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VERSIONS OF FUGLEDE-PUTNAM THEOREM ON p-w-HYPONORMAL OPERATORS

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Abstract. The aim of the article is the presentation of certain extensions of the famous Fuglede-Putnam Theorem on the class of p-w-hyponormal operators, which generalize some results proved by authors in [10].

1. Introduction and Preliminaries

Throughout this work, $B(H)$ denotes the Banach algebra of bounded linear operators on a complex separable Hilbert space H. By $\ker(T)$ and $ran(T)$ respectively, we mean the null space and the range of an operator $T \in B(H)$. Given $T, S \in B(H)$, the generalized derivation $\delta_{T,S}$ induced by T and S is defined for all $X \in B(H)$ by $\delta_{T,S}(X) = TX - XS$. Recall that T is said to be normal if T commutes with its adjoint $T^\star$. The well-known Fuglede-Putnam Theorem states that $\ker(\delta_{T,S}) \subset \ker(\delta_{T^\star,S^\star})$ whenever T and S are normal operators, see [5, 6, 7] and [15] where several generalizations of this result are given for operators T and S belonging to some classes of non normal operators. For $0 < p \leq 1$, an operator $T \in B(H)$ is said to be p-hyponormal if $|T|^{2p} - |T^\star|^{2p} \geq 0$, where $|T| = (T^\star T)^{\frac{1}{2}}$ is the module of T. A 1-hyponormal operator is hyponormal and $\frac{1}{2}$-hyponormal is semi-hyponormal. Reader can find many interesting spectral properties of this class in [1, 2, 12, 15]. In [1], it is defined the Aluthge transform of an operator $T = U|T|$ by $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$, and in [2], it is shown that if T is p-hyponormal, then $\widetilde{T}$ is $(p + \frac{1}{2})$-hyponormal for $0 < p \leq \frac{1}{2}$ and hyponormal for $\frac{1}{2} \leq p \leq 1$. Also, T is said to be log-hyponormal if T is invertible and $\log(T^\star T) \geq \log(TT^\star)$. The operator $T \in B(H)$ is said to be dominant if $ran(T - \lambda) \subset ran(T - \lambda)^\star$ for each λ in the spectrum $\sigma(T)$ of T. Also, if there exists $M > 0$ such that $(T - \lambda)(T - \lambda)^\star \leq M(T - \lambda)^\star(T - \lambda)$ for each $\lambda \in \sigma(T)$, then T is said to be M-hyponormal. Clearly,

$$\text{Hyponormal} \subset M\text{-hyponormal} \subset \text{dominant}$$

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In [15], it is presented an example of an M-hyponormal operator that is not hyponormal. An operator T is said to be w-hyponormal if $\left|\widetilde{T}\right| \geq |T| \geq \left|\widetilde{T^\star}\right|$ [2, 4, 8]. Useful results of the class of w-hyponormal operators are presented in [2, 8, 9], and it was proved that it contains the class of p-hyponormal operators. The following inclusions hold

$$\text{Hyponormal} \subset p\text{-hyponormal} \subset w\text{-hyponormal}$$

The operator T is said to be p-w-hyponormal for certain $0 < p \leq 1$, if

$$\left|\widetilde{T}\right|^p \geq |T|^p \geq \left|\widetilde{T^\star}\right|^p$$

[10, 16]. A 1-w-hyponormal is w-hyponormal, and w-hyponormal operators are evidently p-w-hyponormal. In this article, we'll extend the Fuglede-Putnam theorem for p-w-hyponormal with p-hyponormal operators or with log-hyponormal operators. Other spectral related results are also added.

2. Known Results

The following known results will be needed for the rest of the paper.

Lemma 2.1. [13] *Let T be in $B(H)$ and S be in $B(K)$. The following assertions are equivalent*

1. *The pair (T, S) satisfies the Fuglede-Putnam theorem.*
2. *If $TX = XS$ for some X in $B(K, H)$, then $\overline{ran(X)}$ reduces T, $(\ker(X))^\perp$ reduces S, and restrictions $T\left|\overline{ran(X)}\right.$, $S\left|(\ker(X))^\perp\right.$ are unitarily equivalent normal operators.*

Lemma 2.2. [2] *Let $T \in B(H)$ be a w-hyponormal operator and let $M \subset H$ be an invariant subspace under T. Then $T\left|M\right.$ is w-hyponormal.*

Lemma 2.3. [2] *Let $T \in B(H)$ be a w-hyponormal operator. Then $\widetilde{T}$ is semi-hyponormal.*

Lemma 2.4. [1] *If T is a p-hyponormal operator, then $\widetilde{T}$ is $(p + \frac{1}{2})$-hyponormal for $0 < p \leq \frac{1}{2}$ and hyponormal for $\frac{1}{2} \leq p \leq 1$.*

Lemma 2.5. [16] *Let $T \in B(H)$ be p-w-hyponormal, and let $M \subset H$ be a T-invariant subspace. Then $T\left|M\right.$ is p-w-hyponormal.*

Lemma 2.6. [16] *Let $T \in B(H)$ be a p-w-hyponormal operator. Then $\widetilde{T}$ is $\frac{p}{2}$-hyponormal.*

3. Main results

The familiar Fuglede-Putnam Theorem asserts that for normal operators T and S on H, equation $\delta_{T,S}(X) = 0$ implies $\delta_{T^\star,S^\star}(X) = 0$ for all X in $B(H)$. Extensions of this result for certain classes of non normal operators are presented in many papers, see [5, 6] and [7]. Authors in [11] showed that this result remains true for an M-hyponormal operator T and a dominant operator S.

The following result gives an extension of the Fuglede-Putnam property for M-hyponormal and p-hyponormal operators.

Proposition 1. For an M-hyponormal operator T and for a p-hyponormal operator $S^\star$ in $B(H)$, $\ker(\delta_{T,S}) \subset \ker(\delta_{T^\star,S^\star})$.

Proof. Due to [7] and since an M-hyponormal operator is dominant, the pair (T, S) satisfies the Fuglede-Putnam property. $\square$

Theorem 3.1. *Let T be M-hyponormal and let $S^\star$ be w-hyponormal operators in $B(H)$. Then, $\delta_{T,S}(X) = 0$ entails $\delta_{T^\star,S^\star}(X) = 0$ for all X in $B(H)$. Moreover, $\overline{ran(X)}$ reduces T, $(\ker(X))^\perp$ reduces S and restrictions $T\left|\overline{ran(X)}\right.$, $S\left|(\ker(X))^\perp\right.$ are unitarily equivalent normal operators.*

Proof. Subspaces $\overline{ran(X)}$ and $(\ker(X))^\perp$ are invariant for T and S respectively since $\delta_{T,S}(X) = 0$. Then, we can write

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}, \; S = \begin{pmatrix} S_1 & 0 \\ S_2 & S_3 \end{pmatrix} \text{ and } X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} : H_2 \longrightarrow H_1$$

under the decompositions

$$\begin{aligned} H &= H_1 = \overline{ran(X)} \oplus ran(X)^\perp \\ H &= H_2 = (\ker X)^\perp \oplus \ker X \end{aligned}$$

From $\delta_{T,S}(X) = 0$ we get

$$T_1 X_1 = X_1 S_1 \tag{3.1}$$

where T_1 is M-hyponormal, and S_1 is w-hyponormal by Lemma 2.2. Let $S_1 = U\,|S_1|$ be the polar decomposition of T_1. Since $U\,|S_1| = |S_1^*|\,U$, equality (3.1) can be writen

$$T_1 X_1 = X_1\,|S_1^\star|\,U \tag{3.2}$$

Multiplying the two sides of (3.2) at right by $|S_1^\star|^{\frac{1}{2}}$, we obtain

$$T_1(X_1\,|S_1^\star|^{\frac{1}{2}}) = X_1\,|S_1^\star|\,U\,|S_1^\star|^{\frac{1}{2}} = (X_1\,|S_1^\star|^{\frac{1}{2}})\widetilde{S_1^\star}$$

The Aluthge transform $\widetilde{S_1^\star}$ of $S_1^\star$ is semi-hyponormal by Lemma 2.3. Hence, the pair $(T_1, \widetilde{S_1^\star})$ satisfies the Fuglede-Putnam property by Proposition 1. Thus, restrictions $T_1\left|_{\overline{ran(X_1|S_1^\star|^{\frac{1}{2}})}}\right.$ and $\widetilde{S_1^\star}\left|_{(\ker((X_1|S_1^\star|^{\frac{1}{2}})^\perp}\right.$ are equivalent normal operators by Lemma 2.1. Since X_1 is quasiaffinity, i.e., one-to-one with dense range, and $|S_1^\star|^{\frac{1}{2}}$ is injective,

$$\overline{ran(X_1\,|S_1^\star|^{\frac{1}{2}})} = \overline{ranX_1} = \overline{ranX}$$

and

$$\ker(X_1\,|S_1^\star|^{\frac{1}{2}}) = \ker X_1 = \ker X$$

Thus, $\widetilde{S_1^\star}$ is normal and then S_1 is normal by [15]. The operator $S^\star$ is M-hyponormal and its restriction $S_1^\star$ on $(\ker X)^\perp$ is normal. Consequently, $\ker X$ reduces $S^\star$. Hence $S_2 = 0$.

Similarly, T is M hyponormal, and its restriction T_1 on $\overline{ranX}$ is normal. Then, $\overline{ranX}$ reduces T. Thus $T_2 = 0$. Since the pair (T_1, S_1) satisfies the Fuglede-Putnam theorem, $T_1^\star X_1 = X_1 S_1^\star$. Finally $T^\star X = X S^\star$. $\square$

Theorem 3.2. *Let T be a p-w-hyponormal operator in $B(H)$. If $|T|$ is invertible, then for all $\lambda \notin \sigma(T)$*

i. $\left\| \left|\widetilde{\widetilde{T}}\right|^{\frac{1}{2}} \left|\widetilde{T}\right|^{\frac{1}{2}} |T|^{\frac{1}{2}} (T-\lambda)^{-1} |T|^{-\frac{1}{2}} \left|\widetilde{T}\right|^{-\frac{1}{2}} \left|\widetilde{\widetilde{T}}\right|^{-\frac{1}{2}} \right\| \leq \frac{1}{dist(\lambda,\sigma(T))}$

ii. $\left\|T^{-1}\right\| \leq \frac{1}{\min(|\lambda|,\lambda\in\sigma(T))}$

Proof. i. $\widetilde{T}$ is $\frac{p}{2}$-hyponormal by Lemma 2.6, and $0 < \frac{p}{2} \leq \frac{1}{2}$. Since $\sigma(T) = \sigma(\widetilde{T})$ by [3],

$$\left\| \left|\widetilde{\widetilde{T}}\right|^{\frac{1}{2}} \left|\widetilde{T}\right|^{\frac{1}{2}} (\widetilde{T}-\lambda)^{-1} \left|\widetilde{T}\right|^{-\frac{1}{2}} \left|\widetilde{\widetilde{T}}\right|^{-\frac{1}{2}} \right\| \leq \frac{1}{dist(\lambda,\sigma(T))}$$

for $\lambda \notin \sigma(T)$ by [1]. The proof derives then from the fact that

$$(\widetilde{T}-\lambda)^{-1} = |T|^{\frac{1}{2}} (T-\lambda)^{-1} |T|^{-\frac{1}{2}}$$

ii. Since $\left\|\widetilde{T}\right\| \leq \|T\|$ for an arbitary operator T in $B(H)$,

$$\left\|T^{-1}\right\| \leq \left\|\widetilde{T}^{-1}\right\| = \frac{1}{\min(|\lambda|\,,\lambda\in\sigma(\widetilde{T}))} = \frac{1}{\min(|\lambda|\,,\lambda\in\sigma(T))}$$

□

As a consequence of the previous result, and since the Aluthge transform of a log-hyponormal operator is semi-hyponormal [14], we can then state the following generalization of the Fuglede-Putnam's Theorem for p-w-hyponormal with log-hyponormal operators as follows

Theorem 3.3. *The Fuglede-Putnam Theorem holds for a p-w-hyponormal operator $T \in B(H)$ with $\ker T \subset \ker T^\star$, and a p-hyponormal operator $S^\star \in B(H)$.*

Proof. Let

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and } X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

according to the decompositions

$$\begin{aligned} H &= H_1 = (\ker T)^{\perp} \oplus (\ker T) \\ H &= H_2 = (\ker S^\star)^{\perp} \oplus (\ker S^\star) \end{aligned}$$

From equation $TX = XS$, we get

$$T_1X_1 = X_1S_1 \tag{3.3}$$

and $T_1X_2 = X_3S_1 = 0$. Since T_1 and S_1 are one-to-one, $X_2 = X_3 = 0$. T_1 is a one-to-one p-w-hyponormal operator by Lemma 2.3, and $S_1^\star$ is p-hyponormal. Let $T_1 = U|T_1|$ be the polar decomposition of T_1. Equation (3.3) can be written

$$U|T_1|X_1 = X_1S_1 \tag{3.4}$$

Multiplying the two sides of (3.4) on the left by $|T_1|^{\frac{1}{2}}$ we get

$$|T_1|^{\frac{1}{2}} U |T_1|^{\frac{1}{2}} |T_1|^{\frac{1}{2}} X_1 = |T_1|^{\frac{1}{2}} X_1S_1$$

So $\widetilde{T_1}(|T_1|^{\frac{1}{2}} X_1) = (|T_1|^{\frac{1}{2}} X_1)S_1$. The Aluthge transform $\widetilde{T_1}$ of T_1 is $\frac{p}{2}$-hyponormal by Lemma 2.6, and $S_1^\star$ is p-hyponormal. By [5], the pair $(\widetilde{T_1}, S_1)$ satisfies the Fuglede-Putnam Theorem. Thus,

$$\widetilde{T_1}^*(|T_1|^{\frac{1}{2}} X_1) = (|T_1|^{\frac{1}{2}} X_1)S_1^\star$$

Consequently, restrictions $\widetilde{T_1}\Big|_{\overline{ran(|T_1|^{\frac{1}{2}} X_1)}}$ and $S_1\Big|_{(\ker(|T_1|^{\frac{1}{2}} X_1)^\perp}$ are unitarily equivalent normal operators by Lemma 2.1. Since the operator $|T_1|^{\frac{1}{2}}$ and X_1 are one-to-one, the operator $|T_1|^{\frac{1}{2}} X_1$ so is. Thus

$$(\ker(|T_1|^{\frac{1}{2}} X_1))^\perp = \{0\}^\perp = (\ker X_1)^\perp = (\ker X)^\perp$$

And

$$\overline{ran(\left|\widetilde{T_1}\right|^{\frac{1}{2}} X_1)} = (\ker |T_1|^{\frac{1}{2}} X_1)^\perp = \{0\}^\perp = \overline{ran(X_1)} = \overline{ran(X)}$$

Thus, $\widetilde{T_1}$ is a normal operator. The operator T_1 so is by [15]. Therefore, $\overline{ran(X)}$ reduces T_1 by Lemma 2.1, and $(\ker X_1)^\perp$ reduces $S_1^\star$ by [17]. Since T_1 is normal, and $S_1^\star$ is p-hyponormal, the Fuglede-Putnam property holds for the pair (T_1, S_1). Thus, $T_1^\star X_1 = X_1 S_1^\star$ and then, $T^\star X = XS^\star$. □

Corollary 3.4. The pair (T, S) satisfies the Fuglede-Putnam Theorem if T is a p-hyponormal operator and $S^\star$ is a p-w-hyponormal with $\ker S \subset \ker S^\star$.

Proof. $TX = XS$ for some X in $B(H)$. Put $A = S^\star$, $B = T^\star$ and $C = X^\star$. Then, $B^\star C^\star = C^\star A^\star$. Hence, $AC = CB$, where A is an injective p-w-hyponormal or a p-w-hyponormal with $\ker A \subset \ker A^\star$, and $B^\star$ is p-hyponormal. By the previous result, $A^\star C = CB^*$. Thus, $SX^\star = X^\star T$. Consequently, $T^\star X = XS^\star$. □

Theorem 3.5. $\delta_{(T,S)} \subset \delta_{(T^\star,S^\star)}$ *for a p-w-hyponormal operator* T *with* $\ker T \subset \ker T^\star$, *and a* log-*hyponormal operator* $S^\star$.

We need the following property of log-hyponormal operators for the proof.

Lemma 3.6. [15] *Let* $T \in B(H)$ *be a* log-*hyponormal operator and let* $M \subset H$ *be a* T*-invariant closed subspace. Then, the restriction* $T|M$ *is* log-*hyponormal.*

Proof. (of Theorem 3.5) Let's consider the decompositions

$$\begin{aligned} H &= H_1 = (\ker T)^\perp \oplus (\ker T) \\ H &= H_2 = (\ker S^\star)^\perp \oplus (\ker S^\star) \end{aligned}$$

Then

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

From equation $\delta_{T,S}(X) = 0$, we get

$$\delta_{T_1,S_1}(X_1) = 0 \tag{3.5}$$

and $T_1X_2 = X_3S_1 = 0$. Since T_1 and S_1 are one-to-one, $X_2 = X_3 = 0$. T_1 is a one-to-one p-w-hyponormal operator by Lemma 2.6, and $S_1^\star$ is an injective log-hyponormal by Lemma 3.6. Let $S_1 = U\,|S_1|$ be the polar decomposition of S_1. Since $S_1 = |S_1^\star|\,U$, equation (3.5) can be written

$$T_1X_1 = X_1\,|S_1^\star|\,U \tag{3.6}$$

Multiplying the two sides of (3.6) at right by $|S_1^\star|^{\frac{1}{2}}$ we get

$$T_1(X_1\,|S_1^\star|^{\frac{1}{2}}) = (X_1\,|S_1^\star|^{\frac{1}{2}})\,|S_1^\star|^{\frac{1}{2}}\,U\,|S_1^\star|^{\frac{1}{2}} = (X_1\,|S_1^\star|^{\frac{1}{2}})\widetilde{S_1^\star}$$

T_1 is p-w-hyponormal, and the Aluthge transform $\widetilde{S_1^\star}$ of $S_1^\star$ is $\frac{1}{2}$-hyponormal by [14]. By Theorem 3.3, the Fuglede-Putnam's Theorem holds for the pair $(T_1, \widetilde{S_1^\star})$. Hence, $T_1^\star(X_1\,|S_1^\star|^{\frac{1}{2}}) = (X_1\,|S_1^\star|^{\frac{1}{2}})\widetilde{S_1^\star}^\star$. Furthermore, and by Lemma 2.1, $T_1\Big|_{\overline{ran(X_1|S_1^\star|^{\frac{1}{2}})}}$ and $\widetilde{S_1^\star}\Big|_{(\ker(X_1|S_1^\star|^{\frac{1}{2}})^\perp}$ are unitarily equivalent normal operators. Since the operator $|S_1^\star|^{\frac{1}{2}}$ and X_1 are one-to-one, the operator $X_1\,|S_1^\star|^{\frac{1}{2}}$ so is. The rest of proof is similar to Theorem 3.1. □

Corollary 3.7. Let $T \in B(H)$ be a pure log-hyponormal operator, and let $S^* \in B(H)$ be a p-w-hyponormal with $\ker S \subset \ker S^*$. Then, equation $TX = XS$ implies $X = 0$.

Proof. By Theorem 3.3, equations $TX = XS$ and $T^\star X = XS^\star$ hold. Hence, restriction $T\Big|(\overline{ran(X)}$ is a normal operator by Lemma 2.1, which contradicts the hypotheses that T is pure. Thus, $X = 0$. □

Corollary 3.8. An invertible p-w-hyponormal operator $T \in B(H)$ is normal if and only if it is log-hyponormal.

Proof. Put $T = X = S$ in the previous Theorem. □

In [9, Lemma 7], it is shown that if T is w-hyponormal with $\ker T \subset \ker T^\star$ and S is normal, and if $X \in B(H)$ has dense range such that $TX = XS$, then T is normal. We give now, an extension of this result for a p-w-hyponormal operator as follows

Lemma 3.9. *Let $T \in B(H)$ be a p-w-hyponormal operator with $\ker T \subset \ker T^\star$, and let S be normal. If $TX = XS$ for some $X \in B(H)$ with dense range, then T is normal.*

Proof. The pair (T, S) verifies the Fuglede-Putnam property by Theorem 3.2. Then, by Lemma 2.1, the restriction $T\Big|(\overline{ran(X)}$ is a normal operator. This achieves the proof since $\overline{ran(X)} = H$. □

Corollary 3.10. Let $T, S^\star \in B(H)$ be p-w-hyponormal operators with $\ker T \subset \ker T^\star$, and $\ker S \subset \ker S^\star$. If $TX = XS$ and $SY = YT$ for certain $X, Y \in B(H)$ with dense ranges, then T and S are normal.

4. Conclusion

In this paper, are shown some versions of Fuglede-Putnam Theorem on classes of p-w-hyponormal operators with log-hyponormal and with p-hyponormal opertaors. Some spectral results in [16] on w-hyponormal operators are also extended to p-w-hyponormal operators.

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