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On Quaternionic Bertrand Curves in Euclidean 3-Space

Aykut Has[1], Beyhan Yilmaz[1,*]

[1]*Department of Mathematics, Faculty of Science and Literature, Kahramanmaras Sutcu Imam University, 46100, Kahramanmaras, Turkey.*

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Abstract. In this article, spatial quaternionic Bertrand curve pairs in the 3-dimensional Euclidean space are examined. Algebraic properties of quaternions, basic definitions and theorems are given. Later, some characterizations of spatial quaternionic Bertrand curve pairs are obtained in the 3-dimensional Euclidean space.

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1. Introduction

Quaternions are a four-dimensional number system defined by Sir William Rowan Hamilton in 1843 based on the idea of generalizing complex numbers [6]. Hamilton firstly tried to identify the three-dimensional complex numbers. However, it has noticed that this set of numbers does not have a closed property relative to the multiplication process. In this way, realizing that a 3-dimensional number system cannot exist Hamilton described a four-dimensional number system known as quaternions. Later, the set of quaternions find its place in many fields such as geometry, physics, kinematics, mechanics, vector analysis, computer, animation and robotics technology [5, 7, 10].

Curves and curve pairs often have been studied differential geometry. The first curve pair that comes to mind when talking about curve pair is a Bertrand curve pair. These curve pairs have been studied by many authors in different spaces. By C. Bioche obtained a Bertrand curve pair using C_1 and C_2 curves in 3-dimensional Euclidean space in 1888 [2]. Afterward, in 1960, J.F. Burke gave a new theorem about Biocheís theorem on Bertrand curves. In [3], it is studied. Serret-Frenet formulas for quaternionic curves in R^3 are firstly described by K. Bharathi and M. Nagaraj in 1987. Then, they obtained the Serret-Frenet formulas for the quaternionic curves in R^3 and R^4 by using these formulas, [1]. Later, many studies based on these mentioned studies are published. One of these studies is some characterizations of a quaternionic curve in the semi-Euclidean space E_2^4 obtained by A.C. Çöken and A. Tuna [4, 9].

*Corresponding Author
Email addresses: ahas@ksu.edu.tr (A. Has), beyhanyilmaz@ksu.edu.tr (B. Yılmaz)

2. Preliminaries

In this section, basic definitions and theorems on quaternions, spatial quaternionic curves and quaternionic curves will be given. In general, the real quaternion q is of the form:

$$q = ae_1 + be_2 + ce_3 + de_4,$$

where a, b, c, d are real numbers and e_i, $(1 \leq i \leq 4)$ are quaternionic units which satisfy the non-commutative multiplication rules

$$e_4 = 1, e_1^2 = e_2^2 = e_3^2 = -1, \quad e_1e_2 = -e_2e_1 = e_3,$$

$$e_2e_3 = -e_3e_2 = e_1, \;\; e_3e_1 = -e_1e_3 = e_2.$$

If we denote $Sq = d$ and $Vq = ae_1 + be_2 + ce_3$, we can rewrite a real quaternion as follows $q = Sq + Vq$ where is Sq and V_q are the scalar part and vectorial part of q, respectively. So, we can show the product of two quaternions as:

$$p \times q = SpSq - \langle Vp, Vq \rangle + SpVq + SqVp + Vp \wedge Vq,$$

where $\langle , \rangle$ and $\wedge$ are inner product and cross product in E^3, respectively. The conjugate of q is denoted by γq and defined as:

$$\gamma q = -ae_1 - be_2 - ce_3 + de_4$$

which is called the "Hamilton conjugation". This defines bilinear form h as follows

$$h(p, q) = \frac{1}{2}[p \times \gamma q + q \times \gamma q]$$

which is called the quaternion inner product. The norm of q is given by

$$\|q\|^2 = h(q, q) = q \times \gamma q = \gamma q \times q = a^2 + b^2 + c^2 + d^2.$$

If $\|q\|^2 = 1$, then q is called unit quaternion. Also, q is called a spatial quaternion whenever $q + \gamma q = 0$ and called a temporal quaternion whenever $q - \gamma q = 0$.

Theorem 2.1. *Space of spatial quaternions in three dimensional Euclidean space, it is clearly is identified as* $\{p \in Q_H \mid p + \gamma p = 0\}$. *Let* $I = (0, 1)$ *indicate the unit spacing in the real line* $\mathbb{R}$. *Let*

$$\begin{array}{lcl} \alpha & : & I \subset \mathbb{R} \longrightarrow Q_H \\ s & \longrightarrow & \alpha(s) : \sum_{i=1}^{3} \alpha_i(s)e_i \quad (1 \leq i \leq 3) \end{array}$$

be a curve with nonzero curvatures $\{k(s), r(s)\}$ *and* $\{t(s), n(s), b(s)\}$ *denote the Frenet frame of the curve, [1]. Then,*

$$\begin{pmatrix} t'(s) \\ n'(s) \\ b'(s) \end{pmatrix} = \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & r(s) \\ 0 & -r(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix}.$$

Definition 2.2. Let $\alpha(s)$ and $\beta(s^*)$ be quaternionic curves in E^3 with parameter s and s^*, respectively. Let $\{t(s), n(s), b(s)\}$ and $\{t^*(s^*), n^*(s^*), b^*(s^*)\}$ be Frenet frames of α and β, respectively. If $\{\alpha, \beta\}$ are the Bertrand curve pair, $n(s)$ and $n^*(s^*)$ are linearly dependent [8]. We can write

$$\beta(s^*) = \alpha(s) + \lambda_1 n(s), \tag{2.1}$$

and

$$\alpha(s) = \beta(s^*) + \lambda_2 n^*(s). \tag{2.2}$$

Corollary 2.3 ([8]). *Let* $\alpha(s)$ *and* $\beta(s^*)$ *be quaternionic curves in* E^3. *If* $\{\alpha, \beta\}$ *are the Bertrand curve pair, the following equations are easily visible;*

$$\begin{array}{rcl} t^*(s^*) & = & \cos\theta t(s) + \sin\theta b(s), \\ n^*(s^*) & = & n(s), \\ b^*(s^*) & = & -\sin\theta t(s) + \cos\theta b(s). \end{array} \tag{2.3}$$

Corollary 2.4 ([8]). *Let $\alpha(s)$ and $\beta(s^*)$ be quaternionic curves in E^3. If $\{\alpha,\beta\}$ are the Bertrand curve pair, we can write the following equations;*

$$\begin{aligned} k^* &= k\cos\theta - r\sin\theta, \\ r^* &= k\sin\theta + r\cos\theta. \end{aligned} \tag{2.4}$$

3. Main Results

In this section, some of the known results about Bertrand curve pairs in 3-dimensional Euclidean space were obtained using quaternonic properties. Then, many new and useful characterizations were obtained thanks to the quaternonic interior product in bertrand curves.

Theorem 3.1. *Let $\alpha(s)$ be a spatial quaternionic curve in three-dimensional Euclidean space E^3 with arc-length parameter s and $\beta(s^*)$ be the Bertrand curve pair of $\alpha(s)$ with arc-length parameter s^*. Then, the distance between mutual points is fixed for each $s \in I$.*

Proof. Suppose that $\alpha : I \longrightarrow E^3$ is a given spatial quaternionic curve. Taking the derivative of equation (2.1) according to s and apply Frenet formulas following equations;

$$\frac{d\beta}{ds^*}\frac{ds^*}{ds} = \alpha' + \lambda_1' n + \lambda_1 n',$$

$$t^*\frac{ds^*}{ds} = (1-\lambda_1 k)t + \lambda_1' n + \lambda_1 rb.$$

If the quaternionic internal product is performed with n both sides of the equation, the following result is obtained

$$\frac{ds^*}{ds}h(t^*,n) = h((1-\lambda_1 k)t + \lambda_1' n(s) + \lambda_1 rb, n).$$

If the left side of the above equation is calculated, we have

$$\begin{aligned} \frac{ds^*}{ds}h(t^*,n) &= \frac{1}{2}\left[t^*\frac{ds^*}{ds}\times\gamma n + n\times\gamma(t^*\frac{ds^*}{ds})\right], \\ &= \frac{1}{2}\left[-\frac{ds^*}{ds}(t^*\times n) - (n\times t^*)\frac{ds^*}{ds}\right]. \end{aligned}$$

Since $n(s)$ and $n^*(s^*)$ are linear dependents, the following result can be written.

$$\frac{1}{2}\frac{ds^*}{ds}\left[-(t^*\times n^*) - (n^*\times t^*)\right] = 0. \tag{3.1}$$

Now, if the right side of the same equation is calculated,

$$h((1-\lambda_1 k)t + \lambda_1' n + \lambda_1 rb, n) = \frac{1}{2}[-(1-\lambda_1 k)(t\times n) - \lambda_1'(n\times n) - \lambda_1 r(b\times n) - (1-\lambda_1 k)(n\times t) - \lambda_1'(n\times n) \tag{3.2}$$

result is reached. Finally, the result of is reached as it can be easily seen from equations (3.1) and (3.2),

$$\lambda_1' = 0, \lambda_1 = c,\ c\in\mathbb{R}.$$

□

Theorem 3.2. *The α curve is a spatial quaternionic curve in 3-dimensional Euclidean space, and the β curve is the Bertrand curve pair of the α curve. In that case, the angle between the tangent vectors of the α and β curves is constant.*

Proof. The $\alpha(s)$ and $\beta(s^*)$ curves are a Bertrand curve pair in 3-dimensional Euclidean space. Suppose the angle between the tangent vectors of the α and $\beta(s^*)$ curve is θ. Then, the following equation can be written,

$$h(t,t^*) = \cos\theta.$$

If the derivative of the above equation is obtained according to s,

$$\frac{d}{ds}h(t,t^*) = h(kn,t^*) + h(t,k^*n^*\frac{ds^*}{ds})$$

result is obtained. If the necessary adjustments are made to this equation, we get

$$\begin{aligned} &= \frac{1}{2}\left[kn \times \gamma t^{*} + t^{*} \times \gamma(kn)\right] + \frac{1}{2}\frac{ds^{*}}{ds}\left[t \times \gamma(k^{*}n^{*}) + k^{*}n^{*} \times \gamma t\right] \\ &= \frac{1}{2}\left[-k(n \times t^{*}) - k(t^{*} \times n)\right] + \frac{1}{2}\frac{ds^{*}}{ds}\left[-k^{*}(t \times n^{*}) - k^{*}(n^{*} \times t)\right] \\ &= 0. \end{aligned}$$

As can be easily seen from this equation, the result below can be written

$$\begin{aligned} \frac{d}{ds}h(t, t^{*}) &= \frac{d}{ds}\cos\theta = 0, \\ \cos\theta &= c,\ c \in \mathbb{R}. \end{aligned}$$

□

Theorem 3.3. *The $\alpha(s)$ be a spatial quaternionic curve in 3-dimensional Euclidean space and $\beta(s^{*})$ curve is the Bertrand curve pair of the $\alpha(s)$ curve. k, r and $k(s^{*})$, $r(s^{*})$ with curvature and torsion of the $\alpha(s)$ and $\beta(s^{*})$ curves, respectively. In this case, the following equations are available*

$$\begin{aligned} k^{*} &= k\cos\theta - r\sin\theta, \\ r^{*} &= k\sin\theta + r\cos\theta. \end{aligned} \tag{3.3}$$

Proof. As is known from the definition of curvature and torsion, there is equality of,

$$\begin{aligned} k^{*} &= h((t^{*})', n^{*}), \\ r^{*} &= h((n^{*})', b^{*}), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} t^{*} &= \cos\theta t + \sin\theta b, \\ n^{*} &= n, \\ b^{*} &= -\sin\theta t + \cos\theta n. \end{aligned} \tag{3.5}$$

If the equation (3.4) is used in the equation (3.5),

$$\begin{aligned} k^{*} &= h((\cos\theta t + \sin\theta b)', n), \\ &= \frac{1}{2}[(-\sin\theta\frac{d\theta}{ds}t + (k\cos\theta - r\sin\theta)n + \cos\theta\frac{d\theta}{ds}b) \times \gamma n \\ &\quad + n \times \gamma(-\sin\theta\frac{d\theta}{ds}t + (k\cos\theta - r\sin\theta)n + \cos\theta\frac{d\theta}{ds}b)] \\ &= \frac{1}{2}[\sin\theta\frac{d\theta}{ds}(t \times n) + (k\cos\theta - r\sin\theta)(n \times n) - \cos\theta\frac{d\theta}{ds}(b \times n) \\ &\quad + \sin\theta\frac{d\theta}{ds}(n \times t) + (r\sin\theta - k\cos\theta)(n \times n) - \cos\theta\frac{d\theta}{ds}(n \times b)] \\ &= k\cos\theta - r\sin\theta. \end{aligned}$$

On the other hand, if the necessary adjustments are made, the torsion of the β curve is obtained as follows.

$$\begin{aligned} r^{*} &= h((-kt + rb), (-\sin\theta t + \cos\theta b)) \\ &= \frac{1}{2}\left[(-kt + rb) \times \gamma(-\sin\theta t + \cos\theta b) + (\cos\theta b - \sin\theta t) \times \gamma(rb - kt)\right] \\ &= \frac{1}{2}\left[-k\sin\theta(t \times t) + k\cos\theta(t \times b) + r\sin\theta(b \times t) - r\cos\theta(b \times b)\right] \\ &\quad -k\sin\theta(t \times t) + r\sin\theta(t \times b) + k\cos\theta(b \times t) - r\cos\theta(b \times b) \\ &= \frac{1}{2}\left[2k\sin\theta - k\cos\theta n + r\sin\theta n + 2r\cos\theta - k\sin\theta n + k\cos\theta n\right] \\ &= k\sin\theta + r\cos\theta. \end{aligned}$$

□

Theorem 3.4. *The $\alpha(s)$ be a spatial quaternionic curve in 3-dimensional Euclidean space and $\beta(s^*)$ curve is the Bertrand curve pair of the $\alpha(s)$ curve. k, r with curvature and torsion of the $\alpha(s)$ curves, respectively. In this case, the following equations are available*

$$r = \left(k - \frac{1}{\lambda_1}\tan\theta\right).$$

Proof. If we take the derivative of the equation (2.1) according to s, we get the following

$$t^* \frac{ds^*}{ds} = (1 - \lambda_1 k)t + \lambda_1' n + \lambda_1 r b. \tag{3.6}$$

If quaternionic internal product is performed with b^* on both sides of the above equality

$$\frac{ds^*}{ds} h(b^*, t^*) = ((1 - \lambda_1 k)h(b^* \times t) + \lambda_1' h(b^* \times n) + \lambda_1 r h(b^* \times b))$$

result is reached. If the left side of the equation is calculated, the following result is reached;

$$\begin{aligned} \frac{ds^*}{ds} h(b^*, t^*) &= \frac{1}{2}\left[b^* \times \gamma t^* \frac{ds^*}{ds} + t^* \frac{ds^*}{ds} \times \gamma b^*\right] \\ &= 0, \end{aligned}$$

and if the necessary calculations are made on the right side of the equation, the following result is reached;

$$((1 - \lambda_1 k)h(b^* \times t) + \lambda_1' h(b^* \times n) + \lambda_1 r h(b^* \times b)) = (1 - \lambda_1 k)\sin\theta + \lambda_1 r \cos\theta.$$

So, we can easily see from the above equations;

$$r = \left(k - \frac{1}{\lambda_1}\tan\theta\right)$$

result is obtained and proof is completed. □

Theorem 3.5. *Let $\alpha(s)$ be a spatial quaternionic curves in Euclidean space E^3 and $\beta(s^*)$ be the Bertrand curve pair of $\alpha(s)$. r and r^* torsions of the curve $\alpha(s)$ and $\beta(s^*)$, respectively. r and r^* are the same marked and $r.r^*$ product is constant.*

Proof. If equation (3.5) is used in equation (3.6), then

$$(\cos\theta t - \sin\theta b)\frac{ds^*}{ds} = (1 - \lambda_1 k)t + \lambda_1' n + \lambda_1 r b.$$

If equations are synchronized mutually,

$$\begin{aligned} \cos\theta &= (1 - \lambda_1 k)\frac{ds}{ds^*}, \\ \sin\theta &= (-\lambda_1 r)\frac{ds}{ds^*}, \end{aligned}$$

and

$$\begin{aligned} \cos\theta &= (1 - \lambda_1 k^*)\frac{ds^*}{ds}, \\ \sin\theta &= (-\lambda_1 r^*)\frac{ds^*}{ds}. \end{aligned}$$

The following results can be written considering the equations above,

$$\begin{aligned} r &= \frac{\sin\theta}{-\lambda_1}\frac{ds^*}{ds}, \\ r^* &= \frac{\sin\theta}{-\lambda_1}\frac{ds}{ds^*}, \end{aligned}$$

finally, the result is obtained below,

$$r.r^* = \frac{\sin^2\theta}{\lambda_1^2}.$$

$\frac{1}{\lambda_1^2} = \mu$ multiplied by $r.r^*$ can be written as follows,

$$r.r^* = \mu \sin^2\theta = c, \;\; c \in \mathbb{R}.$$

As a result, because of $\mu \sin^2\theta \geq 0$, r and r^* are the same marked. □

Theorem 3.6. *Let $\alpha(s)$ be a spatial quaternionic curves in Euclidean space E^3 and $\beta(s^*)$ be the Bertrand curve pair of $\alpha(s)$. The following results are available*

i. $\dfrac{ds}{ds^*} = \cos\theta - \lambda_2 k,$

ii. $r = -\dfrac{\sin\theta}{\lambda_2},$

iii. $\tan\theta = \dfrac{-\lambda_2 r^*}{1 - \lambda 2k^*}.$

Proof. i. If both sides of the (2.2) equation receive derivatives according to s^*,

$$\begin{aligned} \frac{d\alpha(s)}{ds}.\frac{ds}{ds^*} &= t^* + \lambda_2' n^* + \lambda_2 (n^*)', \\ t\frac{ds}{ds^*} &= (1 - \lambda_2 k^*) t^* + \lambda_2' n^* + \lambda_2 r^* b^*. \end{aligned} \tag{3.7}$$

result is obtained. If the equation (3.5) is used in the (3.7) equation,

$$\begin{aligned} t\frac{ds}{ds^*} &= (1 - \lambda_2 k^*)(\cos\theta t + \sin\theta b) + \lambda_2' n + \lambda_2 r^*(-\sin\theta t + \cos\theta b) \qquad (3.8) \\ &= [(\cos\theta - \lambda_2 k^* \cos\theta - \lambda_2 r^* \sin\theta) t + \lambda_2' n(\sin\theta - \lambda_2 k^* \sin\theta + \lambda_2 r^* \cos\theta) b] \end{aligned}$$

equality is achieved. If quaternionic inner product is performed with t on both sides of the equality,

$$\begin{aligned} h(t,t)\frac{ds}{ds^*} &= (\cos\theta - \lambda_2 k^* \cos\theta - \lambda_2 r^* \sin\theta) h(t,t) \\ &\quad + \lambda_2' h(t,n) + (\sin\theta - \lambda_2 k^* \sin\theta + \lambda_2 r^* \cos\theta) h(t,b) \end{aligned}$$

and

$$\frac{ds}{ds^*} = \cos\theta - \lambda_2 k^* \cos\theta - \lambda_2 r^* \sin\theta. \tag{3.9}$$

If the equation (3.3) is used in equation (3.9), the first equation is proven.

$$\begin{aligned} \frac{ds}{ds^*} &= \cos\theta - \lambda_2(k\cos\theta - r\sin\theta)\cos\theta - \lambda_2(k\sin\theta + r\cos\theta)\sin\theta \\ &= \cos\theta - \lambda_2 k. \end{aligned} \tag{3.10}$$

ii. If both sides of the equality (3.7) are made the quaternionic inner product with b^*, we have

$$h(b^*, t)\frac{ds}{ds^*} = (1 - \lambda_2 k^*) h(b^*, t^*) + \lambda_2' h(b^*, n^*) + \lambda_2 r^* h(b^*, b^*).$$

As can be easily seen from the equation above,

$$\begin{aligned} -\sin\theta\frac{ds}{ds^*} &= \lambda_2 r^* \\ &= \lambda_2(k\sin\theta + r\cos\theta) \end{aligned}$$

and

$$\frac{ds}{ds^*} = -\lambda_2 k - \lambda_2 r \cot\theta. \tag{3.11}$$

Using equations (3.10) and (3.11), the following result is easily achieved,

$$r = -\frac{\sin\theta}{\lambda_2}.$$

Thus, it is proven in the second equality.

iii. If both sides of the equation (3.8) are made the quaternionic inner product with t^*,

$$h(t^*,t)\frac{ds}{ds^*} = (\cos\theta - \lambda_2 k^*\cos\theta - \lambda_2 r^*\sin\theta)h(t^*,t)$$
$$+\lambda_2' h(t^*,n) + (\sin\theta - \lambda_2 k^*\sin\theta + \lambda_2 r^*\cos\theta)h(t^*,b),$$
$$\cos\theta\frac{ds}{ds^*} = (\cos\theta - \lambda_2 k^*\cos\theta - \lambda_2 r^*\sin\theta)\cos\theta,$$

and

$$\frac{ds}{ds^*} = \cos\theta(1-\lambda_2 k^*) - \lambda_2 r^*\sin\theta$$

result is reached. On the other hand, if quaternionic inner product is performed with b on both sides of equation (3.8),

$$h(b,t)\frac{ds}{ds^*} = (\cos\theta - \lambda_2 k^*\cos\theta - \lambda_2 r^*\sin\theta)h(b,t)$$
$$+\lambda_2' h(b,n) + (\sin\theta - \lambda_2 k^*\sin\theta + \lambda_2 r^*\cos\theta)h(b,b),$$
$$\sin\theta - \lambda_2 k^*\sin\theta + \lambda_2 r^*\cos\theta = 0$$

obtain. Finally, the following result can be written from the above equation,

$$\tan\theta = \frac{-\lambda_2 r^*}{1-\lambda_2 k^*}.$$

□

Thus, the proof of the theorem is completed by proving it in the final equality.

Theorem 3.7. *Let $\alpha(s)$ be a spatial quaternionic curves in Euclidean space E^3 and $\beta(s^*)$ be the Bertrand curve pair of $\alpha(s)$. The following equality exists*

$$\lambda_1 r^* + \lambda_2 r = 0.$$

Proof. Suppose that $\alpha : I \longrightarrow E^3$ be a quaternionic curve. Taking the derivative of equation (2.1) according to s, we have

$$t^*\frac{ds^*}{ds} = (1-\lambda_1 k)t + \lambda_1' n + \lambda r b. \tag{3.12}$$

If we use the equation (2.3) on the right side of the above equation, we obtain

$$t^*\frac{ds^*}{ds} = (1-\lambda_1 k)(\cos\theta t^* - \sin\theta b^*) + \lambda_1' n^* + \lambda r(\sin\theta t^* + \cos\theta b^*)$$

and

$$t^*\frac{ds^*}{ds} = (\cos\theta - \lambda_1 k\cos\theta + \lambda_1 r\sin\theta)t^* + \lambda_1' n^* + (-\sin\theta + \lambda_1 k\sin\theta + \lambda_1 r\cos\theta)b^*.$$

If quaternonic internal product is performed with b^* on both sides of the equality, the following result can be written,

$$\sin\theta = \lambda_1 k\sin\theta + \lambda_1 r\cos\theta.$$

If the equation (2.4) is used in the equation above, the following result is obtained,

$$\lambda_1 = \frac{\sin\theta}{r^*}. \tag{3.13}$$

On the other hand, taking the derivative of equation (2.2) according to s^*,

$$t\frac{ds}{ds^*} = (1-\lambda_2 k^*)t^* + \lambda_2' n^* + \lambda_2 r^* b^*. \tag{3.14}$$

If we use equation (2.3) on the right side of the above equation, we have

$$t\frac{ds}{ds^*} = (1-\lambda_2 k^*)(\cos\theta t + \sin\theta b) + \lambda_2' n + \lambda r^*(-\sin\theta t + \cos\theta b)$$

and

$$t\frac{ds}{ds^*} = (\cos\theta - \lambda_2 k^*\cos\theta - \lambda_2 r^*\sin\theta)t + \lambda_2' n + (\sin\theta - \lambda_2 k^*\sin\theta + \lambda_2 r^*\cos\theta)b.$$

If both sides of the equality are made the quaternionic inner product with b, we obtain

$$\sin\theta = \lambda_2 k^*\sin\theta - \lambda_2 r^*\cos\theta.$$

On the other hand, if we use equation (2.4) in above equation, the following equation find as

$$\lambda_2 = -\frac{\sin\theta}{r}.$$

Finally, if we write the equation (3.12) in equation (3.13), we complete the proof as

$$\lambda_1 r^* + \lambda_2 r = 0.$$

□

Theorem 3.8. *Let $\alpha(s)$ be a spatial quaternionic curves in Euclidean space E^3 and $\beta(s^*)$ be the Bertrand curve pair of $\alpha(s)$. Then, there is equality below;*

$$s^* = \int \frac{\lambda_1 r}{\sin\theta} ds \quad \textit{and} \quad s = -\int \frac{\lambda_2 r^*}{\sin\theta} ds^*.$$

Proof. If both sides of the equation (3.12) are made the quaternionic inner product with b, we can easily see that

$$\begin{aligned} h(b,t^*)\frac{ds^*}{ds} &= \lambda_1 r, \\ \frac{ds^*}{ds} &= \frac{\lambda_1 r}{\sin\theta}, \\ s^* &= \int \frac{\lambda_1 r}{\sin\theta} ds. \end{aligned}$$

On the other hand, if both sides of the equation (3.12) are made the quaternionic inner product with b^*, we can write

$$\begin{aligned} h(t,b^*)\frac{ds}{ds^*} &= \lambda_2 r^*, \\ \frac{ds}{ds^*} &= -\frac{\lambda_2 r^*}{\sin\theta}, \\ s &= -\int \frac{\lambda_2 r^*}{\sin\theta} ds. \end{aligned}$$

□

Theorem 3.9. *Let $\alpha(s)$ be a spatial quaternionic curves in Euclidean space E^3 and $\beta(s^*)$ be the Bertrand curve pair of $\alpha(s)$. The following equality exists*

$$s^* = s + c, \quad c \in \mathbb{R}.$$

Proof. The following equation can be typed because the $\beta(s^*)$ curve is the Bertrand curve pair of the $\alpha(s)$ curve;

$$n(s) = n^*(s^*).$$

If both sides of the above equality are derivative according to s

$$\begin{aligned} \frac{dn}{ds} &= \frac{dn^*}{ds^*}.\frac{ds^*}{ds}, \\ -kt + rb &= (-k^*t^* + rb^*)\frac{ds^*}{ds} \end{aligned}$$

result is reached. If quaternonic internal product with t is performed on both sides of this result,

$$\begin{aligned} -kh(t,t) + rh(t,b) &= (-k^*h(t,t^*) + rh(t,b^*))\frac{ds^*}{ds}, \\ -k &= -(k^*\cos\theta + r^*\sin\theta)\frac{ds^*}{ds}. \end{aligned}$$

If the equation (2.4) is used in the equation above, this can be seen

$$k = k\frac{ds^*}{ds}$$

and

$$s^* = s + c, \quad c \in \mathbb{R}.$$

□

Theorem 3.10. *Let $\alpha(s)$ be a spatial quaternionic curves in Euclidean space E^3 and $\beta(s^*)$ be the Bertrand curve pair of $\alpha(s)$. k, r and k^*, r^* curvature and torsions of $\alpha(s)$ and $\beta(s^*)$, respectively. The following result is available*

$$\lambda_1 = \frac{k - k^*}{k^2 - r^2}.$$

Proof. Suppose that $\alpha : I \longrightarrow E^3$ be a quaternionic curve. If derivative of the (2.1) equation is taken according to s,

$$t^* \frac{ds^*}{ds} = (1 - \lambda_1 k)t + \lambda_1' n + \lambda_1 r b$$

the result is reached. If a derivative of this equation is taken according to s once again,

$$\begin{aligned} k^* n^* (\frac{ds^*}{ds})^2 + t^* \frac{d^2 s^*}{ds^2} &= (-\lambda_1' k - \lambda_1 k')t + (1 - \lambda_1 k)kn + \lambda_1'' n + \lambda_1'(-kt + rb) + \lambda_1' rb + \lambda_1' r' b - \lambda_1' r^2 n \\ &= [(\lambda_1 k)' + \lambda_1' k)]t + (k - \lambda_1 k^2 - \lambda_1 r^2)n + [(\lambda_1 k)' + \lambda_1' r]b. \end{aligned}$$

If both sides of the above equality are made quaternionic inner product by n,

$$k^* (\frac{ds^*}{ds})^2 = k - \lambda_1 (k^2 - r^2).$$

If the equation $\dfrac{ds^*}{ds} = 1$, is used in this equation, the following equation is easily visible,

$$\lambda_1 = \frac{k - k^*}{k^2 - r^2}.$$

□

Theorem 3.11. *Let $\alpha(s)$ be a spatial quaternionic curves in Euclidean space E^3 and $\beta(s^*)$ be the Bertrand curve pair of $\alpha(s)$. k, r and k^*, r^* curvature and torsions of $\alpha(s)$ and $\beta(s^*)$, respectively. The following result is available*

$$\frac{\lambda_1}{\lambda_2} = \frac{k^* + r^* \cot\theta}{k + r\cot\theta}.$$

Proof. The following equation is available from the (3.12) equation,

$$t^* \frac{ds^*}{ds} = (1 - \lambda_1 k)t + \lambda_1' n + \lambda_1 r b.$$

If equations number (2.3) are used in the equation above,

$$\begin{aligned} (\cos\theta t + \sin\theta b)\frac{ds^*}{ds} &= (1 - \lambda_1 k)t + \lambda_1' n + \lambda_1 r b, \\ 1 - \lambda_1 k &= \cos\theta \frac{ds^*}{ds}, \end{aligned}$$

and

$$\lambda_1 r = \sin\theta \frac{ds^*}{ds}.$$

From the above two equations, we obtain

$$\lambda_1 = \frac{1}{k + r\cot\theta}. \tag{3.15}$$

On the other hand, using the equation (3.14) the following result can be written,

$$t\frac{ds}{ds^*} = (1 - \lambda_2 k^*)t^* + \lambda_2' n^* + \lambda_2 r^* b^*.$$

If we use equation (2.3) in the above equation,

$$\begin{aligned} (\cos\theta t^* - \sin\theta b^*)\frac{ds}{ds^*} &= (1 - \lambda_2 k^*)t^* + \lambda_2' n^* + \lambda_2 r^* b^*, \\ 1 - \lambda_2 k^* &= \cos\theta \frac{ds}{ds^*}, \end{aligned}$$

and

$$\lambda_2 r^* = \sin\theta \frac{ds}{ds^*}$$

result is reached. From the above two equations,

$$\lambda_2 = \frac{1}{k^* + r^* \cot\theta} \tag{3.16}$$

is obtained. Using the equations (3.15) and (3.16), the following equality is easily visible,

$$\frac{\lambda_1}{\lambda_2} = \frac{k^* + r^* \cot\theta}{k + r\cot\theta}.$$

□

Example 3.12. Let $\alpha : I \subset \mathbb{R} \to E^3 \to$ and $\beta : I \subset \mathbb{R} \to E^3$ be a regular with unit speed quaternionic Bertrand curve pair in $\mathbb{R}^3$ parameterized by

$$\alpha(s) = \left(\cos\frac{s}{5},\ \sin\frac{s}{5},\ \frac{\sqrt{24}}{5}s\right)$$

and Frenet elements of the α curve

$$\begin{aligned} t &= \left(-\frac{1}{5}\sin\frac{s}{5},\ \frac{1}{5}\cos\frac{s}{5},\ \frac{\sqrt{24}}{5}\right), \\ n &= \left(-\frac{1}{25}\cos\frac{s}{5},\ -\frac{1}{25}\sin\frac{s}{5},\ 0\right), \\ b &= \left(\frac{\sqrt{24}}{125}\sin\frac{s}{5},\ -\frac{\sqrt{24}}{125}\cos\frac{s}{5},\ \frac{1}{125}\right) \end{aligned}$$

are obtained. Considering the equation number (2.1) here, the Bertrand curve pair of the α curve for $\lambda = 1$ is as follows

$$\beta(s) = \left(\frac{24}{25}\cos\frac{s}{5},\ \frac{24}{25}\sin\frac{s}{5},\ \frac{\sqrt{24}}{5}s\right).$$

The view of the quaternionic Bertrand curve pair below.

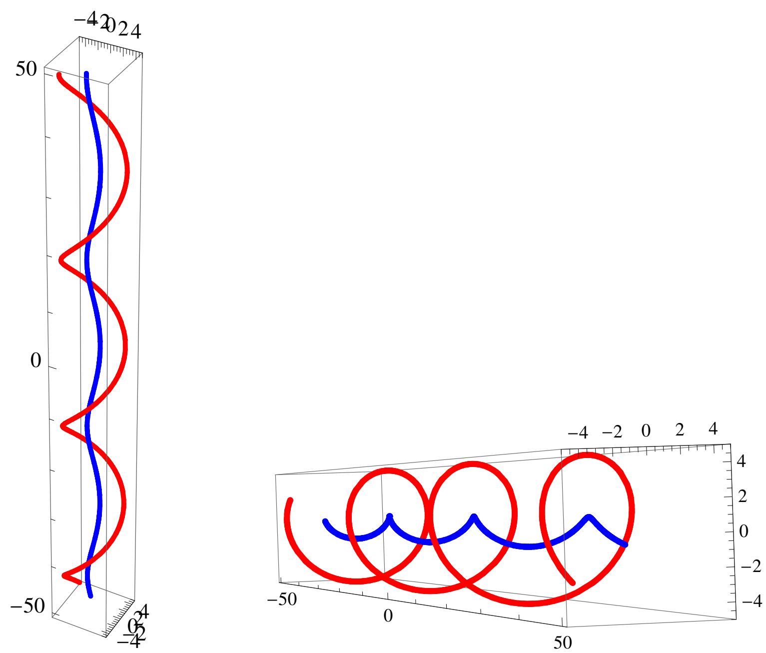

Figure 1. Quaternionic Bertrand curve pair $\alpha(s)$(Red) and $\beta(s)$(Blue)

4. Conclusion

In this study, some characterizations of Bertrand curve pair, which is one of the most popular curve pairs in differential geometry, were obtained using quaternionic properties and finally visualized with the help of examples.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The contribution of all authors is the same.

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