the number of such sequences for a given s. We formulated the periods of these power Horadam sequences in terms of the periods of Horadam sequences for $u = 1, v = 3$ and $u = 3, v = 1$. Finally, we compared that the period formulas of power Horadam sequences which we obtained and the period formulas of power Fibonacci sequences. We found that, the periods formulas of the power Horadam sequences for $u = 3, v = 1$ are the same as the period formulas of the power Fibonacci sequences; but for $u = 1, v = 3$, a certain relationship couldn't be established between the periods of these power sequences.

Keywords: *Horadam sequence, power Fibonacci sequence, Period*

Belirli Horadam Kuvvet Dizileri Üzerine

ÖZ

Bu çalışmada, ilk olarak, kaynak [13]de Ide ve Renault tarafından tanımlanan Fibonacci kuvvet dizilerini inceledik. Sonrasında, sırasıyla $u = 1, v = 3$ ve $u = 3, v = 1$ için modül s de iki tane Horadam kuvvet dizisi tanımladık. Bu iki kuvvet dizisinin var olduğu s modüllerini ve verilen bir s modülü için bu dizilerin sayısını belirledik. $u = 1, v = 3$ ve $u = 3, v = 1$ durumları için tanımladığımız bu Horadam kuvvet dizilerinin periyotlarını, Horadam dizilerinin periyotları cinsinden formülize ettik. Son olarak, Horadam kuvvet dizilerinin elde ettiğimiz periyot formülleri ile Fibonacci kuvvet dizilerinin periyot formüllerini karşılaştırdık. $u = 3, v = 1$ için Horadam kuvvet dizilerinin periyot formülleri Fibonacci kuvvet dizilerinin periyot formülleri ile aynı iken $u = 1, v = 3$ durumunda bu iki kuvvet dizisinin periyotları arasında belirli bir ilişki kurulamadığını elde ettik.

Anahtar Kelimeler: *Horadam dizisi, Fibonacci kuvvet dizisi, Periyot*

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I. INTRODUCTION

If any term of sequence can be calculated with the predecessor terms, these sequences are called recurrence sequence. For example, Fibonacci sequence. The Fibonacci sequence, $\{F_n\}_0^{\infty}$, is a sequence of numbers, beginning with the integer couple 0 and 1, in which the value of any element is computed by taking the sum of the two antecedent numbers. If so, for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$ [1]. This number sequence, which was previously found by Indian mathematicians in the sixth century. But the sequence was introduced by Fibonacci as a result of calculating the problem related to the reproduction of rabbits in 1202. The first terms of this sequence are $1, 1, 2, 3, 5, 8, 13, 21$. Fibonacci has not done any work using these sequences. In fact, the first researches on these sequences were made about 600 years later. However, the subsequent research has increased substantially. There have been many studies in the literature dealing with the quadratic number sequences. Some authors have obtained generalization of the Fibonacci sequence by changing only the first two terms of the sequence or with minor changes only the recurrence relation, while others have obtained generalizations of the Fibonacci sequence by changing both of them. Some of these sequences are chronologically as follows:

Lucas, Pell, Pell Lucas, Horadam, Jacobsthal and Jacobsthal–Lucas, $k-$ Fibonacci and $k-$Lucas, generalized $k-$Fibonacci and generalized $k-$Lucas, generalized $k-$Horadam, power Fibonacci sequence modulo m [2-13].

All of these sequences are based on the Fibonacci sequence. The Fibonacci sequence has many impressive features. Studies on the properties of these impressive sequences are still ongoing. There are quite a lot applications of these number sequences different areas like engineering, nature and cryptography and coding theory.

Also, there are many of study on the period of the Fibonacci sequences and some sequences are based on the Fibonacci sequence modulo m in literature. Authors built some methods and obtained some equations related to the length of period related to Fibonacci numbers modulo m, even though there is no known explicit formula for length of period [4, 14-16].

In this study, firstly, we examined power Fibonacci sequences defined by Ide and Renault in [13]. Then, we described two power Horadam sequences modulo s for $u = 1, v = 3$ and $u = 3, v = 1$, respectively. We determined those modulus s for which the two power Horadam sequences exist and the number of such sequences for a given s. Also, we investigated that the periods of these special power sequences for both $u = 1, v = 3$ and $u = 3, v = 1$. Finally, we compared that the period formulas of power Horadam sequences which we obtained and the period formulas of power Fibonacci sequences. We found that, the periods formulas of the power Horadam sequences for $u = 3, v = 1$ are the same as the period formulas of the power Fibonacci sequences; however for $u = 1, v = 3$, a certain relationship couldn't be established between the periods of these power sequences.

II. MATERIALS AND METHODS

Here, some Horadam sequences, power Fibonacci sequences are used as material, and periodic relations of these sequences with the Fibonacci sequence modulo s are used as method.

Definition 2.1. Horadam sequences are defined by recurrence relation $H_k = uH_{k-1} + vH_{k-2}$ with initial conditions $H_0 = a$, $H_1 = b$ where a, b are real numbers and u, v are non zero numbers [2].

In this study, particularly, we used Horadam sequences for $u = 1, v = 3$ and $u = 3, v = 1$.

Moreover, it can be easily seen that standard Fibonacci sequence obtained when $u = 1$, $v = 1$, $a = 0$, $b = 1$.

Definition 2.2. Let G be a bi-infinite integer sequence providing the recurrence relation $G_k = G_{k-1} + G_{k-2}$. Providing $G \equiv 1, \gamma, \gamma^2, \gamma^3, \ldots (mod\ s)$ for some modulus s, then G is named a power Fibonacci sequence modulo s [13].

Example 2.3. For modulo $s = 31$, the two power Fibonacci sequences are following:

1, 13, 14, 27, 10, 6, 16, 22, 7, 29, 5, 3, 8, 11, 19, 30, 18, 17, 4, 21, 25, 15, 9, 24, 2, 26, 28, 23, 20, 12, 1, 13, … and 1, 19, 20, 8, 28, 5, 2, 7, 9, 16, 25, 10, 4, 14, 18, 1, 19, 20, . . .

Theorem 2.4. There is precisely one power Fibonacci sequence modulo 5. For $s \neq 5$, there exist power Fibonacci sequences modulo s certainly when s has prime factorization $s = {p_1}^{e_1}{p_2}^{e_2} \ldots {p_k}^{e_k}$ or $s = 5{p_1}^{e_1}{p_2}^{e_2} \ldots {p_k}^{e_k}$, where each $p_i \equiv \mp 1 (mod\ 10)$; in either case there are definitely 2^k power Fibonacci sequences modulo s [13].

We know that $\pi(s)$ denote the period of the Fibonacci sequence modulo s and there is no known explicit formula for $\pi(s)$. But, providing $(s, m) = 1$ then $\pi(sm) = [\pi(s), \pi(m)]$ [14]. It is easily seen that, if S is any periodic sequence mod sm and $(s, m) = 1$, then its period is the least common multiple of the period of S taken mod s and the period of S taken mod m. For $u > 2$, $\pi(s)$ is even [14, 15].

In addition, we know that Ide and Renault established a relationship between $\pi(s)$ and the period of power Fibonacci sequences modulo s. And, they obtained following theorems:

Theorem 2.5. Let p be a prime of the form $p \equiv \mp 1 (mod\ 10)$ and let β and σ be two roots of $f(x) \equiv x^2 - x - 1 (mod\ p^e)$. Suppose $|\beta| \geq |\sigma|$.

i. For $\pi(p^e) \equiv 0 (mod\ 4)$, $|\beta| = |\sigma| = \pi(p^e)$.
ii. For $\pi(p^e) \equiv 2 (mod\ 4)$, $|\beta| = 2|\sigma| = \pi(p^e)$ [13].

Theorem 2.6. Let $s = {p_1}^{e_1}{p_2}^{e_2} \ldots {p_k}^{e_k}$ is the product of the primes of the form $p_i \equiv \mp 1 (mod\ 10)$.

i. For $\pi(s) \equiv 0 (mod\ 4)$, the period of each power Fibonacci sequence modulo s is $\pi(s)$.
ii. For $\pi(s) \equiv 2 (mod\ 4)$, the period of one power Fibonacci sequence modulo s is $\frac{\pi(s)}{2}$ and the periods of the others are $\pi(s)$.
iii. For $\pi(s) \equiv 0 (mod\ 4)$, the period of each power Fibonacci sequence modulo $5s$ is $\pi(s)$.
iv. For $\pi(s) \equiv 2 (mod\ 4)$, the period of each power Fibonacci sequence modulo $5s$ is $2\pi(s)$ [13].

III. RESULTS AND DISCUSSION

A. SOME SPECIAL POWER HORADAM SEQUENCES

Here, firstly, we defined two special power Horadam sequences modulo s.
Definition 3.1. Let H' be a bi-infinite integer sequence satisfying the recurrence relation $H'_n = H'_{n-1} + 3H'_{n-2}$ (or $H'_n = 3H'_{n-1} + H'_{n-2}$). If $H' \equiv 1, \gamma, \gamma^2, \gamma^3, \ldots (mod\ s)$ for some modulus s, then H' is called a special power Horadam sequence modulo s.
In this definition, we used Horadam sequences for $u = 1, v = 3$ and $u = 3, v = 1$.

Example 3.2. For modulo $s = 17$, there are two special power Horadam sequences for $u = 1, v = 3$ as follows:

$1,5,8,6,13,14,2,10,16,12,9,11,4,3,15,7,1,5,\ldots$ and 1,13,16,4,1,13, ...

Example 3.3. For modulo $s = 53$, there are two special power Horadam sequences for $u = 3, v = 1$ as follows:
$1,9,28,40,42,7,10,37,15,29,49,17,47,52,44,25,13,11,46,43,16,38,24,4,36,6,1,9,\ldots$ and
1, 47, 36, 49, 24, 15, 16, 10, 46, 42, 13, 28, 44, 1, 47, …

Then, we determined those modulus s for which these special power Horadam sequences for $u = 1, v = 3$ and $u = 3, v = 1$ exist and the number of such sequences for a given s by following theorem. And we obtained that these special power Horadam sequences exist for the same modulus.

Theorem 3.4. There is exactly one special power Horadam sequence modulo $s = 13$ in both cases for $u = 1, v = 3$ and $u = 3, v = 1$. For $s \neq 13$, there exist special power Horadam sequence modulo s precisely when s has prime factorization $s = {p_1}^{e_1}{p_2}^{e_2} \ldots {p_k}^{e_k}$ or $s = 13p_1^{e_1}p_2^{e_2} \ldots p_k^{e_k}$, where each $p_i \equiv \pm 1, \pm 3, \pm 4 (mod 13)$; in either case there are exactly 2^k special power Horadam sequences modulo s.

Proof. If $h(x) = x^2 - 3x - 1$ (or $h(x) = x^2 - x - 3$) where γ is a root of $h(x)$, $1, \gamma, \gamma^2 \ldots$ is a special power Horadam sequences modulo s. The roots of $h(x)$ are those residues of the form $2^{-1}(3 + u)$ (or $2^{-1}(1 + u)$) where $u^2 \equiv 13(mods)$, then s is odd. Let $f(x) = x^2 - 13$. Counting the number of solutions to $f(x) \equiv 0(mods)$. For $s = 13$, the only solution to $x^2 \equiv 13(mod 13)$ is 0 and there are no solutions to $x^2 \equiv 13(mod 169)$. Thus, $x^2 \equiv 13(mod 13^e)$ has a solution only when $e = 1$ and that solution $x \equiv 0(mod 13)$. The corresponding special power Horadam sequence is 1,8,12,5,1, 8, … (or 1,7, 10, 5, 9, 11, 12, 6, 3, 8, 4, 2, 1, 7, …). For $s \neq 13$, by use of the law of quadratic reciprocity, is found that 13 is a quadratic residue modulo primes of the form $p \equiv \pm 1, \pm 3, \pm 4(mod 13)$. Thus, if $p \equiv \pm 1, \pm 3, \pm 4(mod 13)$, then $f(x)(mods)$ has two distinct roots. Moreover, with $f(x) = x^2 - 13$ and p is a prime of the form $p \equiv \pm 1, \pm 3, \pm 4(mod 13)$, if x_1 is a root of $f(x)(modp)$, then $f'(x_1) = 2x_1 \not\equiv 0(modp)$. By Hensel's Lemma [17], we obtained that $f(x)(modp^e)$ has two distinct roots for every positive integer e.

Lastly, if s and s' are relatively prime, if $f(x) \equiv 0(mods)$ has z solutions and $f(x) \equiv 0(mods')$ has t solutions, by Chinese Remainder Theorem, then $f(x) \equiv 0(mod\ s.s')$ has $z.t$ solutions ■.

A. 1. The Periods of Special Power Horadam Sequences

In this section, we studied on the period of two special power Horadam sequences which we described and obtained some results. Here, the period of Horadam sequences for $u = 1, v = 3$ and $u = 3, v = 1$ modulo s is denoted by $\rho(s)$.

Firstly, we investigated the period of special power Horadam sequences for $u = 3, v = 1$. And we obtained following results:
Lemma 3.5. The prime number p is in the form of $p \equiv \pm 1, \pm 3, \pm 4(mod 13)$, and and let θ and μ be the two roots of $h(x) \equiv x^2 - 3x - 1(modp^e)$. Suppose $|\theta| \geq |\mu|$.

i. For $\rho(p^e) \equiv 0(mod\ 4)$, $|\theta| = |\mu| = \rho(p^e)$.
ii. For $\rho(p^e) \equiv 2(mod\ 4)$, $|\theta| = 2|\mu| = \rho(p^e)$.

Proof. Due to θ and μ are roots of $h(x) = x^2 - 3x - 1(modp^e)$, $\theta.\mu \equiv -1$ and
$$(\theta.\mu)^n \equiv \theta^n\mu^n \equiv (-1)^n(modp^e)$$
for any n (n is a positive integer). So, the proof of Lemma 3.5. can be easily obtained similarly to the proof of Lemma 3.2. in [13].

Theorem 3.6. Suppose that $s = p_1^{e_1}p_2^{e_2} \ldots p_k^{e_k}$ in the form of $p_i \equiv \pm 1, \pm 3, \pm 4(mod 13)$.

i. For $\rho(s) \equiv 0(mod4)$, the period of each special power Horadam sequence modulo s is $\rho(s)$.
ii. For $\rho(s) \equiv 2(mod4)$, the period of one of special power Horadam sequences modulo s is $\frac{1}{2}\rho(s)$, and the periods of others are $\rho(s)$.
iii. For $\rho(s) \equiv 0(mod4)$, the period of each special power Horadam sequence modulo $13s$ is $\rho(s)$.
iv. For $\rho(s) \equiv 2(mod4)$, the period of each special power Horadam sequence modulo $13s$ is $2\rho(s)$.

Proof. Firstly, if $s = p^e$, then it can be easily seen that the conditions *i* and *ii* has been provided according to Lemma 3.5. Let s and m be relatively prime, and for induction, suppose that theorem is provided for modulo s and m. Now, if we show that it holds for modulo $s.m$, then the conditions *i* and *ii* will be proved.

Denote the roots of $h(x) \equiv x^2 - 3x - 1(mods)$ by $a_1, a_2, \dots a_k$ and denote the roots of $h(x)(modm)$ by $b_1, b_2, \dots b_t$. Then $k.t$ roots of $h(x)(mod\ s.m)$ are represented d_{ij} for $1 \leq i \leq k, 1 \leq j \leq t$ with root d_{ij} satisfying the following congruences:

$$d_{ij} \equiv a_i(mods)$$
$$d_{ij} \equiv b_j(modm)$$

One easily sees that $|d_{ij}|_{s.m} = \left[|a_i|_s, |b_j|_m\right]$.

To build *i* in theorem, assume that either $\rho(s) \equiv 0(mod4)$ or $\rho(m) \equiv 0(mod4)$. Then $|d_{ij}|_{s.m} = \left[|a_i|_s, |b_j|_m\right]$ or $\left[\rho(s), \frac{1}{2}\rho(m)\right]$ or $\left[\frac{1}{2}\rho(s), \rho(m)\right]$, but from all these results it is seen that $|d_{ij}|_{s.m} = \rho(s.m) \equiv 0(mod4)$.

To prove *ii* in theorem, suppose that either $\rho(s) \equiv 2(mod4)$ or $\rho(m) \equiv 2(mod4)$. If $|d_{ij}|_{s.m} = \left[|a_i|_s, |b_j|_m\right]$ or $\left[\rho(s), \frac{1}{2}\rho(m)\right]$ or $\left[\frac{1}{2}\rho(s), \rho(m)\right]$ then we have $|d_{ij}|_{s.m} = \rho(s.m) \equiv 2(mod4)$ for all these cases. The one remaining case is $|d_{ij}|_{s.m} = \left[\frac{1}{2}\rho(s), \frac{1}{2}\rho(m)\right] = \frac{1}{2}\rho(s.m)$, which is odd.
For *iii* and *iv* in theorem, let show the roots of $h(x) \equiv x^2 - 3x - 1(mods)$ by $a_1, a_2, \dots, a_k$ and obtain that the only root of $h(x)(mod13)$ is 8. In addition, let the roots of $h(x)\ mod\ 13s$ are represented d_i.

$$d_i \equiv 8(mod13)$$
$$d_i \equiv a_i(mods)$$

Now, $|d_i|_{13s} = [|8|_{13}, |a_i|_s] = [4, |a_i|_s]$. If $\rho(s) \equiv 0(mod4)$, then $|a_i|_s = \rho(s) \equiv 0(mod4)$. Therefore $|d_i|_{13s} = [4, |a_i|_s] = \rho(s)$.

Finally, if $\rho(s) \equiv 2(mod4)$, then either $|a_i|_s = \rho(s) \equiv 2(mod4)$ or $|a_i|_s = \frac{1}{2}\rho(s) \equiv 1(mod2)$. And, for the both cases, we obtained $|d_i|_{13s} = 2\rho(s)$■.

According to the Theorem 3.5 and Theorem 3.6, we obtained the periodic relations between special power Horadam sequence and Horadam sequence for $u = 3,\ v = 1$ are the same as the periodic relations between power Fibonacci sequence and the Fibonacci sequence. Then, we examined the periods of special power Horadam sequences for $u = 1, v = 3$. And we obtained following results:

If $u = 1,\ v = 3$, then we obtained that the periods of the special power Horadam sequences can't been formulated similar to the period of power Fibonacci sequences. And even, we obtained that periods of the special power Horadam sequences for $u = 1,\ v = 3$ can't been characterized based on Horadam sequences in a certain formula. Let illustrate this situation as follows:

Example 3.7. For modulo $s = 43$, $s \equiv +4(13)$, Horadam sequence is

0,1,1,3,6,15,33,35,5,24,39,25,13,2,41,4,41,10,4,34,3,19,28,42,40,37,28,10,8,38,19,4,18,30,41
2,39,2,33,39,9,40,24,15,1,3 ... , then $\rho(43) = 42$.

There are two special power Horadam sequences:

- 1,12,15,8,10,34,21,37,14,39,38,26,11,3,36,2,24,30,16,20,25,42,31,28,35,33,9,27,6,29,
4,5,17,32,40,7,41,19,13,27,23,18,1,12 ... , then $|12|_{43} = 42$
- 1,32,35,2,21,27,4,42,11,8,41,22,16,39,1,32 ... , then $|32|_{43} = 14$.

In this situation, we obtained $|12|_{53} = 3|32|_{43} = \rho(43)$.

Example 3.8. For modulo $s = 17$, $s \equiv +4(13)$, Horadam sequence is
0,1,1,4,7,2,6,12, 13, 15, 3, 14, 6, 14, 15, 6, 0,1,1, then $\rho(17) = 16$.
There are two special power Horadam sequences:

- 1,5,8,6,13,14,2,10,16,12,9,11,4,3,15,7,1,5, ... , then $|5|_{17} = 16$
- 1,13,16,4,1,13, ..., then $|13|_{17} = 4$.

In this situation, we obtained $|5|_{17} = 4|13|_{17} = \rho(17)$.

It is easily seen that $s \equiv +4(13)$ for both Example 3.7. and Example 3.8. And, we know that there are two special power Horadam sequence modulo $s = p^e$ according to Theorem 3.4. For the periods of the two sequences, we obtained that while the period of one of these sequences is $\frac{1}{3}\rho(s)$ for modulo $s = 43$, that is $\frac{1}{4}\rho(s)$ for modulo $s = 17$. And, the period of the other special power Horadam sequence is equal to $\rho(s)$ for both $s = 43$ and $s = 17$.

Example 3.9. For modulo $= 61$, $s \equiv -4(13)$, Horadam sequence is
0,1,1,4,7,19,11,10,14,15,28,15,12,28,6,3,21,1,6,9,27,25,19,7,6,27,16,10,0,1 ... . So, $\rho(61) = 20$.
And, there are two special power Horadam sequences:

- 1,24,27,38,58,50,41,8,9,33,60,37,34,23,3,11,20,53,52,28,1,24 ... , then $|24|_{61} = 20$
- 1,37,40,29,27,53,12,49,24,49,60,24,21,32,34,8,49,12,37,12,1,37 ... , then $|37|_{61} = 20$

So, we obtained that the period of both power sequences is 20. In this situation, we obtained $|24|_{61} = |37|_{61} = \rho(61)$.

For $\rho(61) \equiv 0(mod\ 4)$, the period of both of special power Horadam sequences is equal to $\rho(61)$. But for $\rho(17) \equiv 0(mod\ 4)$, the period of one of these sequences is $\frac{1}{4}\rho(17)$, while the period of the others is $\rho(17)$. And, similarly, for $\rho(43) \equiv 2(mod\ 4)$, the period of one of these sequences is $\frac{1}{3}\rho(43)$, while the period of the other is $\rho(43)$. But for $\rho(23) \equiv 2(mod\ 4)$, the period of one of these sequences is $\frac{1}{2}\rho(23)$, while the period of the other is $\rho(23)$. From these examples, we obtained the periodic relations between special power Horadam sequence and Horadam sequence for $u = 1$, $v = 3$ are not alike the periodic relations between power Fibonacci sequence and the Fibonacci sequence.

Also, when we compare the periods of these special power Horadam sequences for both for $u = 1$, $v = 3$ and for $u = 3$, $v = 1$ with the periods of the Horadam sequences in the same modulo, we can easily see that the period of one of these special power Horadam sequences modulo s is definitely equal to the period of Horadam sequence modulo s. Moreover, the periods of the other power Horadam sequences are a divisor of $\rho(s)$.

IV. CONCLUSION

In here, first of all, we analyzed power Fibonacci sequence modulo s which is defined by Ide and Renault [13] and the numbers and periods of these sequences. Then, we described two special power Horadam sequences modulo s for $u = 1, v = 3$ and $u = 3, v = 1$, respectively. And, for both cases of

these sequences, we determined those modulo s for which these power Horadam sequences exist and the number of such sequences for a given s. Also, we obtained that these special power Horadam sequences exist for the same modulus in both cases. Then, we investigated that the periods of these special power sequences for both $u = 1, v = 3$ and $u = 3, v = 1$. And, we obtained for $u = 3, v = 1$ as follows:
i. For $\rho(s) \equiv 0(mod4)$, the period of each special power Horadam sequence modulo s is $\rho(s)$.
ii. For $\rho(s) \equiv 2(mod4)$, the period of one of special power Horadam sequences modulo s is $\frac{1}{2}\rho(s)$, and the periods of others are $\rho(s)$.
iii. For $\rho(s) \equiv 0(mod4)$, the period of each special power Horadam sequence modulo $13s$ is $\rho(s)$.
iv. For $\rho(s) \equiv 2(mod4)$, the period of each special power Horadam sequence modulo $13s$ is $2\rho(s)$.

In addition, we obtained for $u = 1, v = 3$, periods of the special power Horadam sequences can't been characterized based on power Horadam sequences in a certain way. But we obtained that the period of one of these special power Horadam sequences is definitely equal to the period of Horadam sequence, while the periods of the other power Horadam sequences are a divisor of $\rho(s)$.

Finally, we compared that the period formulas of power Horadam sequences which we obtained and the period formulas of power Fibonacci sequences. We found that, the periods formulas of the power Horadam sequences for $u = 3, v = 1$ are the same as the period formulas of the power Fibonacci sequences while those defined for $u = 1, v = 3$ are different from the periods of the power Fibonacci sequences.

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