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* Ricci Solitons and Symmetries of Type *D* Gravitational Fields in Spacetime Manifolds

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Abstract

In the present research paper we study * Ricci solitons with a physical interpretation of the notion of the vector field associated with * Ricci solitons. We investigate the geometrical symmetries of Petrov type *D* gravitational fields along the vector field also associated with * Ricci solitons.

Keywords: *Ricci solitons, Petrov type, *D* gravitational field, Weyl curvature tensor.

1. INTRODUCTION

In general theory of relativity, the curvature tensor describing the gravitational field mainly consists of two parts viz, the matter part and the free gravitational part. The interaction between these two parts is described through Bianchi identities. For a given distribution of matter, the construction of gravitational potential satisfying Einstein's field equations is the principal goal of all studies in gravitational physics and this has often been achieved by imposing symmetries on the geometry compatible with the dynamics of the chosen distribution of matter. The geometrical symmetries of the space time are expressible through the vanishing of the Lie derivative of certain tensors with respect to a vector.

In differential geometry and theoretical physics, the **Petrov classification** (also known as Petrov–Pirani–Penrose classification) describes the possible algebraic symmetries of the Weyl tensor at each event in a Lorentzian manifold.

It is most often applied in studying exact solutions of Einstein's field equations, but strictly speaking the classification is a theorem in pure mathematics applying to any Lorentzian manifold, independent of any physical interpretation. The classification was found in 1954 by A. Z. Petrov and independently by Felix Pirani in 1957.

The following Figure 1 show the Penrose diagram of the possible degeneration of the Petrov type of the Weyl tenosr.

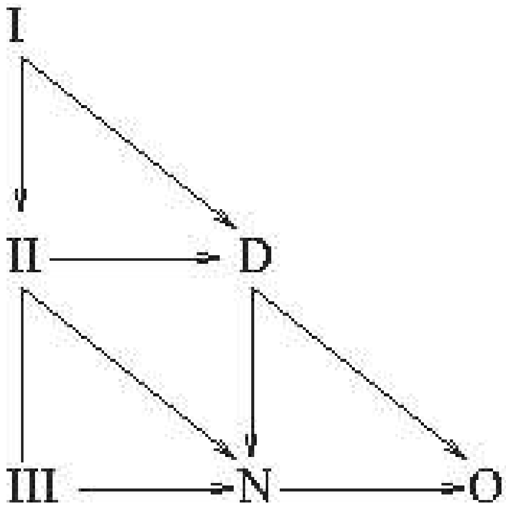

Figure: 1. Penrose Diagram

The nature of the gravitational radiation from a bounded source is an important physical problem. Even reasonably far from the source, however, twisting type *D* solutions of the vacuum field equations are required for an exact description of that gravitation. It is well known fact that Petrov type *D* solutions of the Einstein vacuum equations are among the most interesting of all empty spacetime metrics [15]. The physical importance, they represent spacetime with gravitational radiation while mathematically they form a class of solution of Einstein equations which should be possible to be determined explicitly.

Recently geometric flows have become important tools in Riemannian geometry and general relativity. In [8] B. List has studied a geometric flow whose fixed points corresponds to static Ricci flat spacetime which is nothing but Ricci flow pullback by a certain diffeomorphism. The association of each Ricci flat spacetime gives notion of local Ricci soliton in one higher dimension. The importance of geometric flow in Riemannian geometry is due to Hamilton who has given the flow equation and B. List generalized Hamilton's equation and extend it to sapcetime for static metric [8]. He has given system of flow equations whose fixed points solve the Einstein free-scalar field system [8]. This observation is useful for the correspondence of solutions of system i.e., Ricci soliton and symmetry property of sapcetime, that how Riemannian space (or spacetime) with Ricci Soliton deals different kind of symmetry properties.

Ricci solitons generate self-similar solutions to Ricci flow. Ricci solitons is the generalization of Einstein metrics

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{1.1}$$

In 1959, S. Tachibana [17] introduced the notion of * Ricci tensor

$$R^*_{ij} = trace(\phi \circ \qquad)) \tag{1.2}$$

Definition. A pseudo Riemannian merric g on M is called * Ricci solitons if

$$L_V g + 2R^*_{ij} + 2\lambda g_{ij} = 0 \tag{1.3}$$

Now, we have following lemma[18]

Lemma. [18] In a pseudo Riemannian manifold (M^n, g), $(n > 2)$ a (0, 2) symmetric tensor is a generalized *R tensor [18]

(

$$R^*_{ii} = R_{ij} + \varphi g_{ij} \tag{1.4}$$

where R and *R are Ricci and *Ricci tensor of type (0,2) respectively and φ is an arbitrary scalar function.

The role of symmetries in general theory of relativity has been introduced by Katzin, Levine and Davis in a series of papers [10,12]. These symmetries, also known as collineations, were further studied by Ahsan [1-4], Ahsan and Ali [5,6]. The perfect fluid spacetime including electromagnetic field which admit symmetry mapping belonging to the family of contracted Ricci colloneation, have been studied by Norris et al. [13]. The role of geometrical symmetries in the study of fluid spacetime, with an empgasis on conformal collineation has been studied by Duggal [11] and others. The different types of symmetries of Petrov type *D* gravitational fields has been subject of interest since last few decades (cf,[4]).

Motivated by the role of symmetries and Einstein solitons, a study of vector field involved in the definition of Einstein solitons and symmetries of spacetime is made. The main results on the relation between the symmetries of Petrov type *D* gravitational fields and * Ricci solitons has been studied.

2. PRELIMINARIES

(a) *Ricci Solitons. A family $g_\lambda = (\lambda - \varphi, x)$ of the Riemannian metrics on a *n*-dimensional $(n \geq 3)$ smooth manifold *M* with parameter λ ranging in the time interval $I \subset$ including zero Equation (1.3) of Ricci flow for $g_0 = g(0)$ and the * Ricci tensor R^*_{ij} and curvature tensor R of the g_0 satisfied. Corresponding to self similar solution of equation (1.3) is the notion of the local * Ricci soliton, defined as a metric g_0 satisfying equation

$$-2R^*_{ij} = L_\xi g_0 + 2(\lambda - \varphi) g_0 \tag{2.1}$$

For vector field ξ on V_n and a constant λ. The * Ricci solitons is said to be steady (static) if $\lambda = 0$, shrinking $\lambda < 0$ and expanding if $\lambda > 0$. The metric g_0 is called a gradient * Ricci solitons if $\xi = \nabla \phi$ i.e., gradient of some function ϕ . Schwarzschild metric, Akbar and Woolger [7] have derived the expressions around this notion; while Ali and Ahsan [6] have studied this concept for obtaining the Gaussian curvature of Schwarzschild solitons ad we have.

For *n*-dimensional Riemannian manifold equation (2.1) a can be written in general as

$$R^*{}_{ij} - \frac{1}{2} L_\xi g_{ij} = (\lambda - \varphi) g_{ij} \tag{2.2}$$

So far more than twentyseven different types of collineations have been studied and the literature on such collineations is very large abd still with results of elegance (see [4]). However, here we shall mention only those symmetry assumptions that are required for subsequent investigation and we have

(b) Motion. A spacetime is said to admit motion if there exist a vector field ξ^t such that

$$L_\xi g_{ij} = \xi_{i;j} + \xi_{j;i} = 0 \tag{2.3}$$

equation (2.3) is known as Killing equation and vector ξ^t is called a Killing vector field [16].

(c) Conformal Motion. If

$$L_\xi g_{ij} = \sigma g_{ij} \tag{2.4}$$

Where σ is a scalar, then the spacetime is said to admit conformal motion and vector field ξ is called a conformally Killing vector field.

(d) Special Conformal Motion. A spacetime admits if special conformal motion

$$L_\xi g_{ij} = \sigma g_{ij}, \quad \sigma_{ij} = 0 \tag{2.5}$$

(e) Curvature Collineation. A spacetime admits curvature collineation if there is a vector field ξ^i such that

$$L_\xi R^i{}_{jkl} = 0 \tag{2.6}$$

Where $R^i{}_{jkl}$ is the Riemannian curvature tensor.

(f) Ricci Collineation. A spacetime is said to admit Ricci collineation if there is a vector filed ξ^i such that

$$L_\xi R_{ij} = 0 \tag{2.7}$$

Where R_{ij} is the Ricci tensor.

(g) Affine Collineation. If

$$L_\xi \Gamma^i{}_{jk} = \xi^i{}_{jk} + R^i{}_{jmk}\xi^m = 0 \tag{2.8}$$

Then spacetime is said to admit an affine collineation.

(h) Weyl Projective Collineation. A symmetry property of a spacetime is called Weyll projective collineation if and only if

$$L_\xi W^i{}_{jkl} = 0 \quad (n > 2) \tag{2.9}$$

Where $W^i{}_{jkl}$ is Weyl tensor.

3. MAIN RESULTS

In this section, we shall discuss the role of * Ricci solitons in the study of Einstein spaces and Petrov type *D* gravitational fields. In 4-dimensioanl spacetime, the Weyl tensor is related to the Riemannian and Ricci tensors through the equation

$$C_{ijkl} = R_{ijkl} - \frac{1}{2}\left(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{jk}R_{il} - g_{il}R_{jk}\right) + \frac{1}{6}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)R \tag{3.1}$$

In NP-formalism (cf. [15]), the components of Weyl tensor are expressed by five complex scalars $\Psi_0, \Psi_1, \Psi_2, \Psi_3$ and Ψ_4. Through these components the gravitational field has been classified into six categories type *I, II, D, III and O* (cf [15]). The Weyl scalar along with Goldberg-Sachs theorem declares type *N* pure radiation field follow the conditions

$$\Psi_4 = \Psi \neq 0, \quad \Psi_i = 0, \quad i = 0, 1, 2, 3 \tag{3.2}$$

$$\kappa = \sigma = \varepsilon = 0 \tag{3.3}$$

where $\kappa, \sigma, \varepsilon$ are the spin-coefficients [15]. Ali and Ahsan [6] have obtained symmetries for Weyl conformal tensor. Using equations (3.1) to (3.3) and the definitions (b)-(c), we can write the following:

Lemma 3.1. In type D pure gravitational fields every conformal motion, special conformal motion and homothetic motion, all degenerate to motion.

From equations (1.3), (1.4) (2.1) and (2.2), we have

$$2R_{ij} = L_\xi g_{ij} + 2(\lambda - \varphi)g_{ij} \quad \text{3.4)}$$

$$= \xi_{i;j} + \xi_{j;i} + (\lambda - \varphi)g_{ij} \tag{3.5}$$

Contracting this equation with g^{ij}, we get

$$R = \xi^i{}_{;j} + (\lambda - \varphi)n \tag{3.6}$$

Which can be expressed as

$$div\,\xi = \nabla_i \xi^i = [R - (\lambda - \varphi)n] \tag{3.7}$$

where $R = g^{ij}R_{ij}$ is the scalar curvature. From equations (3.3) and (3.7), we get

$$\left(\frac{1}{n}Rg_{ij} - R_{ij}\right) = -\frac{1}{2}L_\xi g_{ij} + \frac{1}{n}(div\,\xi)g_{ij} - \varphi g_{ij} \tag{3.8}$$

Now for g_{ij} to be Einstein metric i.e., $R^*{}_{ij} = \sigma g_{ij}$ where σ can be chosen as $\frac{R}{n}$, equation (3.3) together with the definition of conformal motion gives the following results:

Lemma 3.2. **[14]** The vector field ξ associated with * Ricci solitons (M,g) is conformally Killing if and only if (M,g) is an Einstein manifold of dimension $(n \geq 3)$.

Now, using Lemmas 3.1 and 3.2, we can state the following theorem:

Theorem 3.3. Type *D* pure gravitational field admit motion along a vector field ξ associated to * Ricci solitons (M,g) if and only if *M* is an Einstein space.

For Killing vector field ξ **,** equation (1.4) reduces to

$$R^{*}_{ij} = (\lambda - \varphi) g_{ij} \tag{3.9}$$

Taking Lie derivative with respect to vector field ξ

$$L_{ij} R_{ij} = \left(\lambda - \frac{R}{2}\right) L_{ij} g_{ij} = 0 \tag{3.10}$$

Thus, we have the following theorem:

Theorem 3.4. A vector field ξ associated to Einstein solitons (M,g) is Ricci collineation vector field in Type *N* pure radiation field if g is Einstein metric.

Taking the Lie derivative of Christoffel symbol $\Gamma^{i}_{\ jk} = \frac{1}{2} g^{il}\left(\frac{\partial g_{jl}}{\partial \xi^{k}} - \frac{\partial g_{jk}}{\partial \xi^{l}} + \frac{\partial g_{kl}}{\partial \xi^{j}}\right)$

along the vector field ξ, after the calculation we get

$$L_{\xi}\Gamma^{i}_{\ jk} = \xi^{i}_{\ ;jk} + R^{i}_{\ jmk}\xi^{m} \tag{3.11}$$

Now if ξ is Killing vector field, then

$$\xi^{i}_{\ ;jk} + R^{i}_{\ jmk}\xi^{m} = 0 \tag{3.12}$$

Where

$$R^{h}_{\ ijk} = -\frac{\partial \Gamma^{h}_{\ ij}}{\partial x^{k}} + \frac{\partial \Gamma^{h}_{\ ik}}{\partial x^{j}} - \Gamma^{a}_{\ ij}\Gamma^{h}_{\ ak} + \Gamma^{b}_{\ ik}\Gamma^{h}_{\ bj} \tag{3.13}$$

is the Riemannian curvature tensor.

Using (3.11) and (3.12) along with the definition of affine collineation, we can have the following result:

Theorem 3.5. Type *N* pure radiation field admit affine collineation along a Killing vector field ξ associated to Einstein solitons (M,g) if and only if *M* is an Einstein space.

By the definition of Lie derivative

$$L_\xi R^i{}_{jkl} = \xi^h R^i{}_{jkl;h} - R^h{}_{jkl}\xi^i{}_{;h} + R^i{}_{hkl}\xi^h{}_{;j} + R^i{}_{jhl}\xi^h{}_{;k} + R^i{}_{jkh}\xi^h{}_{;l} \quad (3.14)$$

Using the definition of Christoffel symbol and Killing vector ξ, we have

$$L_\xi R^i{}_{jkl} = 0$$

which establishes the curvature collineation, so we have.

Theorem 3.6. A killing vector field ξ associated to Einstein solitons (M,g) is Curvature collineation vector field in type N pure radiation field if g is Einstein metric.

The Weyl projective tensor is given by

$$W^i{}_{jkl} = R^i{}_{jkl} - \frac{1}{3}(R_{jk}\delta^i{}_j - R_{jl}\delta^i{}_k) \text{ for } R_{ij} = 0, \ W^i{}_{jkl} = R^i{}_{jkl} \ or \ W_{ijkl} = R_{ijkl} \quad (3.15)$$

From equation (1.8) and (3.15), we can easily write.

Lemma 3.7. [12] In a Riemannian manifold curvature collineation implies the Weyl projective collineation but converse is true for empty spacetimes.

So, the theorem 3.6 and Lemma 3.7 constitute the following:

Corollary 3.8. A Killing vector field ξ associated to Einstein solitons (M,g) is Weyl Projective collineation vector field in type *N* pure radiation if g is Einstein metric.

CONCLUSIONS

For Einstein space different kind of symmetry properties for *N* pure radiation field*s* are established with the help of vector field associated with Einstein solitons. There are other symmetries for type *N* which can be obtained through the existence of Killing vectors corresponding to Einstein solitons.

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