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Two New Versions of the Pasting Lemma via Soft Mixed Structure

Nihal Taş

Balıkesir University, Department of Mathematics, 10145 Balıkesir, Turkey

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Abstract

In this paper, we present two new generalizations of the pasting lemma using soft mixed structure. To do this, we introduce the notions of a (τ_1,τ_2)-g-closed soft set and a (τ_1,τ_2)-gpr-closed soft set. We establish the notions of mixed g-soft continuity and mixed gpr-soft continuity between two soft topological spaces (X,τ_1,Δ_1), (X,τ_2,Δ_1) and a soft topological space (X,τ,Δ_2). Finally we prove two new versions of the pasting lemma using the mixed g-soft continuous mapping and the mixed gpr-soft continuous mapping.

1. Introduction and motivation

"*Soft set theory"* was introduced as a general mathematical tool for dealing with encountered difficulties and problems in medical science, social science, engineering, economics etc. [1]. Many researchers have been studying some topological concepts with basic properties and some generalizations of a soft topological space via different approaches (for example, see [2]-[15]). Also some applications of the soft set theory were obtained to other sciences such as medical science, food science, insurance, investment etc. (see [16]-[26] for some examples). Recently, different decision making applications have been studied (for example, see [27]-[30]).

"*Mixed structure* has"been studied on various topological spaces such as a soft topological space, a generalized topological space etc. Using the mixed structure, some topological notions have been generalized with a new approach. For example, some mixed sets and mixed continuities were defined on a generalized topological space (resp. on a soft topological space) (see [31]-[37]).

"*Pasting lemma"* is one of the most important notions on a topological space for continuous functions. Especially, it has a significant place in algebraic topology. Recent years, some new forms of the pasting lemma have been introduced by many mathematicians (for example, see [15], [38]-[42] and the references therein).

Motivated by the above studies, we present two new version of the pasting lemma using mixed structure on a soft topological space. For this purpose, we introduce the notions of (τ_1,τ_2)-g-closed soft set, a (τ_1,τ_2)-gpr-closed soft set, mixed soft pre closure and mixed soft pre interior. We prove some topological properties of these new notions. Also we give some counter examples for necessary relationships. We define the notions of mixed g-soft continuity and mixed gpr-soft continuity between two soft topological spaces (X,τ_1,Δ_1), (X,τ_2,Δ_1) and a soft topological space (X,τ,Δ_2). Finally, we establish two new versions of the pasting lemma for mixed g-soft continuous functions and mixed gpr-soft continuous functions on a soft topological space.

Email address and ORCID number: nihaltas@balikesir.edu.tr, 0000-0002-4535-4019

2. Preliminaries

In this section, we recall some basic concepts related to soft set theory. Throughout this paper, we assume that X is an initial universal set, Δ is a nonempty set of parameters and $\Delta_1, \Delta_2 \subseteq \Delta$.

Definition 2.1. *[1] Let $\phi : \Delta_1 \to P(X)$ be a mapping. Then a pair (ϕ, Δ_1) is called a soft set over X. $SS(X)_\Delta$ denotes the family of all soft sets on X.*

Definition 2.2. *[7] Let (ϕ, Δ_1) be a soft set over X.*
(1) (ϕ, Δ_1) is called a null soft set if $\phi(e) = \emptyset$ for all $e \in \Delta_1$. It is denoted by $\widetilde{\emptyset}$.
(2) (ϕ, Δ_1) is called an absolute soft set if $\phi(e) = X$ for all $e \in \Delta_1$. It is denoted by $\widetilde{X}$.

Definition 2.3. *[7] Let $(\phi, \Delta_1) \in SS(X)_{\Delta_1}$ and $(\varphi, \Delta_2) \in SS(X)_{\Delta_2}$.*
(1) (ϕ, Δ_1) is called a soft subset of (φ, Δ_2) if $\Delta_1 \subseteq \Delta_2$ and $\phi(e) \subseteq \varphi(e)$ for all $e \in \Delta_1$. It is denoted by

$$(\phi, \Delta_1) \widetilde{\subseteq} (\varphi, \Delta_2).$$

(2) (ϕ, Δ_1) is called soft equal to (φ, Δ_2) if $(\phi, \Delta_1) \widetilde{\subseteq} (\varphi, \Delta_2)$ and $(\varphi, \Delta_2) \widetilde{\subseteq} (\phi, \Delta_1)$. It is denoted by

$$(\phi, \Delta_1) = (\varphi, \Delta_2).$$

Definition 2.4. *[10] Let (ϕ, Δ_1), $(\varphi, \Delta_1) \in SS(X)_{\Delta_1}$.*
(1) The complement of (ϕ, Δ_1) is defined as

$$(\phi, \Delta_1)^c = (\phi^c, \Delta_1),$$

where $\phi^c(e) = (\phi(e))^c = X - \phi(e)$ for all $e \in \Delta_1$.
(2) The difference of (ϕ, Δ_1) and (φ, Δ_1) is defined as

$$(\phi, \Delta_1) - (\varphi, \Delta_1) = (\phi - \varphi, \Delta_1),$$

where $(\phi - \varphi)(e) = \phi(e) - \varphi(e)$ for all $e \in \Delta_1$.

Definition 2.5. *[14] Let J be an arbitrary index set and $\{(\phi_i, \Delta)\}_{i \in J}$ be a subfamily of $SS(X)_\Delta$.*
(1) The union of these soft sets is the soft set (φ, Δ), where

$$\varphi(e) = \bigcup_{i \in J} \phi_i(e),$$

for each $e \in \Delta$. It is denoted by $\widetilde{\bigcup_{i \in J}} (\phi_i, \Delta) = (\varphi, \Delta)$.
(2) The intersection of these soft sets is the soft set (θ, Δ), where

$$\theta(e) = \bigcap_{i \in J} \phi_i(e),$$

for each $e \in \Delta$. It is denoted by $\widetilde{\bigcap_{i \in J}} (\phi_i, \Delta) = (\theta, \Delta)$.

Definition 2.6. *[10] Let $(\phi, \Delta_1) \in SS(X)_{\Delta_1}$ and $x \in X$. The point x is called in the soft set (ϕ, Δ_1) if $x \in \phi(e)$ for all $e \in \Delta_1$. It is denoted by $x \in (\phi, \Delta_1)$.*

Definition 2.7. *[10] Let $(\phi, \Delta) \in SS(X)_\Delta$ and Y a nonempty subset of X. The sub soft set of (ϕ, Δ) over Y, denoted by $({}^Y\phi, \Delta)$, is defined by*

$${}^Y\phi(e) = Y \cap \phi(e),$$

for all $e \in \Delta$. In other words, $({}^Y\phi, \Delta) = \widetilde{Y} \cap (\phi, \Delta)$.

Definition 2.8. *[10] Let τ be the collection of soft sets over X. Then τ is called a soft topology on X if the following conditions hold:*
(t_1) $\widetilde{\emptyset}, \widetilde{X} \in \tau$.
(t_2) The intersection of any two soft sets in τ belongs to τ.
(t_3) The union of any number of soft sets in τ belongs to τ.
The triple (X, τ, Δ) is called a soft topological space over X.

Definition 2.9. *[10] The members of τ are said to be τ-soft open sets or soft open sets in X and also a soft set over X is called soft closed in X if its complement belongs to τ.*
$OS(X, \tau, \Delta)$ or $OS(X)$ denotes the set of all soft open sets over X and $CS(X, \tau, \Delta)$ or $CS(X)$ denotes the set of all soft closed sets.

Definition 2.10. *[10] Let (X,τ,Δ) be a soft topological space over X and Y a nonempty subset of X. Then*

$$\tau_Y = \left\{ (^Y\phi,\Delta) : (\phi,\Delta) \in \tau \right\}$$

is called the soft relative topology on Y and (Y,τ_Y,Δ) is called a soft subspace of (X,τ,Δ).

Theorem 2.11. *[10] Let (Y,τ_Y,Δ) be a soft subspace of a soft topological space (X,τ,Δ) and (ϕ,Δ) be a soft set over X. Then*
(1) (ϕ,Δ) is soft open in Y if and only if $(\phi,\Delta) = \widetilde{Y} \cap (\varphi,\Delta)$ for some $(\varphi,\Delta) \in \tau$.
(2) (ϕ,Δ) is soft closed in Y if and only if $(\phi,\Delta) = \widetilde{Y} \cap (\varphi,\Delta)$ for some soft closed set (φ,Δ) in X.

Theorem 2.12. *[10] Let (Y,τ_Y,Δ) be a soft subspace of a soft topological space (X,τ,Δ) and (ϕ,Δ) be a soft set over X. If $\widetilde{Y} \in \tau$ then $(\phi,\Delta) \in \tau$.*

Definition 2.13. *[10] Let (X,τ,Δ) be a soft topological space and $(\phi,\Delta) \in SS(X)_\Delta$. The soft closure of (ϕ,Δ) is the intersection of all soft closed super sets of (ϕ,Δ). It is denoted by $cl(\phi,\Delta)$ or $\tau - cl(\phi,\Delta)$.*

Definition 2.14. *[14] Let (X,τ,Δ) be a soft topological space and $(\phi,\Delta) \in SS(X)_\Delta$.The soft interior of (ϕ,Δ) is the union of all open soft subsets of (ϕ,Δ). It is denoted by $int(\phi,\Delta)$ or $\tau - int(\phi,\Delta)$.*

Theorem 2.15. *[43] Let (X,τ,Δ) be a soft topological space and $(\phi,\Delta),(\varphi,\Delta) \in SS(X)_\Delta$. Then*
(1) $cl\widetilde{\emptyset} = \widetilde{\emptyset}$, $cl\widetilde{X} = \widetilde{X}$, $int\widetilde{\emptyset} = \widetilde{\emptyset}$ and $int\widetilde{X} = \widetilde{X}$.
(2) $(\phi,\Delta)\widetilde{\subseteq} cl(\phi,\Delta)$ and $int(\phi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$.
(3) $cl(cl(\phi,\Delta)) = cl(\phi,\Delta)$ and $int(int(\phi,\Delta)) = int(\phi,\Delta)$.
(4) (ϕ,Δ) is a closed soft set if and only if $(\phi,\Delta) = cl(\phi,\Delta)$.
(5) (ϕ,Δ) is a soft open set if and only if $(\phi,\Delta) = int(\phi,\Delta)$.
(6) $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ implies both $cl(\phi,\Delta)\widetilde{\subseteq} cl(\varphi,\Delta)$ and $int(\phi,\Delta)\widetilde{\subseteq} int(\varphi,\Delta)$.
(7) $cl((\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)) = cl(\phi,\Delta)\widetilde{\cup} cl(\varphi,\Delta)$ and $int((\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)) = int(\phi,\Delta)\widetilde{\cap} int(\varphi,\Delta)$.
(8) $cl((\phi,\Delta)\widetilde{\cap}(\varphi,\Delta))\widetilde{\subseteq} cl(\phi,\Delta)\widetilde{\cap} cl(\varphi,\Delta)$ and $int((\phi,\Delta)\widetilde{\cup}(\varphi,\Delta))\widetilde{\supseteq} int(\phi,\Delta)\widetilde{\cup} int(\varphi,\Delta)$.

Definition 2.16. *[14, 44] Let $SS(X)_{\Delta_1}$, $SS(Y)_{\Delta_2}$ be two families of soft sets, $u : X \to Y$ and $p : \Delta_1 \to \Delta_2$ mappings. Then the mapping $f_{pu} : SS(X)_{\Delta_1} \to SS(Y)_{\Delta_2}$ is defined as:*
(1) Let $(\phi,\Delta_1) \in SS(X)_{\Delta_1}$. The image of (ϕ,Δ_1) under f_{pu}, written as $f_{pu}(\phi,\Delta_1) = (f_{pu}(\phi), p(\Delta_1))$, is a soft set in $SS(Y)_{\Delta_2}$ such that

$$f_{pu}(\phi)(y) = \begin{cases} \bigcup\limits_{x \in p^{-1}(y)\cap\Delta_1} u(\phi(x)) & \text{if} \quad p^{-1}(y)\cap\Delta_1 \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases},$$

for all $y \in \Delta_2$.
(2) Let $(\varphi,\Delta_2) \in SS(Y)_{\Delta_2}$. The inverse image of (φ,Δ_2) under f_{pu}, written as $f^{-1}_{pu}(\varphi,\Delta_2) = (f^{-1}_{pu}(\varphi), p^{-1}(\Delta_2))$, is a soft set in $SS(X)_{\Delta_1}$ such that

$$f^{-1}_{pu}(\varphi)(x) = \begin{cases} u^{-1}(\varphi(p(x))) & \text{if} \quad p(x) \in \Delta_2 \\ \emptyset & \text{otherwise} \end{cases},$$

for all $x \in \Delta_1$.

Definition 2.17. *[15] Let $f_{pu} : SS(X)_{\Delta_1} \to SS(Y)_{\Delta_2}$ be a soft mapping and $Z \subseteq X$. Then the restriction of f_{pu} to $SS(Z)_{\Delta_1}$ is the soft mapping $f_{pu}\Big|_{SS(Z)_{\Delta_1}}$ from $SS(Z)_{\Delta_1}$ to $SS(Y)_{\Delta_2}$ which defined by the functions $p : \Delta_1 \to \Delta_2$ and $u|_Z : Z \to Y$ where $u|_Z$ is the restriction of u to Z.*

Definition 2.18. *[36] Let τ_1, τ_2 be two soft topologies over X and $(\phi,\Delta) \in SS(X)_\Delta$. Then (ϕ,Δ) is said to be*
(1) (τ_1,τ_2)-semi open soft if $(\phi,\Delta)\widetilde{\subseteq}\tau_2 - cl(\tau_1 - int(\phi,\Delta))$,
(2) (τ_1,τ_2)-pre open soft if $(\phi,\Delta)\widetilde{\subseteq}\tau_1 - int(\tau_2 - cl(\phi,\Delta))$,
(3) (τ_1,τ_2)-α-open soft if $(\phi,\Delta)\widetilde{\subseteq}\tau_1 - int(\tau_2 - cl(\tau_1 - int(\phi,\Delta)))$,
(4) (τ_1,τ_2)-β-open soft if $(\phi,\Delta)\widetilde{\subseteq}\tau_2 - cl(\tau_1 - int(\tau_2 - cl(\phi,\Delta)))$,
(5) (τ_1,τ_2)-regular open soft if $(\phi,\Delta) = \tau_1 - int(\tau_2 - cl(\phi,\Delta))$.
The complement of a (τ_1,τ_2)-semi open soft set ((τ_1,τ_2)-pre open soft set, (τ_1,τ_2)-α-open soft set, (τ_1,τ_2)-β-open soft set, (τ_1,τ_2)-regular open soft set) is called a (τ_1,τ_2)-semi closed soft set ((τ_1,τ_2)-pre closed soft set, (τ_1,τ_2)-α-closed soft set, (τ_1,τ_2)-β-closed soft set, (τ_1,τ_2)-regular closed soft set).

3. Main results

In this section, we present two new versions of the pasting lemma on a soft topological space.

3.1. (τ_1,τ_2)-g-closed soft sets and a pasting lemma

In this subsection we introduce the notion of a (τ_1,τ_2)-g-closed soft set and investigate some properties of this new notion to obtain a new pasting lemma on a soft topological space.

Definition 3.1. *Let τ_1, τ_2 be two soft topologies over X and $(\phi,\Delta)\in SS(X)_\Delta$. Then (ϕ,Δ) is called a (τ_1,τ_2)-generalized closed soft if $\tau_2-cl(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ whenever $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ and (φ,Δ) is τ_1-soft open. It is denoted by (τ_1,τ_2)-g-closed soft. The complement of a (τ_1,τ_2)-g-closed soft set is (τ_1,τ_2)-g-open soft.*

Example 3.2. *Let $X=\{a,b,c\}$, $\Delta=\{e_1,e_2\}$, $\tau_1=\left\{\widetilde{\emptyset},\widetilde{X},(\phi,\zeta)\right\}$ and $\tau_2=\left\{\widetilde{\emptyset},\widetilde{X}\right\}$ where (ϕ,ζ) is a soft set over X defined as*

$$(\zeta,\Delta)=\{(e_1,\{a\}),(e_2,\{b\})\}.$$

Then the soft set $(\phi,\Delta)=\{(e_1,\{a,b\}),(e_2,\{a,c\})\}$ is a (τ_1,τ_2)-g-closed soft set. Indeed, if we take $(\varphi,\Delta)=\widetilde{X}\in\tau_1$ then we have

$$\tau_2-cl(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$$

and

$$(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta).$$

Theorem 3.3. *Let τ_1, τ_2 be two soft topologies over X such that $\tau_2\subset\tau_1$. If $(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\subseteq}\widetilde{X}$, (φ,Δ) is a (τ_1,τ_2)-g-closed soft set relative to (ϕ,Δ) and (ϕ,Δ) is a (τ_1,τ_2)-g-closed soft set in X, then (φ,Δ) is (τ_1,τ_2)-g-closed soft relative to $\widetilde{X}$.*

Proof. Let $(\varphi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$ and (θ,Δ) is τ_1-soft open. Then, using the hypothesis $(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\subseteq}\widetilde{X}$, we have

$$(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\cap}(\theta,\Delta)$$

and

$$\tau_{2_{(\phi,\Delta)}}-cl(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\cap}(\theta,\Delta).$$

It follows that

$$(\phi,\Delta)\widetilde{\cap}(\tau_2-cl(\varphi,\Delta))\widetilde{\subseteq}(\phi,\Delta)\widetilde{\cap}(\theta,\Delta)$$

and

$$(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)\widetilde{\cup}(\tau_2-cl(\varphi,\Delta))^c.$$

Since (ϕ,Δ) is a (τ_1,τ_2)-g-closed soft set and $\tau_2\subset\tau_1$, then we have

$$\tau_2-cl(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)\widetilde{\cup}(\tau_2-cl(\varphi,\Delta))^c.$$

Therefore, we obtain

$$\tau_2-cl(\varphi,\Delta)\widetilde{\subseteq}\tau_2-cl(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)\widetilde{\cup}(\tau_2-cl(\varphi,\Delta))^c$$

and so

$$\tau_2-cl(\varphi,\Delta)\widetilde{\subseteq}(\theta,\Delta).$$

Consequently, (φ,Δ) is (τ_1,τ_2)-g-closed soft relative to $\widetilde{X}$. □

In the following theorem, we see that the union of two (τ_1,τ_2)-g-closed soft sets is a (τ_1,τ_2)-g-closed soft set.

Theorem 3.4. *Let τ_1, τ_2 be two soft topologies over X and $(\phi,\Delta),(\varphi,\Delta)\in SS(X)_\Delta$. If (ϕ,Δ) and (φ,Δ) are two (τ_1,τ_2)-g-closed soft sets then $(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)$ is (τ_1,τ_2)-g-closed soft.*

Proof. If $(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$ and (θ,Δ) is a τ_1-soft open set, then using the hypothesis, we get

$$\tau_2-cl\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]=\tau_2-cl(\phi,\Delta)\widetilde{\cup}\tau_2-cl(\varphi,\Delta)\widetilde{\subseteq}(\theta,\Delta).$$

Hence $(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)$ is (τ_1,τ_2)-g-closed soft. □

The intersection of two (τ_1,τ_2)-g-closed soft sets is generally not a (τ_1,τ_2)-g-closed soft set as seen in the following example.

Example 3.5. *Let $X=\{a,b,c\}$, $\Delta=\{e_1,e_2\}$, $\tau_1=\left\{\widetilde{\emptyset},\widetilde{X},(\phi,\Delta)\right\}$ and $\tau_2=\left\{\widetilde{\emptyset},\widetilde{X}\right\}$ where (ϕ,Δ) is a soft set over X defined as*

$$(\phi,\Delta)=\{(e_1,\{a\}),(e_2,\{a\})\}.$$

Then the soft sets $(\varphi,\Delta)=\{(e_1,\{a,b\}),(e_2,\{a,c\})\}$ *and* $(\theta,\Delta)=\{(e_1,\{a,c\}),(e_2,\{a,b\})\}$ *are two* (τ_1,τ_2)*-g-closed soft sets. We get*

$$(\varphi,\Delta)\widetilde{\cap}(\theta,\Delta)=\{(e_1,\{a\}),(e_2,\{a\})\}$$

and so $(\varphi,\Delta)\widetilde{\cap}(\theta,\Delta)$ *is not a* (τ_1,τ_2)*-g-closed soft set.*

Proposition 3.6. *Let* τ_1, τ_2 *be two soft topologies over* X *such that* $\tau_2\subset\tau_1$. *Let* (ϕ,Δ) *be a* (τ_1,τ_2)*-g-closed soft set and* (φ,Δ) *a* τ_2*-soft closed set. Then* $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)$ *is a* (τ_1,τ_2)*-g-closed soft set.*

Proof. Since (φ,Δ) is τ_2-soft closed, then $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)$ is a τ_2-soft closed set in (ϕ,Δ) and so it is (τ_1,τ_2)-g-closed soft. From Theorem 3.3, $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)$ is a (τ_1,τ_2)-g-closed soft set. □

Theorem 3.7. *Let* $(\phi,\Delta)\widetilde{\subseteq}\widetilde{Y}\widetilde{\subseteq}\widetilde{X}$ *and* (ϕ,Δ) *be a* (τ_1,τ_2)*-g-closed soft set in* X. *Then* (ϕ,Δ) *is* (τ_1,τ_2)*-g-closed soft relative to* (Y,E).

Proof. Let $(\phi,\Delta)\widetilde{\subseteq}\widetilde{Y}\widetilde{\cap}(\varphi,\Delta)$ and (φ,Δ) be a τ_1-soft open set in X. Then $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ and so by the hypothesis, we get

$$\tau_2-cl(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta).$$

It follows that $\widetilde{Y}\widetilde{\cap}[\tau_2-cl(\phi,\Delta)]\widetilde{\subseteq}\widetilde{Y}\widetilde{\cap}(\varphi,\Delta)$. Consequently, (ϕ,Δ) is (τ_1,τ_2)-g-closed soft relative to (Y,E). □

Theorem 3.8. *Let* τ_1, τ_2 *be two soft topologies over* X *such that* $\tau_2\subset\tau_1$. *If a soft set* (ϕ,Δ) *is* (τ_1,τ_2)*-g-closed soft then* $[\tau_2-cl(\phi,\Delta)]-(\phi,\Delta)$ *contains no nonempty* τ_2*-soft closed set.*

Proof. Let (φ,Δ) be a τ_2-soft closed set of $[\tau_2-cl(\phi,\Delta)]-(\phi,\Delta)$. So we get $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)^c$. Since (ϕ,Δ) is (τ_1,τ_2)-g-closed soft, we have

$$\tau_2-cl(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)^c$$

or

$$(\varphi,\Delta)\widetilde{\subseteq}[\tau_2-cl(\phi,\Delta)]^c.$$

Thus we obtain

$$(\varphi,\Delta)\widetilde{\subseteq}[\tau_2-cl(\phi,\Delta)]\widetilde{\cap}[\tau_2-cl(\phi,\Delta)]^c=\widetilde{\emptyset},$$

that is, (φ,Δ) is a null soft set. □

As a consequence of Theorem 3.8, we give the following corollary.

Corollary 3.9. *Let* τ_1, τ_2 *be two soft topologies over* X *such that* $\tau_2\subset\tau_1$. *A* (τ_1,τ_2)*-g-closed soft set* (ϕ,Δ) *is* τ_2*-soft closed if and only if* $[\tau_2-cl(\phi,\Delta)]-(\phi,\Delta)$ *is* τ_2*-soft closed.*

Proof. If (ϕ,Δ) is τ_2-soft closed, then we have $[\tau_2-cl(\phi,\Delta)]-(\phi,\Delta)=\widetilde{\emptyset}$. Conversely, assume that $[\tau_2-cl(\phi,\Delta)]-(\phi,\Delta)$ is τ_2-soft closed. But (ϕ,Δ) is (τ_1,τ_2)-g-closed soft and $[\tau_2-cl(\phi,\Delta)]-(\phi,\Delta)$ is a τ_2-soft closed subset of itself. From Theorem 3.8, we have $[\tau_2-cl(\phi,\Delta)]-(\phi,\Delta)=\widetilde{\emptyset}$ and so $\tau_2-cl(\phi,\Delta)=(\phi,\Delta)$. □

We introduce the notion of mixed g-soft continuity as follows:

Definition 3.10. *Let* X,Y *be two initial universe sets,* $\Delta_1,\Delta_2\subseteq\Delta$ *two sets of parameters,* τ_1, τ_2 *two soft topologies over* X *and* τ *a soft topology over* Y. *Assume that* $u:X\to Y$, $p:\Delta_1\to\Delta_2$ *are two mappings and* $f_{pu}:SS(X)_{\Delta_1}\to SS(Y)_{\Delta_2}$ *is a function. Then* f_{pu} *is called mixed g-soft continuous (briefly,* $(\tau_1\tau_2,\tau)$*-g-soft cts) if* $f_{pu}^{-1}(\varphi,\Delta_2)$ *is a* (τ_1,τ_2)*-g-closed soft set for every* τ*-soft closed set* (φ,Δ_2) *in* Y.

Now we present a new version of the pasting lemma in the following theorem.

Theorem 3.11. *(Pasting lemma for* (τ_1,τ_2)*-g-closed soft sets) Let* $\widetilde{X}=\widetilde{A}\widetilde{\cup}\widetilde{B}$ *be a soft topological space with two soft topologies* τ_1, τ_2 *and* Y *a soft topological space with a soft topology* τ. *Let* $f_{p_1u_1}:SS(A)_{\Delta_1}\to SS(Y)_{\Delta_2}$ *and* $f_{p_2u_2}:SS(B)_{\Delta_1}\to SS(Y)_{\Delta_2}$ *be two mixed g-soft continuous mappings where* $p_1=p_2:\Delta_1\to\Delta_2$, $u_1:A\to Y$ *and* $u_2:B\to Y$ *are functions. Assume that* $\widetilde{A}$, $\widetilde{B}$ *are two* (τ_1,τ_2)*-g-closed soft sets and* $\tau_2\subset\tau_1$. *If* $u_1(x)=u_2(x)$ *for every* $x\in A\cap B$, *then* $f_{p_1u_1}$ *and* $f_{p_2u_2}$ *combine to give a mixed g-soft continuous mapping* $f_{pu}:SS(X)_{\Delta_1}\to SS(Y)_{\Delta_2}$ *defined by the functions* $p=p_1=p_2$ *and* $u(x)=u_1(x)$ *if* $x\in A$ *and* $u(x)=u_2(x)$ *if* $x\in B$.

Proof. Let (φ,Δ_2) be a τ-soft closed set in Y. Then we can easily seen that

$$f_{pu}^{-1}(\varphi,\Delta_2)=f_{p_1u_1}^{-1}(\varphi,\Delta_2)\widetilde{\cup}f_{p_2u_2}^{-1}(\varphi,\Delta_2).$$

From the mixed g-soft continuity of $f_{p_1u_1}$, then $f_{p_1u_1}^{-1}(\varphi,\Delta_2)$ is a (τ_1,τ_2)-g-closed soft set in A. Since $\widetilde{A}$ is (τ_1,τ_2)-g-closed soft, by Theorem 3.3, $f_{p_1u_1}^{-1}(\varphi,\Delta_2)$ is a (τ_1,τ_2)-g-closed soft set relative to $\widetilde{X}$. Similarly, $f_{p_2u_2}^{-1}(\varphi,\Delta_2)$ is a (τ_1,τ_2)-g-closed soft set relative to $\widetilde{X}$. Also using Theorem 3.4, we get that $f_{pu}^{-1}(\varphi,\Delta_2)$ is (τ_1,τ_2)-g-closed soft in X. Therefore, f_{pu} is a mixed g-soft continuous mapping. □

3.2. (τ_1,τ_2)-gpr-closed soft sets and a pasting lemma

In this subsection, we define the notion of a (τ_1,τ_2)-gpr-closed soft set. To do this, we introduce the notion of a mixed soft pre closure and a mixed soft pre interior. We investigate some basic properties of these new notions.

Definition 3.12. *Let X be a soft topological space with two soft topologies τ_1, τ_2 and $(\phi,\Delta)\in SS(X)_\Delta$.*
(1) *The mixed soft pre closure of (ϕ,Δ) is defined by*

$$\tau_1\tau_2-pcl(\phi,\Delta)=\widetilde{\cap}\left\{(\varphi,\Delta):(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta) \text{ and } (\varphi,\Delta) \text{ is } (\tau_1,\tau_2)\text{-pre closed soft}\right\}.$$

(2) *The mixed soft pre interior of (ϕ,Δ) is defined by*

$$\tau_1\tau_2-pint(\phi,\Delta)=\widetilde{\cup}\left\{(\varphi,\Delta):(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta) \text{ and } (\varphi,\Delta) \text{ is } (\tau_1,\tau_2)\text{-pre open soft}\right\}.$$

We give some properties of (τ_1,τ_2)-pre open soft sets to obtain some basic theorems related to mixed soft pre closure and mixed soft pre interior.

Theorem 3.13. *Arbitrary union of (τ_1,τ_2)-pre open soft sets is a (τ_1,τ_2)-pre open soft set.*

Proof. Let $\mathscr{A}=\{(\phi,\Delta)_i : i\in I\}$ be a collection of (τ_1,τ_2)-pre open soft sets. Then we have

$$(\phi,\Delta)_i\widetilde{\subseteq}\tau_1-int(\tau_2-cl(\phi,\Delta)_i),$$

for each $(\phi,\Delta)_i\in\mathscr{A}$. Therefore, we get

$$\widetilde{\cup}(\phi,\Delta)_i\widetilde{\subseteq}\widetilde{\cup}[\tau_1-int(\tau_2-cl(\phi,\Delta)_i)]\widetilde{\subseteq}\tau_1-int(\widetilde{\cup}[\tau_2-cl(\phi,\Delta)_i])\widetilde{\subseteq}\tau_1-int\left(\tau_2-cl\left(\widetilde{\cup}(\phi,\Delta)_i\right)\right).$$

Consequently, $\widetilde{\cup}(\phi,\Delta)_i$ is a (τ_1,τ_2)-pre open soft set. □

As a result of Theorem 3.13, we give the following corollary.

Corollary 3.14. *Arbitrary intersection of (τ_1,τ_2)-pre closed soft sets is a (τ_1,τ_2)-pre closed soft set.*

Finite intersection of (τ_1,τ_2)-pre open soft sets is not always a (τ_1,τ_2)-pre open soft set as seen in the following example.

Example 3.15. *Let $X=\{a,b,c\}$, $\Delta=\{e_1,e_2\}$, $\tau_1=\left\{\widetilde{\emptyset},\widetilde{X},(\phi_1,\Delta),(\phi_2,\Delta)\right\}$ and $\tau_2=\left\{\widetilde{\emptyset},\widetilde{X},(\varphi,\Delta)\right\}$ where (ϕ_1,Δ), (ϕ_2,Δ) and (φ,Δ) are soft sets over X defined as*

$$(\phi_1,\Delta)=\{(e_1,\{a\}),(e_2,\{b,c\})\},$$

$$(\phi_2,\Delta)=\{(e_1,\{b,c\}),(e_2,\{a\})\}$$

and

$$(\varphi,\Delta)=\{(e_1,X),(e_2,\{a,b\})\}.$$

Then the soft sets $(\theta,\Delta)=\{(e_1,\{a\}),(e_2,\{a,c\})\}$ and $(\psi,\Delta)=\{(e_1,\{b\}),(e_2,\{b,c\})\}$ are two (τ_1,τ_2)-pre open soft sets. We get

$$(\theta,\Delta)\widetilde{\cap}(\psi,\Delta)=\{(e_1,\emptyset),(e_2,\{c\})\}$$

and so $(\theta,\Delta)\widetilde{\cap}(\psi,\Delta)$ is not a (τ_1,τ_2)-pre open soft set.

Now we prove the following theorems.

Theorem 3.16. *Let X be a soft topological space with two soft topologies τ_1, τ_2 and $(\phi,\Delta)\in SS(X)_\Delta$. Then the followings hold*:
(1) *(ϕ,Δ) is (τ_1,τ_2)-pre closed soft if and only if $(\phi,\Delta)=\tau_1\tau_2-pcl(\phi,\Delta)$.*
(2) *(ϕ,Δ) is (τ_1,τ_2)-pre open soft if and only if $(\phi,\Delta)=\tau_1\tau_2-pint(\phi,\Delta)$.*
(3) *$\tau_1\tau_2-pcl\widetilde{\emptyset}=\widetilde{\emptyset}$ and $\tau_1\tau_2-pcl\widetilde{X}=\widetilde{X}$.*
(4) *$\tau_1\tau_2-pint\widetilde{\emptyset}=\widetilde{\emptyset}$ and $\tau_1\tau_2-pint\widetilde{X}=\widetilde{X}$.*
(5) *$\tau_1\tau_2-pcl[\tau_1\tau_2-pcl(\phi,\Delta)]=\tau_1\tau_2-pcl(\phi,\Delta)$.*
(6) *$\tau_1\tau_2-pint[\tau_1\tau_2-pint(\phi,\Delta)]=\tau_1\tau_2-pint(\phi,\Delta)$.*
(7) *$[\tau_1\tau_2-pcl(\phi,\Delta)]^c=\tau_1\tau_2-pint(\phi^c,\Delta)$.*
(8) *$[\tau_1\tau_2-pint(\phi,\Delta)]^c=\tau_1\tau_2-pcl(\phi^c,\Delta)$.*

Proof. (1) Let (ϕ,Δ) be a (τ_1,τ_2)-pre closed soft set. Since (ϕ,Δ) is the smallest (τ_1,τ_2)-pre closed soft set containing itself, using Definition 3.12 (1), we have $(\phi,\Delta)=\tau_1\tau_2-pcl(\phi,\Delta)$. The converse statement of the proof is clear from Corollary 3.14.
(2) Let (ϕ,Δ) be a (τ_1,τ_2)-pre open soft set. Since (ϕ,Δ) is the largest (τ_1,τ_2)-pre open soft set contained (ϕ,Δ), using Definition 3.12 (2), we have $(\phi,\Delta)=\tau_1\tau_2-pint(\phi,\Delta)$. The converse part of the proof can be easily from Theorem 3.13.
(3) Since $\widetilde{\emptyset}$ and $\widetilde{X}$ are (τ_1,τ_2)-pre closed soft sets, then using (1), we get $\tau_1\tau_2-pcl\widetilde{\emptyset}=\widetilde{\emptyset}$ and $\tau_1\tau_2-pcl\widetilde{X}=\widetilde{X}$.
(4) Since $\widetilde{\emptyset}$ and $\widetilde{X}$ are (τ_1,τ_2)-pre open soft sets, then using (2), we get $\tau_1\tau_2-pint\widetilde{\emptyset}=\widetilde{\emptyset}$ and $\tau_1\tau_2-pint\widetilde{X}=\widetilde{X}$.
(5) Using (1), we obtain

$$\tau_1\tau_2-pcl\,[\tau_1\tau_2-pcl(\phi,\Delta)]=\tau_1\tau_2-pcl(\phi,\Delta),$$

since $\tau_1\tau_2-pcl(\phi,\Delta)$ is (τ_1,τ_2)-pre closed soft.
(6) Using (2), we get

$$\tau_1\tau_2-pint\,[\tau_1\tau_2-pint(\phi,\Delta)]=\tau_1\tau_2-pint(\phi,\Delta),$$

since $\tau_1\tau_2-pint(\phi,\Delta)$ is (τ_1,τ_2)-pre open soft.
(7) Using Definition 2.4 (1) and Definition 3.12, we get

$$\begin{aligned}
& [\tau_1\tau_2-pcl(\phi,\Delta)]^c \\
= & \left[\widetilde{\cap}\left\{(\varphi,\Delta):(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta) \text{ and } (\varphi,\Delta) \text{ is } (\tau_1,\tau_2)\text{-pre closed soft}\right\}\right]^c \\
= & \widetilde{\cup}\left\{(\varphi^c,\Delta):(\varphi^c,\Delta)\widetilde{\subseteq}(\phi^c,\Delta) \text{ and } (\varphi^c,\Delta) \text{ is } (\tau_1,\tau_2)\text{-pre open soft}\right\} \\
= & \tau_1\tau_2-pint(\phi^c,\Delta).
\end{aligned}$$

(8) By the similar arguments used in the proof of (7), it can be easily proved. □

Theorem 3.17. *Let X be a soft topological space with two soft topologies τ_1, τ_2 and $(\phi,\Delta),(\varphi,\Delta)\in SS(X)_\Delta$. Then the followings hold*:
(1) *If $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ then $\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pint(\varphi,\Delta)$.*
(2) *If $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ then $\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\varphi,\Delta)$.*
(3) $\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]=\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\cup}\tau_1\tau_2-pcl(\varphi,\Delta)$.
(4) $\tau_1\tau_2-pint\left[(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\right]=\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\cap}\tau_1\tau_2-pint(\varphi,\Delta)$.
(5) $\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\right]\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\cap}\tau_1\tau_2-pcl(\varphi,\Delta)$.
(6) $\tau_1\tau_2-pint\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]\widetilde{\supseteq}\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\cup}\tau_1\tau_2-pint(\varphi,\Delta)$.

Proof. (1) Using the hypothesis, we have

$$\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)\Longrightarrow\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta).$$

Since $\tau_1\tau_2-pint(\phi,\Delta)$ is the largest (τ_1,τ_2)-pre open soft set contained in (φ,Δ). Therefore, we get

$$\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pint(\varphi,\Delta).$$

(2) Since $(\phi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta)$ and $(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\varphi,\Delta)$, we have

$$(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\varphi,\Delta)\Longrightarrow(\phi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\varphi,\Delta).$$

Because $\tau_1\tau_2-pcl(\phi,\Delta)$ is the smallest (τ_1,τ_2)-pre closed soft set containing (ϕ,Δ), then we obtain

$$\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\varphi,\Delta).$$

(3) We have

$$(\phi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta) \text{ and } (\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta).$$

By the condition (2), we get

$$\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right],$$

$$\tau_1\tau_2-pcl(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]$$

and so

$$\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\cup}\tau_1\tau_2-pcl(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]. \tag{3.1}$$

Conversely, we have

$$(\phi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta),\ (\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\varphi,\Delta)$$

and so

$$(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\cup}\tau_1\tau_2-pcl(\varphi,\Delta),$$

that is, $\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\cup}\tau_1\tau_2-pcl(\varphi,\Delta)$ is a (τ_1,τ_2)-pre closed soft set containing $(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)$. Since $\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]$ is the smallest (τ_1,τ_2)-pre closed soft set containing $(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)$, we obtain

$$\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\cup}\tau_1\tau_2-pcl(\varphi,\Delta). \tag{3.2}$$

From the inequalities (3.1) and (3.2), we get

$$\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]=\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\cup}\tau_1\tau_2-pcl(\varphi,\Delta).$$

(4) By the similar arguments used in the proof of (3), we prove

$$\tau_1\tau_2-pint\left[(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\right]=\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\cap}\tau_1\tau_2-pint(\varphi,\Delta).$$

(5) Since $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$ and $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$, we get

$$\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\right]\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta),$$

$$\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\right]\widetilde{\subseteq}\tau_1\tau_2-pcl(\varphi,\Delta)$$

and so

$$\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\right]\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\cap}\tau_1\tau_2-pcl(\varphi,\Delta).$$

(6) By the similar arguments used in the proof of (5), we obtain

$$\tau_1\tau_2-pint\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]\widetilde{\supseteq}\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\cup}\tau_1\tau_2-pint(\varphi,\Delta).$$

□

Theorem 3.18. *Let X be a soft topological space with two soft topologies τ_1, τ_2 and $(\phi,\Delta)\in SS(X)_\Delta$. Then the followings hold*:
(1) $\tau_1\tau_2-pcl(\phi,\Delta)=(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta))$.
(2) $\tau_1\tau_2-pint(\phi,\Delta)=(\phi,\Delta)\widetilde{\cap}\tau_1-int(\tau_2-cl(\phi,\Delta))$.

Proof. (1) We have

$$\begin{aligned}
&\tau_1-cl\left[\tau_2-int\left[(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta))\right]\right]\\
&\widetilde{\subseteq}\tau_1-cl\left[\tau_2-int(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta))\right]\\
=\ &\tau_1-cl(\tau_2-int(\phi,\Delta))\widetilde{\subseteq}(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta)).
\end{aligned}$$

Therefore, $(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta))$ is a (τ_1,τ_2)-pre closed soft set whence

$$\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta)). \tag{3.3}$$

Conversely, since $\tau_1\tau_2-pcl(\phi,\Delta)$ is (τ_1,τ_2)-pre closed soft, we get

$$\begin{aligned}
\tau_1-cl(\tau_2-int(\phi,\Delta))&\widetilde{\subseteq}\tau_1-cl(\tau_2-int(\tau_1\tau_2-pcl(\phi,\Delta)))\\
&\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta)
\end{aligned}$$

and so

$$(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta))\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta). \tag{3.4}$$

By the inequalities (3.3) and (3.4), we obtain

$$\tau_1\tau_2-pcl(\phi,\Delta)=(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta)).$$

(2) It is a consequence of (1). □

Proposition 3.19. *Let τ_1, τ_2 be two soft topologies over X and $(\phi,\Delta)\in SS(X)_\Delta$. If $(\phi,\Delta)\widetilde{\subseteq}\widetilde{Y}\widetilde{\subseteq}\widetilde{X}$ and $\widetilde{Y}\in\tau_2$ then we have*

$$\tau_1\tau_2 - pcl_Y(\phi,\Delta) = \tau_1\tau_2 - pcl_X(\phi,\Delta)\widetilde{\cap}\widetilde{Y}.$$

Proof. From Theorem 3.18, we get

$$\begin{aligned}
\tau_1\tau_2 - pcl_Y(\phi,\Delta) &= (\phi,\Delta)\widetilde{\cup}[\tau_1 - cl_Y(\tau_2 - int_Y(\phi,\Delta))] \\
&= (\phi,\Delta)\widetilde{\cup}[\tau_1 - cl_Y(\tau_2 - int(\phi,\Delta))] \\
&= (\phi,\Delta)\widetilde{\cup}\left[\tau_1 - cl\left(\tau_2 - int(\phi,\Delta)\widetilde{\cap}\widetilde{Y}\right)\right] \\
&= \left[(\phi,\Delta)\widetilde{\cup}\tau_1 - cl(\tau_2 - int(\phi,\Delta))\right]\widetilde{\cap}\left[(\phi,\Delta)\widetilde{\cup}\widetilde{Y}\right] \\
&= \tau_1\tau_2 - pcl_X(\phi,\Delta)\widetilde{\cap}\widetilde{Y}.
\end{aligned}$$

□

Proposition 3.20. *Let τ_1, τ_2 be two soft topologies over X and $(\phi,\Delta)\in SS(X)_\Delta$. If $\widetilde{Y}\in\tau_2$ and $\widetilde{Y}$ is a (τ_1,τ_2)-pre closed soft set then we have*

$$\tau_1\tau_2 - pcl_Y(\phi,\Delta) = \tau_1\tau_2 - pcl_X(\phi,\Delta).$$

Proof. From Proposition 3.19, we have

$$\tau_1\tau_2 - pcl_Y(\phi,\Delta) = \tau_1\tau_2 - pcl_X(\phi,\Delta)\widetilde{\cap}\widetilde{Y}.$$

Since $\widetilde{Y}$ is a (τ_1,τ_2)-pre closed soft set, we get

$$\tau_1\tau_2 - pcl_X(\phi,\Delta)\widetilde{\subseteq}\widetilde{Y}.$$

Consequently, we obtain

$$\tau_1\tau_2 - pcl_Y(\phi,\Delta) = \tau_1\tau_2 - pcl_X(\phi,\Delta).$$

□

We introduce the notion of a (τ_1,τ_2)-gpr-closed soft set.

Definition 3.21. *Let τ_1, τ_2 be two soft topologies over X and $(\phi,\Delta)\in SS(X)_\Delta$. Then (ϕ,Δ) is called a (τ_1,τ_2)-generalized pre regular closed soft if $\tau_1\tau_2 - pcl(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ whenever $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ and (φ,Δ) is (τ_1,τ_2)-regular open soft. It is denoted by (τ_1,τ_2)-gpr-closed soft. The complement of a (τ_1,τ_2)-gpr-closed soft set is (τ_1,τ_2)-gpr-open soft.*

Example 3.22. *Let $X=\{a,b,c,d\}$, $\Delta=\{e_1,e_2\}$, $\tau_1=\left\{\widetilde{\emptyset},\widetilde{X},(\phi_1,\Delta),(\phi_2,\Delta)\right\}$ and $\tau_2=\left\{\widetilde{\emptyset},\widetilde{X},(\phi_2,\Delta)\right\}$ where (ϕ_1,Δ) and (ϕ_2,Δ) are two soft sets over X defined as*

$$(\phi_1,\Delta)=\{(e_1,\{a,b\}),(e_2,\{c,d\})\}$$

and

$$(\phi_2,\Delta)=\{(e_1,\{c,d\}),(e_2,\{a,b\})\}.$$

Then the soft set $(\varphi,\Delta)=\{(e_1,\{a\}),(e_2,\{c\})\}$ is a (τ_1,τ_2)-gpr-closed soft set.

Now we give the following implications:

$$\begin{array}{c}
(\tau_1,\tau_2)\text{-regular closed soft} \\
\Downarrow \\
(\tau_1,\tau_2)\text{-pre closed soft} \\
\Downarrow \\
(\tau_1,\tau_2)\text{-}gpr\text{-closed soft}
\end{array}$$

The inverse implications of these are not always true as seen in the following example.

Example 3.23. *Let $X=\{a,b,c\}$, $\Delta=\{e\}$, $\tau_1=\left\{\widetilde{\emptyset},\widetilde{X},(\phi_1,\Delta),(\phi_2,\Delta),(\phi_3,\Delta)\right\}$ and $\tau_2=\left\{\widetilde{\emptyset},\widetilde{X},(\phi_3,\Delta)\right\}$ where (ϕ_1,Δ), (ϕ_2,Δ) and (ϕ_3,Δ) are soft sets over X defined as*

$$(\phi_1,\Delta)=\{(e,\{b\})\},\ (\phi_2,\Delta)=\{(e,\{c\})\}$$

and

$$(\phi_3,\Delta)=\{(e,\{b,c\})\}.$$

Then the soft set $(\varphi_1,\Delta)=\{(e,\{b,c\})\}$ is a (τ_1,τ_2)-gpr-closed soft set, but it is not (τ_1,τ_2)-pre closed soft. Also the soft set $(\varphi_2,\Delta)=\{(e,\{a\})\}$ is a (τ_1,τ_2)-pre closed soft set, but it is not (τ_1,τ_2)-regular closed soft.

Now we prove some necessary properties and theorems related to the notion of a (τ_1,τ_2)-gpr-open soft set.

Theorem 3.24. *Let X be a soft topological space with two soft topologies τ_1, τ_2. $(\phi,\Delta)\in SS(X)_E$ is (τ_1,τ_2)-gpr-open soft if and only if $(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pint(\phi,\Delta)$ whenever (φ,Δ) is (τ_1,τ_2)-regular closed soft and $(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$.*

Proof. Let (ϕ,Δ) be a (τ_1,τ_2)-gpr-open soft set, (φ,Δ) be a (τ_1,τ_2)-regular closed soft set and $(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$. Then we have $\widetilde{X}-(\phi,\Delta)\widetilde{\subseteq}\widetilde{X}-(\varphi,\Delta)$ where $\widetilde{X}-(\varphi,\Delta)$ is (τ_1,τ_2)-regular open soft. Since $\widetilde{X}-(\phi,\Delta)$ is (τ_1,τ_2)-gpr-closed soft, then we get $\tau_1\tau_2-pcl(\widetilde{X}-(\phi,\Delta))\widetilde{\subseteq}\widetilde{X}-(\varphi,\Delta)$. Hence we obtain

$$\widetilde{X}-\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\subseteq}\widetilde{X}-(\varphi,\Delta)$$

and so $(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pint(\phi,\Delta)$. Conversely, we suppose that (φ,Δ) is (τ_1,τ_2)-regular closed soft and $(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$ implies $(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pint(\phi,\Delta)$. Let $\widetilde{X}-(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$ where (θ,Δ) is (τ_1,τ_2)-regular open soft. Then we have $\widetilde{X}-(\theta,\Delta)\widetilde{\subseteq}(\phi,\Delta)$ where $\widetilde{X}-(\theta,\Delta)$ is (τ_1,τ_2)-regular closed soft. By the hypothesis, we get $\widetilde{X}-(\theta,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pint(\phi,\Delta)$, that is, $\widetilde{X}-\widetilde{\subseteq}\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$. Hence we obtain

$$\tau_1\tau_2-pcl(\widetilde{X}-(\phi,\Delta))\widetilde{\subseteq}(\theta,\Delta)$$

and so $\widetilde{X}-(\phi,\Delta)$ is (τ_1,τ_2)-gpr-closed soft, that is, (ϕ,Δ) is (τ_1,τ_2)-gpr-open soft. □

Theorem 3.25. *Let X be a soft topological space with two soft topologies τ_1, τ_2. If (ϕ,Δ) is (τ_1,τ_2)-gpr-closed soft and $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta)$, then (φ,Δ) is (τ_1,τ_2)-gpr-closed soft.*

Proof. Let $(\varphi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$ where (θ,Δ) is (τ_1,τ_2)-regular open soft. Then $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ implies $(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$. Since (ϕ,Δ) is (τ_1,τ_2)-gpr-closed soft, we get $\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$. Also $(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta)$ implies

$$\tau_1\tau_2-pcl(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta).$$

Thus we obtain

$$\tau_1\tau_2-pcl(\varphi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$$

and so (φ,Δ) is (τ_1,τ_2)-gpr-closed soft. □

Theorem 3.26. *Let X be a soft topological space with two soft topologies τ_1, τ_2. If (ϕ,Δ) is (τ_1,τ_2)-gpr-open soft and $\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$, then (φ,Δ) is (τ_1,τ_2)-gpr-open soft.*

Proof. $\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$ implies

$$\widetilde{X}-(\phi,\Delta)\widetilde{\subseteq}\widetilde{X}-(\varphi,\Delta)\widetilde{\subseteq}\widetilde{X}-[\tau_1\tau_2-pint(\phi,\Delta)],$$

that is,

$$\widetilde{X}-(\phi,\Delta)\widetilde{\subseteq}\widetilde{X}-(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\widetilde{X}-(\phi,\Delta)).$$

Since $\widetilde{X}-(\phi,\Delta)$ is a (τ_1,τ_2)-gpr-closed soft set, from Theorem 3.25, $\widetilde{X}-(\varphi,\Delta)$ is (τ_1,τ_2)-gpr-closed soft and so (φ,Δ) is (τ_1,τ_2)-gpr-open soft. □

The union and the intersection of two (τ_1,τ_2)-gpr-closed soft sets can not be always (τ_1,τ_2)-gpr-closed soft as seen in the following examples, respectively.

Example 3.27. *Let $X=\{a,b,c,d,e\}$, $\Delta=\{e'\}$, $\tau_1=\tau_2=\left\{\widetilde{\emptyset},\widetilde{X},(\phi_1,\Delta),(\phi_2,\Delta),(\phi_3,\Delta)\right\}$ where (ϕ_1,Δ), (ϕ_2,Δ) and (ϕ_3,Δ) are soft sets over X defined as*

$$(\phi_1,\Delta)=\left\{(e',\{a,c\})\right\},\ (\phi_2,\Delta)=\left\{(e',\{b,d\})\right\}$$

and

$$(\phi_3,\Delta)=\left\{(e',\{a,b,c,d\})\right\}.$$

Then the soft set $(\varphi,\Delta)=\{(e',\{a\})\}$ and $(\theta,\Delta)=\{(e',\{c\})\}$ are two (τ_1,τ_2)-gpr-closed soft set, but $(\varphi,\Delta)\widetilde{\cup}(\theta,\Delta)=\{(e',\{a,c\})\}$ is not (τ_1,τ_2)-gpr-closed soft.

Example 3.28. *Let $X=\{a,b,c\}$, $\Delta=\{e'\}$, $\tau_1=\tau_2=\left\{\widetilde{\emptyset},\widetilde{X},(\phi_1,\Delta),(\phi_2,\Delta),(\phi_3,\Delta)\right\}$ where (ϕ_1,Δ), (ϕ_2,Δ) and (ϕ_3,Δ) are soft sets over X defined as*

$$(\phi_1,\Delta)=\left\{(e',\{b\})\right\},\ (\phi_2,\Delta)=\left\{(e',\{c\})\right\}$$

and

$$(\phi_3,\Delta)=\left\{(e',\{b,c\})\right\}.$$

Then the soft set $(\varphi,\Delta)=\{(e',\{b,c\})\}$ and $(\theta,\Delta)=\{(e',\{a,b\})\}$ are two (τ_1,τ_2)-gpr-closed soft set, but $(\varphi,\Delta)\widetilde{\cap}(\theta,\Delta)=\{(e',\{b\})\}$ is not (τ_1,τ_2)-gpr-closed soft.

Proposition 3.29. *Let X be a soft topological space with two soft topologies τ_1, τ_2 and $(\phi,\Delta),(\varphi,\Delta)\in SS(X)_\Delta$. If (φ,Δ) is (τ_1,τ_2)-gpr-open soft and $\tau_1\tau_2-pint(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$ then $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)$ is (τ_1,τ_2)-gpr-open soft.*

Proof. Since (φ,Δ) is (τ_1,τ_2)-gpr-open soft and $\tau_1\tau_2-pint(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$ then we have

$$\tau_1\tau_2-pint(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\widetilde{\subseteq}(\varphi,\Delta).$$

From Theorem 3.26, $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)$ is (τ_1,τ_2)-gpr-open soft. □

The class of all (τ_1,τ_2)-pre-open soft sets is denoted by $PO(X,\tau_1,\tau_2)$.

Proposition 3.30. *Let X be a soft topological space with two soft topologies τ_1, τ_2, $(\phi,\Delta),(\varphi,\Delta)\in SS(X)_\Delta$ and $PO(X,\tau_1,\tau_2)$ closed under finite intersections. If (ϕ,Δ) and (φ,Δ) are two (τ_1,τ_2)-gpr-open soft sets, then $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)$ is (τ_1,τ_2)-gpr-open soft.*

Proof. Let us consider

$$\widetilde{X}-\left[(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\right]=\left[\widetilde{X}-(\phi,\Delta)\right]\widetilde{\cup}\left[\widetilde{X}-(\varphi,\Delta)\right]\widetilde{\subseteq}(\theta,\Delta),$$

where (θ,Δ) is (τ_1,τ_2)-regular open soft. Then we have $\widetilde{X}-(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$ and $\widetilde{X}-(\varphi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$. Since (ϕ,Δ) and (φ,Δ) are two (τ_1,τ_2)-gpr-open soft sets, we have

$$\tau_1\tau_2-pcl\left(\widetilde{X}-(\phi,\Delta)\right)\widetilde{\subseteq}(\theta,\Delta)$$

and

$$\tau_1\tau_2-pcl\left(\widetilde{X}-(\varphi,\Delta)\right)\widetilde{\subseteq}(\theta,\Delta).$$

By the hypothesis, we find

$$\begin{aligned}&\tau_1\tau_2-pcl\left[\left(\widetilde{X}-(\phi,\Delta)\right)\widetilde{\cup}\left(\widetilde{X}-(\varphi,\Delta)\right)\right]\\&\widetilde{\subseteq}\tau_1\tau_2-pcl\left(\widetilde{X}-(\phi,\Delta)\right)\widetilde{\cup}\tau_1\tau_2-pcl\left(\widetilde{X}-(\varphi,\Delta)\right)\widetilde{\subseteq}(\theta,\Delta),\end{aligned}$$

that is,

$$\tau_1\tau_2-pcl\left[\widetilde{X}-\left((\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\right)\right]\widetilde{\subseteq}(\theta,\Delta).$$

Consequently, $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)$ is (τ_1,τ_2)-gpr-open soft. □

The following lemma will be used in the proof of a proposition related to a (τ_1,τ_2)-gpr-closed soft set in a soft subspace.

Lemma 3.31. *Let $\widetilde{Y}\widetilde{\subseteq}\widetilde{X}$, X be a soft topological space with two soft topologies τ_1, τ_2 and $(\phi,\Delta)\in SS(X)_\Delta$. If $\widetilde{Y}$ is a τ_2-soft open set and $\tau_2\subset\tau_1$, then $(\phi,\Delta)\widetilde{\cap}\widetilde{Y}$ is a (τ_1,τ_2)-regular open soft set relative to $\widetilde{Y}$ for some (ϕ,Δ) which is a (τ_1,τ_2)-regular open soft set relative to $\widetilde{X}$.*

Proof. Let (ϕ,Δ) be a (τ_1,τ_2)-regular open soft set and $(\varphi,\Delta)=(\phi,\Delta)\widetilde{\cap}\widetilde{Y}$. Then we have

$$\begin{aligned}\tau_1-int\left(\tau_2-cl\left((\varphi,\Delta)\widetilde{\cap}\widetilde{Y}\right)\right) &= \tau_1-int\left(\tau_2-cl\,(\phi,\Delta)\,\widetilde{\cap}\widetilde{Y}\right)\\ &= \tau_1-int\,(\tau_2-cl\,(\phi,\Delta))\,\widetilde{\cap}\widetilde{Y}\\ &= (\phi,\Delta)\,\widetilde{\cap}\widetilde{Y}=(\varphi,\Delta).\end{aligned}$$

Hence (φ,Δ) is a (τ_1,τ_2)-regular open soft set relative to $\widetilde{Y}$. □

Proposition 3.32. *Let X be a soft topological space with two soft topologies τ_1, τ_2 such that $\tau_2\subset\tau_1$ and $(\phi,\Delta)\widetilde{\subseteq}\widetilde{Y}\widetilde{\subseteq}\widetilde{X}$. Then the followings hold:*
(1) If $\widetilde{Y}$ is a τ_2-soft open set and (ϕ,Δ) is a (τ_1,τ_2)-gpr-closed soft set in X then (ϕ,Δ) is a (τ_1,τ_2)-gpr-closed soft set in Y.
(2) If $\widetilde{Y}$ is a τ_2-soft open set and a (τ_1,τ_2)-pre closed soft set in X and (ϕ,Δ) is a (τ_1,τ_2)-gpr-closed soft set in Y then (ϕ,Δ) is a (τ_1,τ_2)-gpr-closed soft set in X.

Proof. (1) Let (ϕ,Δ) be a (τ_1,τ_2)-gpr-closed soft set in X and $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ where (φ,Δ) is a (τ_1,τ_2)-regular open soft set in Y. By Lemma 3.31, we have $(\varphi,\Delta)=(\theta,\Delta)\widetilde{\cap}\widetilde{Y}$ where (θ,Δ) is a (τ_1,τ_2)-regular open soft set in X, that is, $(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$. Since (ϕ,Δ) is a (τ_1,τ_2)-gpr-closed soft set in X then we get

$$\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta),$$

which implies

$$\tau_1\tau_2-pcl_X(\phi,\Delta)\widetilde{\cap}\widetilde{Y}\widetilde{\subseteq}(\theta,\Delta)\widetilde{\cap}\widetilde{Y}.$$

By Lemma 3.20, we have

$$\tau_1\tau_2-pcl_Y(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta).$$

Therefore, (ϕ,Δ) is a (τ_1,τ_2)-gpr-closed soft set in Y.
(2) Let (ϕ,Δ) be a (τ_1,τ_2)-gpr-closed soft set in Y. Then $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ where (φ,Δ) is a (τ_1,τ_2)-regular open soft set in X. Hence we get

$$(\phi,\Delta)=(\phi,\Delta)\widetilde{\cap}\widetilde{Y}\widetilde{\subseteq}(\varphi,\Delta)\widetilde{\cap}\widetilde{Y},$$

where $(\varphi,\Delta)\widetilde{\cap}\widetilde{Y}$ is (τ_1,τ_2)-regular open soft in Y by Lemma 3.31. Using the hypothesis, we get

$$\tau_1\tau_2-pcl_Y(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)\widetilde{\cap}\widetilde{Y}.$$

By Lemma 3.20, we obtain

$$\tau_1\tau_2-pcl_X(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)\widetilde{\cap}\widetilde{Y}\widetilde{\subseteq}(\varphi,\Delta),$$

that is, (ϕ,Δ) is a (τ_1,τ_2)-gpr-closed soft set in X. □

We introduce the notion of mixed gpr-soft continuity as follows:

Definition 3.33. *Let X,Y be two initial universe sets, $\Delta_1,\Delta_2\subseteq\Delta$ two sets of parameters, τ_1, τ_2 two soft topologies over X and τ a soft topology over Y. Assume that $u:X\to Y$, $p:\Delta_1\to\Delta_2$ are two mappings and $f_{pu}:SS(X)_{\Delta_1}\to SS(Y)_{\Delta_2}$ is a function. Then f_{pu} is called mixed gpr-soft continuous (briefly, $(\tau_1\tau_2,\tau)$-gpr-soft cts) if $f_{pu}^{-1}(\varphi,\Delta_2)$ is a (τ_1,τ_2)-gpr-closed soft set for every τ-soft closed set (φ,Δ_2) in Y.*

Using the concept of mixed gpr-soft continuity, we present a new version of the pasting lemma in the following theorem.

Theorem 3.34. *(Pasting lemma for (τ_1,τ_2)-gpr-closed soft sets) Let $\widetilde{X}=\widetilde{A}\widetilde{\cup}\widetilde{B}$ be a soft topological space with two soft topologies τ_1, τ_2, Y a soft topological space with a soft topology τ and the family of all (τ_1,τ_2)-gpr-open soft sets closed under finite intersections. Let $f_{p_1u_1}:SS(A)_{\Delta_1}\to SS(Y)_{\Delta_2}$ and $f_{p_2u_2}:SS(B)_{\Delta_1}\to SS(Y)_{\Delta_2}$ be two mixed gpr-soft continuous mappings where $p_1=p_2:\Delta_1\to\Delta_2$, $u_1:A\to Y$ and $u_2:B\to Y$ are functions. Suppose that $\widetilde{A}$, $\widetilde{B}$ are τ_2-soft open and (τ_1,τ_2)-pre closed soft and $\tau_2\subset\tau_1$. If $u_1(x)=u_2(x)$ for every $x\in A\cap B$, then $f_{p_1u_1}$ and $f_{p_2u_2}$ combine to give a mixed gpr-soft continuous mapping $f_{pu}:SS(X)_{\Delta_1}\to SS(Y)_{\Delta_2}$ defined by the functions $p=p_1=p_2$ and $u(x)=u_1(x)$ if $x\in A$ and $u(x)=u_2(x)$ if $x\in B$.*

Proof. Let (φ,Δ_2) be a τ-soft closed set in Y. Then we can easily seen that

$$f_{pu}^{-1}(\varphi,\Delta_2)=f_{p_1u_1}^{-1}(\varphi,\Delta_2)\widetilde{\cup}f_{p_2u_2}^{-1}(\varphi,\Delta_2).$$

Since $f_{p_1u_1}$ is mixed gpr-soft continuous then $f_{p_1u_1}^{-1}(\varphi,\Delta_2)$ is a (τ_1,τ_2)-gpr-closed soft set in A. Since $\widetilde{A}$ is τ_2-soft open and (τ_1,τ_2)-pre closed soft, then $f_{p_1u_1}^{-1}(\varphi,\Delta_2)$ is a (τ_1,τ_2)-gpr-closed soft set in $\widetilde{X}$ by Proposition 3.32 (2). Similarly, $f_{p_2u_2}^{-1}(\varphi,\Delta_2)$ is a (τ_1,τ_2)-gpr-closed soft set in $\widetilde{X}$. Also we get that $f_{pu}^{-1}(\varphi,\Delta_2)$ is (τ_1,τ_2)-gpr-closed soft in X from the hypothesis. Therefore, f_{pu} is a mixed gpr-soft continuous mapping. □

4. Conclusion and future work

In this paper, two new versions of the pasting lemma for mixed g-soft continuous functions and mixed gpr-soft continuous functions are presented on a soft topological space. As a future work, some applications of these pasting lemmas can be investigated to analytic continuation on a complex plane.

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