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Existence of ground state solutions of elliptic system in Fractional Orlicz-Sobolev Spaces

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Abstract

We employing a minimization arguments on appropriate Nehari manifolds, we obtain ground state solutions for a non-local elliptic system driven by the fractional $a(.)$-Laplacian operator, with Dirichlet boundary conditions type.

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1. Introduction

The aim of this paper is to study the existence of ground state solutions for the following non-local problem

$$\begin{cases} (-\Delta)^s_{a_1(.)}u = H_u(x,u,v) & in\,\Omega, \\ (-\Delta)^s_{a_2(.)}v = H_v(x,u,v) & in\,\Omega, \\ u = v = 0 & on\ \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is bounded open subset with Lipschitz boundary $\partial\Omega$, $s \in (0,1)$, H_u, H_v denote the partial derivatives of H with respect to the second variable and the third variable and $(-\Delta)^s_{a_i(.)}$ represents the non-local fractional $a_i(.)$-Laplacian operator of elliptic type introduced in [12] and defined as,

$$(-\Delta)^s_{a_i(.)}u(x) = P.V\int_{\mathbb{R}^N} a_i\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|x-y|^s}\frac{dy}{|x-y|^N}, \tag{2}$$

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for all $x \in \mathbb{R}^N$, where $P.V$ is the principal value and $a_{i=1,2} : \mathbb{R}^+ \to \mathbb{R}^+$ are two functions given later. We recall that $a_1(t) = |t|^{p-2}t$, $a_2(t) = |t|^{q-2}t$ (for $1 < q \le p < N$, $N > ps$, $t > 0$), satisfies (ϕ_1)-(ϕ_2) in hypothesis, and the operator (2) is reduced to the well know non-local fractional p-Laplacian operator. Moreover, for the variable exponent case, we find the fractional $p(x,.)$-Laplacian operator which is given by

$$(-\Delta)^s_{p(.)}u(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \setminus B_\epsilon} \frac{|u(x)-u(y)|^{p(x)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}dy \quad for\ x \in \mathbb{R}^N,$$

and the system (1) reduces to the fractional $(p(.), q(.))$-Laplacian system studied in [9] and given as:

$$\begin{cases} (-\Delta)^s_{p(.)}u = H_u(x,u,v) & in\, \Omega, \\ (-\Delta)^s_{q(.)}v = H_v(x,u,v) & in\, \Omega, \\ u = v = 0 & on\ \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3}$$

There has also been a great deal of interest in the existence of solutions for systems like (3). For this reason, we have many researchers in the litterature who studied this type of systems by using some important methods, such as variational method, Nehari manifolds and fibering method, three critical points theorem (see for instance [2, 3, 13]). In addition, this type of operator can be used for many purposes., such as, phase transition phenomena, population dynamics, continuum mechanics, see for example [19, 17, 16]. The fractional operator $(-\Delta)^s_{a_i(.)}$ have also been employed in a number of problems, like, a non-local Kirchhoff problem (see [6]):

$$\begin{cases} K\Big(\int_{\Omega\times\Omega} A\Big(\frac{|u(x)-u(y)|}{|x-y|^s}\Big)\frac{dsdy}{|x-y|^N}\Big)(-\Delta)^s_{a(.)}u = H_u(x,u) & in\, \Omega, \\ u = 0 & on\ \mathbb{R}^N \setminus \Omega, \end{cases}$$

where K is the Kirchhoff function and A is an N-function. As well Azroul et al in [5], by means of Ekeland's variational principal and direct variational approach, they investigate the existence of nontrivial weak solutions for the following non-local problems:

$$\begin{cases} (-\Delta)^s_{a(.)}u + a(|u|)u = \lambda h(x,u) & in\, \Omega, \\ u = 0 & on\ \mathbb{R}^N \setminus \Omega. \end{cases}$$

In [20] we studied the fractional A-Kirchhoff system type, we used the Mountain pass theorem to get a weak solution to the following system

$$\begin{cases} K_1\big[\mathscr{F}_1(u)\big](-\Delta)^s_{a_1(.)}u = H_u(x,u,v) & in\, \Omega, \\ K_2\big[\mathscr{F}_2(v)\big](-\Delta)^s_{a_2(.)}v = H_u(x,u,v) & in\, \Omega, \\ u = v = 0 & on\ \mathbb{R}^N \setminus \Omega, \end{cases}$$

where the functional $\mathscr{F}_{i=1,2} : X^{s,A_i}_0(\Omega) \to \mathbb{R}$ is defined by

$$\mathscr{F}_i(u) = \int_{\Omega^2} A_i\Big(\frac{|u(x)-u(y)|}{|x-y|^s}\Big)\frac{dxdy}{|x-y|^N}, \tag{4}$$

where the space $X^{s,A_i}_0(\Omega)$ is defined as follows

$$X^{s,A_i}_0(\Omega) = \big\{u \in X^{s,A_i}(\mathbb{R}^N) : u = 0\ a.e\ \mathbb{R}^N \setminus \Omega\big\},$$

which equipped by the Gagliardo semi-norm $[.]_{s,A_i}$, given by

$$[u]_{s,A_i} = \inf\Big\{\lambda > 0 : \int_\Omega\int_\Omega A_i\Big(\frac{|u(x)-u(y)|}{\lambda|x-y|^s}\Big)|x-y|^{-N}dxdy \le 1\Big\},$$

and $X^{s,A_i}(\Omega)$ is the fractional Orlicz-Sobolev spaces (see [12]) defined by

$$X^{s,A_i}(\Omega) = \Big\{ u \in L_{A_i}(\Omega) : \int_\Omega \int_\Omega A_i\Big(\frac{\lambda |u(x)-u(y)|}{|x-y|^s}\Big)|x-y|^{-N} dxdy < \infty \Big\}.$$

This space is equipped with the norm,

$$||u||_{s,A_i} = ||u||_{A_i} + [u]_{s,A_i},$$

where $L_A(\Omega)$ is the Orlicz space given by

$$L_A(\Omega) = \Big\{ u : \Omega \to \mathbb{R}\, \text{mesurable function such that } \int_\Omega A\Big(\lambda |u(x)|\Big)\, \mathrm{d}x < +\infty, \lambda > 0 \Big\},$$

equipped with the usual norm $\|u\|_A = \inf\Big\{ \lambda > 0 \, : \, \int_\Omega A\Big(\frac{|u(x)|}{\lambda}\Big)\, \mathrm{d}x \leq 1 \Big\}$.

In this paper, we use the fibering map analysis and the Nehari manifold approach to solve problem (1). The approach is not new but the obtained results are. Our work is motivated by the work of Corrêa et al [15]. The main difficulty in this work arises from the non-homogonitie of the operator (2).

This work is structured as follows. In Section. 2 we briefly recall some properties of Orlicz and fractional Orlicz Sobolev spaces. Section. 3 is devoted to specify the assumptions on data and showing our existence results of problem (1) and its proof.

2. Some preliminary results

The reader is refered to [1, 4, 10, 12, 21] for more details on Orlicz and fractional Orlicz-Sobolev space.

Let's $A \colon \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e. A is a convex even function and is represented as follows:

$$A(t) = \int_0^t a(r)\, \mathrm{d}r,$$

where $a \colon \mathbb{R}^+ \to \mathbb{R}^+$ is satisfies:

$$a(t) > 0 \quad \text{for } t > 0, \qquad \lim_{t\to\infty} a(t) = \infty, \qquad a(0) = 0, \tag{5}$$

and is a non-decreasing and right continuous function.

Let's remember that an N-function A is satisfied a Δ_2-condition ($A \in \Delta_2$), if for some constant $k > 0$,

$$A(2t) \leq k\, A(t) \quad \text{for every } t > 0. \tag{6}$$

Let A and B be two N-functions. The notation $B \ll A$ means that, for each $\varepsilon > 0$,

$$\frac{B(t)}{A(\varepsilon t)} \to 0 \quad \text{as } t \to \infty.$$

Recall that, the Hölder inequality holds

$$\int_\Omega |u(x)v(x)|\, \mathrm{d}x \leq ||u||_A ||v||_{(\overline{A})} \text{ for all } u \in L_A(\Omega) \text{ and } v \in L_{\overline{A}}(\Omega),$$

where $\overline{A}$ is the complementary function of A and given by this relationship

$$\overline{A}(t) := \sup_{r\geq 0}\{tr - A(r)\}.$$

The Young inequality reads as follows:

$$st \leq A(s) + \overline{A}(t) \text{ for all } t,\ s \geq 0. \tag{7}$$

The fact that, $A \in \Delta_2$ implies that

$$u_k \to u \ \ in \ \ L_A(\Omega) \Leftrightarrow \int_\Omega A(u_k - u)dx \to 0. \tag{8}$$

We assume that

$$(H_0)\ \int_0^1 \frac{A_i^{-1}(t)}{t^{1+\frac{s}{N}}}\,\mathrm{d}t < \infty \ \text{ and } (H_\infty) \int_1^{+\infty} \frac{A_i^{-1}(t)}{t^{1+\frac{s}{N}}}\,\mathrm{d}t = +\infty \quad \text{for } s \in (0,1\ ).$$

Under (H_0) and (H_∞), we introduce the conjugate N-function, denoted A^*, given by the following expression of its inverse in $\mathbb{R}^+$:

$$(A_i^*)^{-1}(t) = \int_0^t \frac{A_i^{-1}(r)}{r^{\frac{N+s}{N}}}dr \quad \text{for } t \geq 0. \tag{9}$$

Now, we set some propriaties on the fractional Orlicz-Sobolev spaces.

Theorem 2.1 (Generalized Poincaré inequality)**.** *[7] Let A_i be an N-function. Then there exists a positive constant M such that,*

$$||u||_{A_i} \leq M[u]_{s,A_i} \quad \forall u \in X_0^{s,A_i}(\Omega). \tag{10}$$

Remark 1. If Ω is bounded and A_i be an N-function, then $[u]_{s,A_i}$ is a norm of $X_0^{s,A_i}(\Omega)$ equivalent to $||u||_{s,A_i}$. Moreover, $(X_0^{s,A_i}(\Omega), [u]_{s,A_i})$ is a separable, reflexive Banach space, if and only if $A_i \in \Delta_2$ and $\overline{A}_i \in \Delta_2$ (see [12]). Furthermore if $A_i(\sqrt{t})$ is convex, then the space $(X_0^{s,A_i}(\Omega), [u]_{s,A_i})$ is uniformly convex.

Theorem 2.2. *([7]) Let Ω be a bounded open set in $\mathbb{R}^N$, and $C^{0,1}-$regularity with bounded boundary. If (H_0), (H_∞) and (50) (in appendix) hold true, then the embedding $X^{s,A_i}(\Omega) \hookrightarrow L_{A_i^*}(\Omega)$ is continuous, and the embedding $X^{s,A_i}(\Omega) \hookrightarrow L_B(\Omega)$ is compact for any N-function $B \ll A_i^*$.*

Remark 2. By Lemma 3.12, Lemma 3.14 and (50), we show that A_i, $\overline{A}_i \in \Delta_2$.

Remark 3. Under hypotheses (H_1) and by Theorem 2.2 the following embeddings $X_0^{s,A_i}(\Omega) \hookrightarrow L^{\Psi_i}(\Omega)$ and $X_0^{s,A_i}(\Omega) \hookrightarrow L^{\overline{\Psi}_i}(\Omega)$ are compact.

Remark 4. Based on the Young's inequality (7), $H(x, 0, 0) = 0$ and the fact

$$H(x, u, v) = \int_0^u H_r(x, r, v)dr + \int_0^v H_t(x, 0, t)dt \quad \forall (x, u, v) \in \Omega \times \mathbb{R} \times \mathbb{R}. \tag{11}$$

By (14) and Lemma 3.12 (in appendix), there exists a constant $c_4 > 0$ such that

$$|H(x, u, v)| \leq c_4(\Psi_1(u) + \Psi_2(v)), \quad \forall (x, u, v) \in \Omega \times \mathbb{R} \times \mathbb{R}. \tag{12}$$

We have now all the required tools to examine our problem (1). To do this, we will need to define our work space $X := X_0^{s,A_1}(\Omega) \times X_0^{s,A_2}(\Omega)$ under the norm

$$\begin{aligned} ||(u, v)|| &:= [u]_{s,A_1} + [v]_{s,A_2} \\ &\simeq ||u||_{s,A_1} + ||v||_{s,A_2} \end{aligned}$$

By Remark (1), we can show that X is a separable and reflexive Banach space.

3. Hypothesis and Nehari manifolds approach.

To state our result, we assume the following condition on the function a_i and H:
The functions $a_i : \mathbb{R}^+ \to \mathbb{R}^+$ are of class C^2 and satisfies the following conditions:
(ϕ_1) $\lim_{t\to 0} ta_i(t) = 0$, and $\lim_{t\to\infty} ta_i(t) = \infty$.
(ϕ_2) $t \to ta_i(t)$ is strictly increasing.
(ϕ_3):

$$-1 < l_i - 2 := \inf_{t>0} \frac{(ta_i(t))''t}{(ta_i(t))'} \le \sup_{t>0} \frac{(ta_i(t))''t}{(ta_i(t))'} := n_i - 2 \le \min\{N-2, l_{\Psi_i} - 2, l_{\overline{\Psi}_i} - 2\}.$$

It's pretty obviously if we applying arguments as in [14], we show easily that (ϕ_3) implies the following condition:
$(\phi_3)'$: $1 < l_i := \inf_{t>0} \frac{t^2 a_i(t)}{A_i(t)} \le \sup_{t>0} \frac{t^2 a_i(t)}{A_i(t)} := n_i < \min\{N, l_{\Psi_i}, l_{\overline{\Psi}_i}\}$.
and H satisfies:
(H_0): $H : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function such that $H(x, 0, 0) = 0$ for all $x \in \Omega$.
(H_1): There exists a complementary functions $(\Psi_i, \overline{\Psi}_i)$ increasing essentially more slowly than $A^*_{i=1,2}$ near infinity such that

$$n_i < l_{\Psi_i} \le \inf_{t>0} \frac{t\psi_i(t)}{\Psi_i(t)} \le \sup_{t>0} \frac{t\psi_i(t)}{\Psi_i(t)} := n_{\Psi_i} < \infty, \tag{13}$$

where $\Psi_i(t) := \int_0^{|t|} \psi_i(r)dr$, $\overline{\Psi}_i(t) := \int_0^{|t|} \overline{\psi}_i(r)dr \; for \; all \; t \in \mathbb{R}$. Moreover,

$$\begin{cases} |H_u(x,u,v)| \le c_1(\psi_1(|u|) + \overline{\Psi}_1^{-1}(\Psi_2(v))), \\ |H_v(x,u,v)| \le c_1(\psi_2(|v|) + \overline{\Psi}_2^{-1}(\Psi_1(u))), \end{cases} \tag{14}$$

for all $(x, u, v) \in \Omega \times \mathbb{R} \times \mathbb{R}$, where $c_1 > 0$.
(H_2): $\lim_{|(u,v)|\to+\infty} \frac{H(x,u,v)}{|u|^{n_1} + |v|^{n_2}} = +\infty$, uniformly for all x in Ω.
(H_3): The function

$$(u,v) \mapsto \frac{\min\{H_u(x,u,v), H_v(x,u,v)\}}{|u|^{n_1-2}u + |v|^{n_2-2}v}$$

is increasing on $\mathbb{R}^2 \backslash \{(0,0)\}$.

Let $\mathscr{F}_{i=1;2.}$ be the function defined in (4), then we have the following Lemmas:

Lemma 3.1. *([10] Lemma 4.1) The following properties hold true:*
1)

$$\mathscr{F}_i\left(\frac{u}{[u]_{s,A_i}}\right) \le 1, \quad for\, all\, u \in X_0^{s,A_i} \setminus \{0\},$$

2)

$$\xi_0([u]_{s,A_i}) \le \mathscr{F}(u) \le \xi_1([u]_{s,A_i}), \quad for \quad all \quad u \in X_0^{s,A_i}.$$

Lemma 3.2. *[7] The functional $\mathscr{F}_i$ is weak lower semi-continuous.*

At that point the fractional $a_i(.)$-Laplacian operator defined in (2) is well defined between $X_0^{s,A_i}(\Omega)$ and its dual Space $(X_0^{s,A_i}(\Omega))^*$. In fact, in ([12], Theorem 6.12) the following representation formula is provided

$$\begin{aligned} \langle \mathscr{F}_i'(u), v\rangle &= \int_\Omega \int_\Omega a_i\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{2s}} \frac{dxdy}{|x-y|^N} \\ &= \langle (-\Delta)^s_{a_i} u, v\rangle, \end{aligned} \tag{15}$$

for all $u, v \in X_0^{s,A_i}(\Omega)$.

Lemma 3.3. *[11] Suppose that $A_i(\sqrt{t})$ is convex. Moreover, we assume that the sequence (w_k) converges weakly to w in $X_0^{s,A_i}(\Omega)$ and*

$$\limsup \langle \mathscr{F}_i'(w_k), w_k - w\rangle \leq 0. \tag{16}$$

Then (w_k) converge strongly to $w \in X_0^{s,A_i}(\Omega)$.

We can see that the energy functional I on X corresponding to system (1) is as follows

$$I(u,v) := \mathscr{F}_1(u) + \mathscr{F}_2(v) - \Upsilon(u,v), \qquad for\ \ all \quad (u,v) \in X,$$

where $\Upsilon(u,v) = \displaystyle\int_\Omega H(x,u,v)dx$. Under the assumptions (50) and (H_1), by similar arguments as ([8], lemma 3.2 ;[12], Theorem 6.12) we can prove that I is well-defined and of class $C^1(X,\mathbb{R})$ and

$$\begin{aligned} \langle I'(u,v),(\overline{u},\overline{v})\rangle &= \int_{\Omega\times\Omega} a_1(|h_u|)h_u h_{\overline{u}} d\mu + \int_{\Omega\times\Omega} a_2(|h_v|)h_v h_{\overline{v}} d\mu \\ &- \int_\Omega H_u(x,u,v)\overline{u}dx - \int_\Omega H_v(x,u,v)\overline{v}dx, \end{aligned} \tag{17}$$

for all $(\overline{u},\overline{v}) \in X$, where $h_u = \frac{u(x)-u(y)}{|x-y|^s}$ and $d\mu = \frac{dxdy}{|x-y|^N}$. Then, the critical points of I on X are weak solutions of system (1). However, the energy function I is not bounded below on the whole space X. In fact, using (12) and when $||(v,u)|| > 1$ by Lemma 3.1 and Poincaré's inequality we infer that

$$\begin{aligned} I(u,v) \geq \quad & \min\{||u||_{s,A_1}^{l_1}, ||u||_{s,A_1}^{n_1}\} + \min\{||v||_{s,A_2}^{l_2}, ||v||_{s,A_2}^{n_2}\} \\ - \quad & c_4 \int_\Omega \Psi_1(v)dx - c_4\int_\Omega \Psi_2(v)dx \\ \geq \quad & ||u||_{s,A_1}^{l_1} + ||v||_{s,A_2}^{l_2} - c_4 \max\{||u||_{\Psi_1}^{l_{\Psi_1}}; ||u||_{\Psi_1}^{n_{\Psi_1}}\} - c_4 \max\{||v||_{\Psi_2}^{l_{\Psi_2}}; ||v||_{\Psi_2}^{n_{\Psi_2}}\} \\ \geq \quad & ||u||_{s,A_1}^{l_1} + ||v||_{s,A_2}^{l_2} - c_4||u||_{\Psi_1}^{n_{\Psi_1}} - c_4||v||_{\Psi_2}^{n_{\Psi_2}} \\ \geq \quad & ||u||_{s,A_1}^{l_1} + ||v||_{s,A_2}^{l_2} - c_5||u||_{s,A_1}^{n_{\Psi_1}} - c_5||v||_{s,A_2}^{n_{\Psi_2}}. \end{aligned}$$

Since $1 < l_i < n_{\Psi_i}$, then I is not bounded below on the whole space X. But is bounded below on an appropriate subset $\mathscr{N}$ of X. For that we will minimize the energy functional I on the constraint of Nehari manifold

$$\mathscr{N} = \big\{(u,v) \in X\backslash\{0,0\} : \langle I'(u,v),(u,v)\rangle = 0\big\}.$$

Main result.

The main result of this section is:

Theorem 3.4. *Assume that (ϕ_1)-(ϕ_3) and (H_0)-(H_3) hold. Then the system (1) possesses a nontrivial ground state solution (u,v) in the sense that there is $(u,v) \in \mathscr{N}$ such that*

$$\int_{\Omega\times\Omega} a_1(|h_u|)h_u h_{\overline{u}} d\mu + \int_{\Omega\times\Omega} a_2(|h_v|)h_v h_{\overline{v}} d\mu = \int_\Omega H_u(x,u,v)\overline{u}dx + \int_\Omega H_v(x,u,v)\overline{v}dx,$$

$$I(u,v) = \inf_{(\hat{u},\hat{v})\in\mathscr{N}} I(\hat{u},\hat{v}).$$

In order to prove Theorem 3.4, we require the following Lemmas

Lemma 3.5. *Suppose that (50) and (H_1) holds then we have*

$$(i) \quad \int_\Omega H_u(x,u_k,v_k)(u_k-u)dx \to 0 \quad and \quad (ii) \quad \int_\Omega H_v(x,u_k,v_k)(v_k-v)dx \to 0.$$

Proof. Let $\{(u_k, v_k)\}$ be sequence in X, such that $(u_k, v_k) \rightharpoonup (u, v)$, by Remark 3 we infer that:
$(*)$ $u_k \rightharpoonup u$ in $X_0^{s,A_1}(\Omega)$, $\quad u_k \to u$ in $L^{\Psi_1}(\Omega)$, $\quad u_k \to u$ a.e Ω and $u_k \le h_1$ for some $h_1 \in L^{\Psi_1}(\Omega)$,
$(**)$ $v_k \rightharpoonup v$ in $X_0^{s,A_2}(\Omega)$, $\quad v_k \to v$ in $L^{\Psi_2}(\Omega)$, $\quad v_k \to v$ a.e Ω and $u_k \le h_2$ for some $h_2 \in L^{\Psi_1}(\Omega)$.
Then we have by (H_1) and Höder's inequality

$$\left| \int_\Omega H_u(x, u_k, v_k)(u_k - u)dx \right| \le \quad 2c_1 ||\psi_1(|u_k|) + \overline{\Psi}_1^{-1}(\Psi_2(v_k))||_{\overline{\Psi}_1} ||u_k - u||_{\Psi_1}. \tag{18}$$

Assumption (H_1) shows that the complement $(\Psi_1, \overline{\Psi}_1) \in \Delta_2$, which together with the convexity of N-function, (54), Remark 3, $(*)$ and $(**)$ implies that

$$\int_\Omega \overline{\Psi}_1(\psi_1(|u_k|) + \overline{\Psi}_1^{-1}(\Psi_2(|v_k|)))dx \le C,$$

which, together with (54) again, shows that

$$||\psi_1(|u_k|) + \overline{\Psi}_1^{-1}(\Psi_2(v_k))||_{\overline{\Psi}_1} \le C, \tag{19}$$

for some $C > 0$. Moreover, $(*)$ and (8) shows that

$$||u_k - u||_{\Psi_1} \to 0.$$

Then, combining (19), (40) and (47) (see appendix), we get the item (i). Similarly, we proof item (ii). □

Lemma 3.6. *Assume that (H_3) is hold, then the functions $r \mapsto \frac{1}{G} H_r(x, r, z)r - \overline{H}_r(x, r, z)$ and $z \mapsto \frac{1}{G} H_z(x, r, z)z - \overline{H}_z(x, 0, z)$ are increasing in $(0; \infty)$ for each $x \in \Omega$, where $\overline{H}_r(x, r, z) = \int_0^r H_p(x, p, z)dp$, $\overline{H}_z(x, 0, z) = \int_0^z H_q(x, 0, q)dq$ and $G = \min\{n_1, n_2\}$.*

Proof. By using (H_3) we have

$$\frac{d}{dr}\left(\frac{1}{G} H_r(x, r, z)r - \overline{H}_r(x, r, z)\right) = \frac{t^G}{G}\frac{d}{dr}\left(\frac{H_r(x, r, z)}{r^{G-1}}\right) \ge \frac{t^G}{G}\frac{d}{dr}\left(\frac{H_r(x, r, z)}{r^{G-1} + z^{G-1}}\right) > 0, \tag{20}$$

for all $r > 0$ and $x \in \Omega$. Similary,

$$\frac{d}{dz}\left(\frac{1}{G} H_z(x, r, z)z - \overline{H}_z(x, 0, z)\right) = \frac{t^G}{G}\frac{d}{dz}\left(\frac{H_z(x, 0, z)}{z^{G-1}}\right) > 0, \quad z > 0,\, x \in \Omega, \tag{21}$$

□

At this point, aiming to determited the behavior of I on $\mathscr{N}$ we introduce the fibering map $\Phi_{u,v} : \mathbb{R}^+ \to \mathbb{R}$ associated to the Nehari manifold given by

$$\Phi_{u,v}(t) = I(tu, tv), \quad \textit{for all } (u, v) \in X.$$

Then we get the following properties:

Proposition 5. *Suppose that $(\phi_1) - (\phi_3)$ and $(H_0) - (H_3)$ hold. Then*

$$\frac{\Phi_{u,v}(t)}{t^{n_1} + t^{n_2}} > 0 \quad as \quad t \to 0 \quad and \quad \frac{\Phi_{u,v}(t)}{t^{n_1} + t^{n_2}} = -\infty \quad as \quad t \to \infty.$$

Proof. Let's claim the first limit. By equation (12) there exists $c_4 > 0$ such that

$$|H(x,u,v)| \leq c_4(\Psi_1(u) + \Psi_2(v)),$$

for a given $\epsilon, t \in (0,1)$. By using Lemma 3.1, Lemma 2.3 in [6] and (51) (in appendix), we infer

$$\begin{aligned}\Phi_{u,v}(t) > \ & \frac{t^{n_1}\epsilon}{\lambda_1}\int_\Omega A_1(|u|)dx + \frac{t^{n_2}\epsilon}{\lambda_2}\int_\Omega A_2(|v|)dx - c_4(t^{l_{\Psi_1}} + t^{l_{\Psi_2}}) \\ \times & \ \max\left\{\int_\Omega \Psi_1(u)dx, \int_\Omega \Psi_2(v)dx\right\}.\end{aligned}$$

By the arguments above, we have for all $(u,v) \in X\backslash\{0,0\}$

$$\begin{aligned}\frac{\Phi_{u,v}(t)}{t^{n_1}+t^{n_2}} > & \ \min\left\{\frac{\epsilon}{\lambda_1}\int_\Omega A_1(|u|)dx, \frac{\epsilon}{\lambda_2}\int_\Omega A_2(|v|)dx\right\} - c_4\frac{t^{l_{\Psi_1}} + t^{l_{\Psi_2}}}{t^{n_1}+t^{n_2}} \\ \times & \ \max\left\{\int_\Omega \Psi_1(u)dx, \int_\Omega \Psi_2(v)dx\right\}.\end{aligned}$$

Using the fact that $n_i < l_{\Psi_i}$, then the last inequality rewrites as

$$\frac{\Phi_{u,v}(t)}{t^{n_1}+t^{n_2}} > \ \min\left\{\frac{\epsilon}{\lambda_1}\int_\Omega A_1(|u|)dx, \frac{\epsilon}{\lambda_2}\int_\Omega A_2(|v|)dx\right\} + o(1).$$

Hence

$$\lim_{t\to 0}\frac{\Phi_{u,v}(t)}{t^{n_1}+t^{n_2}} > \ \min\left\{\frac{\epsilon}{\lambda_1}\int_\Omega A_1(|u|)dx, \frac{\epsilon}{\lambda_2}\int_\Omega A_2(|v|)dx\right\} \geq 0.$$

Now for the second limit. By using (51), we have

$$\frac{\Phi_{u,v}(t)}{t^{n_1}+t^{n_2}} \leq \ \max\left\{\int_\Omega A_1(|u|)dx, \int_\Omega A_2(|v|)dx\right\} - \int_\Omega \frac{H(x,tu,tv)}{t^{n_1}+t^{n_2}}dx. \tag{22}$$

Moreover, we can see that

$$\int_\Omega \frac{H(x,tu,tv)}{t^{n_1}+t^{n_2}}dx \ \geq \int_\Omega \frac{H(x,tu,tv)}{|tu|^{n_1}+|tv|^{n_2}} \times \min\{|u|^{n_1} + |v|^{n_2}\}dx \to +\infty. \tag{23}$$

In fact that condition (H_3) and (23) we infer that

$$\int_\Omega \frac{H(x,tu,tv)}{t^{n_1}+t^{n_2}}dx \to +\infty. \tag{24}$$

Combining (24) in (22) we proof the second limit. □

We have already seen that I is of class $C^1(X,\mathbb{R})$, then by using (17), the first derivative of the map $\Phi_{u,v}$ is given by

$$\begin{aligned}\Phi'_{u,v}(t) &= \langle I'(tu,tv),(u,v)\rangle \\ &= \int_{\Omega\times\Omega} ta_1(|th_u|)h_u^2 d\mu + \int_{\Omega\times\Omega} ta_2(|th_v|)h_v^2 d\mu \\ &- \int_\Omega H_u(x,tu,tv)udx - \int_\Omega H_v(x,tu,tv)vdx \quad \text{for all } t > 0.\end{aligned} \tag{25}$$

Then we have the following properties:

Proposition 6. *Suppose that* $(\phi_1)-(\phi_3)$ *and* $(H_0)-(H_3)$ *hold. Then*

$$\frac{\Phi'_{u,v}(t)}{t^{n_1}+t^{n_2}}>0 \quad as \quad t\to 0 \quad and \quad \frac{\Phi'_{u,v}(t)}{t^{n_1}+t^{n_2}}=-\infty \quad as \quad t\to\infty.$$

Proof. For the first limit, let's first show that, by equation (12) there exists $c_1>0$ such that

$$\begin{aligned}\int_\Omega |H_u(x,tu,tv)||u|dx+\int_\Omega |H_v(x,tu,tv)||v|dx \ &\leq c_1\int_\Omega\Big(\psi_1(|tu|)+\overline{\Psi}_1^{-1}(\Psi_2(tv))\Big)|u|dx\\ &+c_1\int_\Omega\Big(\psi_2(|tv|)+\overline{\Psi}_2^{-1}(\Psi_1(tu)\Big)|v|dx,\end{aligned} \tag{26}$$

using Hölder inequality for right side integral we have that,

$$\begin{aligned}\int_\Omega\Big(\psi_1(|tu|)+\overline{\Psi}_1^{-1}(\Psi_2(|tv|))\Big)udx &\leq \frac{1}{t}\int_\Omega\Big(\Psi_1(\psi_1(|tu|))+\overline{\Psi}_1(|tu|)+\Psi_2(|tv|)+\Psi_1(|tu|)\\ &\leq \frac{1}{t}\int_\Omega\Big((n_{\Psi_1}-1)\Psi_1(|tu|)+\overline{\Psi}_1(|tu|)+\Psi_2(|tv|)+\Psi_1(|tu|)\\ &\leq \frac{1}{t}\int_\Omega\Big(n_{\Psi_1}\Psi_1(|tu|)+\overline{\Psi}_1(|tu|)+\Psi_2(|tv|)\Big)dx,\end{aligned}$$

for $0<t<1$ and by applying (51), we infer that

$$\begin{aligned}\int_\Omega\Big(\psi_1(|tu|) \ &+\overline{\Psi}_1^{-1}(\Psi_2(|tv|))\Big)|u|dx \leq n_{\Psi_1}t^{l_{\Psi_1}-1}\int_\Omega\Psi_1(|u|)dx+t^{l_{\overline{\Psi}_1}-1}\int_\Omega\overline{\Psi}_1(|u|)\\ &+t^{l_{\Psi_2}-1}\int_\Omega\Psi_2(|v|)dx.\end{aligned} \tag{27}$$

By same argument above we have

$$\begin{aligned}\int_\Omega\Big(\psi_2(|tv|) \ &+\overline{\Psi}_2^{-1}(\Psi_1(|tu|))\Big)|v|dx \leq n_{\Psi_2}t^{l_{\Psi_2}-1}\int_\Omega\Psi_2(|v|)dx+t^{l_{\overline{\Psi}_2}-1}\int_\Omega\overline{\Psi}_2(|v|)\\ &+t^{l_{\Psi_1}-1}\int_\Omega\Psi_1(|u|)dx.\end{aligned} \tag{28}$$

Combining (27) and (28) in (26) we get that

$$\begin{aligned}\int_\Omega |H_u(x,tu,tv)||u|dx \ &+\int_\Omega |H_v(x,tu,tv)||v|dx \leq c_1(t^{l_{\Psi_1}-1}+t^{l_{\Psi_2}-1})\\ &\times\max\Big\{(n_{\Psi_1}+1)\int_\Omega\Psi_1(|u|)dx,(n_{\Psi_2}+1)\int_\Omega\Psi_2(|v|)dx\Big\}\\ &+c_1(t^{l_{\overline{\Psi}_1}-1}+t^{l_{\overline{\Psi}_2}-1})\max\Big\{\int_\Omega\overline{\Psi}_1(|u|)dx,\int_\Omega\overline{\Psi}_2(|v|)dx\Big\}.\end{aligned} \tag{29}$$

Now using $(\phi_3)'$ and (29) in (25) we have

$$\begin{aligned}\Phi'_{u,v}(t) &\geq \frac{l_1}{t}\int_{\Omega\times\Omega}A_1(|th_u|)d\mu+\frac{l_2}{t}\int_{\Omega\times\Omega}A_2(|th_v|)d\mu-c_1(t^{l_{\Psi_1}-1}+t^{l_{\Psi_2}-1})\\ &\times\max\Big\{(n_{\Psi_1}+1)\int_\Omega\Psi_1(|u|)dx,(n_{\Psi_2}+1)\int_\Omega\Psi_2(|v|)dx\Big\}\\ &-c_1(t^{l_{\overline{\Psi}_1}-1}+t^{l_{\overline{\Psi}_2}-1})\max\Big\{\int_\Omega\overline{\Psi}_1(|u|)dx,\int_\Omega\overline{\Psi}_2(|v|)dx\Big\}.\end{aligned}$$

Now using Lemma 2.3 in [6] and (51), we infer that

$$
\begin{aligned}
\Phi'_{u,v}(t) \geq & \frac{l_1 t^{n_1-1}}{\lambda_1}\int_\Omega A_1(|u|)dx + \frac{l_2 t^{n_2-1}}{\lambda_2}\int_\Omega A_2(|v|)dx - c_1(t^{l_{\Psi_1}-1}+t^{l_{\Psi_2}-1}) \\
& \times \max\Big\{(n_{\Psi_1}+1)\int_\Omega \Psi_1(|u|)dx, (n_{\Psi_2}+1)\int_\Omega \Psi_2(|v|)dx\Big\} \\
& - c_1(t^{l_{\overline{\Psi}_1}-1}+t^{l_{\overline{\Psi}_2}-1})\max\Big\{\int_\Omega \overline{\Psi}_1(|u|)dx, \int_\Omega \overline{\Psi}_2(|v|)dx\Big\}.
\end{aligned}
$$

then,

$$
\begin{aligned}
\frac{\Phi'_{u,v}(t)}{t^{n_1-1}+t^{n_2-1}} \geq & \min\Big\{\frac{l_1}{\lambda_1}\int_\Omega A_1(|u|)dx, \frac{l_2}{\lambda_2}\int_\Omega A_2(|v|)dx\Big\} - \frac{c_1(t^{l_{\Psi_1}-1}+t^{l_{\Psi_2}-1})}{t^{n_1-1}+t^{n_2-1}} \\
& \times \max\Big\{(n_{\Psi_1}+1)\int_\Omega \Psi_1(|u|)dx, (n_{\Psi_2}+1)\int_\Omega \Psi_2(|v|)dx\Big\} \\
& - \frac{c_1(t^{l_{\overline{\Psi}_1}-1}+t^{l_{\overline{\Psi}_2}-1})}{t^{n_1-1}+t^{n_2-1}}\max\Big\{\int_\Omega \overline{\Psi}_1(|u|)dx, \int_\Omega \overline{\Psi}_2(|v|)dx\Big\},
\end{aligned}
$$

the fact that $n_i < \min\{l_{\Psi_i}, l_{\overline{\Psi}_i}\}$ we conclude the result for the first limit. For the second limit, we have

$$
\begin{aligned}
\frac{\Phi'_{u,v}(t)}{t^{n_1-1}+t^{n_2-1}} & \leq \max\Big\{n_1\int_{\Omega\times\Omega} A_1(|h_u|)d\mu, n_2\int_{\Omega\times\Omega} A_2(|h_v|)d\mu\Big\} \\
& - \int_\Omega \frac{H_u(x,tu,tv)|u|}{t^{n_1-1}+t^{n_2-1}}dx - \int_\Omega \frac{H_v(x,tu,tv)|v|}{t^{n_1-1}+t^{n_2-1}}dx.
\end{aligned} \tag{30}
$$

Using (11) we have

$$
\begin{aligned}
\frac{H(x,u,v)}{|u|^{n_1}+|v|^{n_2}} & = \int_0^u \frac{H_r(x,r,v)}{|u|^{n_1}+|v|^{n_2}}dr + \int_0^v \frac{H_z(x,0,z)}{|u|^{n_1}+|v|^{n_2}}dr \\
& \leq \int_0^u \frac{H_r(x,r,v)}{|r|^{n_1-1}+|v|^{n_2-1}} \times \frac{|r|^{n_1-1}+|v|^{n_2-1}}{|r|^{n_1}+|v|^{n_2}}dr \\
& + \int_0^v \frac{H_z(x,0,z)}{|z|^{n_2-1}} \times \frac{|u|^{n_1-1}+|z|^{n_2-1}}{|u|^{n_1}+|z|^{n_2}}dz.
\end{aligned} \tag{31}
$$

We can easly see that

$$
\frac{|u|^{n_1-1}+|z|^{n_2-1}}{|u|^{n_1}+|z|^{n_2}} \leq 1 \quad \textit{and} \quad \frac{|r|^{n_1-1}+|v|^{n_2-1}}{|r|^{n_1}+|v|^{n_2}} \leq 1. \tag{32}
$$

As a conseqence of (H_3) we infer that

$$
\frac{H_z(x,0,z)}{|z|^{n_2-1}} \leq \frac{H_v(x,u,v)}{|u|^{n_1-1}+|v|^{n_2-1}} \quad \textit{and} \quad \frac{H_r(x,r,v)}{|r|^{n_1-1}+|v|^{n_2-1}} \leq \frac{H_u(x,u,v)}{|u|^{n_1-1}+|v|^{n_2-1}}. \tag{33}
$$

It follows using (33), (32) and inequalities (31) that

$$
\begin{aligned}
\frac{H(x,u,v)}{|u|^{n_1}+|v|^{n_2}} & \leq \int_0^u \frac{H_u(x,u,v)}{|u|^{n_1-1}+|v|^{n_2-1}}dr + \int_0^v \frac{H_v(x,u,v)}{|u|^{n_1-1}+|v|^{n_2-1}}dz \\
& = \frac{H_u(x,u,v)u}{|u|^{n_1-1}+|v|^{n_2-1}} + \frac{H_v(x,u,v)v}{|u|^{n_1-1}+|v|^{n_2-1}}.
\end{aligned} \tag{34}
$$

Using (H_2) we conclude that

$$
\frac{H_u(x,u,v)u}{|u|^{n_1-1}+|v|^{n_2-1}} + \frac{H_v(x,u,v)v}{|u|^{n_1-1}+|v|^{n_2-1}} \to +\infty \quad \textit{when } (u,v) \to +\infty.
$$

At that point, last limit just above and (30) give the result of the second limit. □

Lemma 3.7. *Assume that* (ϕ_3), (H_3) *and* (25), *then* $t \mapsto \dfrac{\Phi'_{u,v}(t)}{t^{K-1}}$ *is a decreasing function, where* $K = \max\{n_1, n_2\}$.

Proof. Let's claim that

$$\frac{d}{dt}\Big[\frac{\Phi'_{u,v}(t)}{t^{K-1}}\Big] < 0.$$

Using (25) then we have that,

$$\begin{aligned}\frac{d}{dt}\Big[\frac{\Phi'_{u,v}(t)}{t^{K-1}}\Big] &= \int_{\Omega\times\Omega} \frac{d}{dt}\Big[\frac{ta_1(|th_u|)h_u^2 + ta_2(|th_v|)h_v^2}{t^{K-1}}\Big]d\mu \\ &- \int_{\Omega} \frac{d}{dt}\Big[\frac{H_u(x,tu,tv)u + H_v(x,tu,tv)v}{t^{K-1}}\Big]dx.\end{aligned} \tag{35}$$

Also,

$$\begin{aligned}\frac{d}{dt}\Big[\frac{ta_1(|th_u|)h_u^2 + ta_2(|th_v|)h_v^2}{t^{K-1}}\Big] &= \frac{h_u^2\big(a_1'(th_u)th_u - (K-2)a_1(th_u)\big)t^{K-3}}{t^{2K-4}} \\ &+ \frac{h_v^2\big(a_2'(th_v)th_v - (K-2)a_2(th_v)\big)t^{K-3}}{t^{2K-4}}.\end{aligned} \tag{36}$$

We remark that (ϕ_3) implies

$$\frac{ta_i'(t)}{a_i(t)} \leq K-2 \quad \text{for all } t > 0,$$

which implies that,

$$ta_i'(t) - (K-2)a_i(t) \leq 0. \tag{37}$$

Using inequality (37), we infer that

$$\frac{d}{dt}\Big[\frac{ta_1(|th_u|)h_u^2 + ta_2(|th_v|)h_v^2}{t^{K-1}}\Big] \leq 0.$$

Hence

$$\frac{d}{dt}\Big[\frac{\Phi'_{u,v}(t)}{t^{K-1}}\Big] \leq -\int_{\Omega} \frac{d}{dt}\Big[\frac{H_u(x,tu,tv)u + H_v(x,tu,tv)v}{t^{K-1}}\Big]dx. \tag{38}$$

Wee can see that

$$\begin{aligned}\frac{H_u(x,tu,tv)u + H_v(x,tu,tv)v}{t^{K-1}} &\geq \frac{H_u(x,tu,tv)u + H_v(x,tu,tv)v}{t^{n_1-1} + t^{n_2-1}} \\ &\geq \frac{\min\{H_u(x,tu,tv), H_u(x,tu,tv)\}}{|ut|^{n_1-2}tu + |vt|^{n_2-2}tv} \\ &\times \min\{2|u|^{n_1-2}v^2, 2|v|^{n_2}, 2|u|^{n_1}, 2|v|^{n_2-2}u^2\}.\end{aligned} \tag{39}$$

According to (H_3) and (38) we get

$$\frac{d}{dt}\Big[\frac{\Phi'_{u,v}(t)}{t^{K-1}}\Big] < 0.$$

□

Lemma 3.8. *Assume* (ϕ_1) − (ϕ_3), (H_0) − (H_3). *Then the function* $J_i : X_0^{s,A_i} \to \mathbb{R}$, *given by*

$$J_i(w) = \int_{\Omega\times\Omega} a_i(|h_w|)|h_w|^2 d\mu \quad \textit{for all}\, w \in X_0^{s,A_i}$$

is $C^1(X_0^{s,A_i}, \mathbb{R})$ *and*

$$\langle J_i'(w), v\rangle = \int_{\Omega\times\Omega} \big(2a_i(|h_w|) + a_i'(h_w)|h_w|\big)h_w h_v d\mu \quad \textit{for all}\, w, v \in X_0^{s,A_i}. \tag{40}$$

Proof. The proof is similar to that given in [Proposition 3.5, [15]]. □

Lemma 3.9. *Assume (ϕ_1)-(ϕ_3), (H_1)-(H_3). Then for each $(u,v) \in X\backslash\{(0,0)\}$, there exists an only $t = t(u,v) > 0$ such that $(tu,tv) \in \mathscr{N}$. Moreover, $I(u,v) > 0$ for each $(u,v) \in \mathscr{N}$.*

Proof. Let $(u,v) \in X\backslash\{(0,0)\}$, by the definition of I we have that $\Phi_{u,v} \in C^1(X_0^{s,A_i}, \mathbb{R})$. Furthermore, by proposition 6 we have for a small t that $\Phi'_{u,v}(t) > 0$ and for a large t that $\Phi'_{u,v}(t) < 0$. On the other hand, the map $t \mapsto \Phi'_{u,v}(t)$ is continuous, then there exists at least one number $t \in (0,\infty)$ such that $\Phi'_{u,v}(t) = 0$. Which means that $(tu,tv) \in \mathscr{N}$. Let's see there is only one $t = t(u,v)$ such that $\Phi'_{u,v}(t) = 0$. By Lemma 3.7 we have that, $t \mapsto \dfrac{\Phi'_{u,v}(t)}{t^{K-1}}$ is a decreasing function that vanishes once in $(0,\infty)$ so that there is an only $t = t(u,v) > 0$ such that $\dfrac{\Phi'_{u,v}(t)}{t^{K-1}} = 0$. Hence the function $\Phi_{u,v}$ admits an only critical point namely $t = t(u,v) > 0$ and $(tu,tv) \in \mathscr{N}$. Moreover it follows by Proposition 5 that $t(u,v)$ is a maximum point of $\Phi_{u,v}$ on $(0,\infty)$. In the proof of Proposition 5, we infer that $\Phi_{u,v}(t(u,v)) > 0$, which implies that $I(t(u,v)u, t(u,v)v) > 0$. The arguments above also show that $\Phi''_{u,v}(t) < 0$ for each $(u,v) \in X\backslash\{(0,0)\}$. Finally, since $(u,v) \in \mathscr{N}$ if only if $t(u,v) = 1$, we deduce that $I(u,v) > 0$ for each $(u,v) \in \mathscr{N}$. This completes the proof. □

Lemma 3.10. *Let $(u_k,v_k) \in \mathscr{N}$ be a minimizing sequence of I over the Nehari manifold $\mathscr{N}$. Assume that $(\phi_1)-(\phi_3)$, $(H_0)-(H_3)$ hold true. Then (u_k,v_k) is bounded in X.*

Proof. Let $(u_k,v_k) \in \mathscr{N}$ be a minimizing sequence that is $(u_k,v_k) \in \mathscr{N}$ and $I(u_k,v_k) \to C_{\mathscr{N}}$. In order to demonstrate the boundedness of (u_k,v_k), we argue by contradiction. Let us suppose that there exists a subsequence of (u_k,v_k), always denoted (u_k,v_k), such that $||(u_k,v_k)|| \to +\infty$. We discussed the problem in two cases.

Case1: Suppose that $||u_k||_{s,A_1} \to +\infty$ and also $||v_k||_{s,A_2} \to +\infty$. Let $\overline{u}_k = u_k||u_k||^{-1}_{s,A_1}$ and $\overline{v}_k = v_k||v_k||^{-1}_{s,A_2}$. Then the sequence $(\overline{u}_k,\overline{v}_k)$ is bounded in separable, reflexive Banach space X. By Remark 3, there exists a point $(\overline{u},\overline{v}) \in X$ such that:

(a) $\overline{u}_k \to \overline{u}$ in $L^{\Psi_1}(\Omega)$ and $\overline{u}_k \to \overline{u}$ in a.e in Ω.

(b) $\overline{v}_k \to \overline{v}$ in $L^{\Psi_2}(\Omega)$ and $\overline{v}_n \to \overline{v}$ in a.e in Ω.

We claim that $[\overline{u} \neq 0] := [x \in \Omega : \overline{u}(x) \neq 0] \neq 0$ and $[\overline{v} \neq 0] := [x \in \Omega : \overline{v}(x) \neq 0] \neq 0$. Firstly, assume by the way of contradiction that $[\overline{u} \neq 0] = 0$ and $[\overline{v} \neq 0] = 0$, that is $\overline{u} = 0$ in $X_0^{s,A_1}(\Omega)$ and $\overline{v} = 0$ in $X_0^{s,A_2}(\Omega)$. Now using the continuity of the function H, (a) and (b), we infer that

$$\int_\Omega H(x,\overline{u}_k,\overline{v}_k)dx \to 0 \qquad x \in \Omega. \tag{41}$$

Since $(u_k,v_k) \in \mathscr{N}$ then

$$I(u_k,v_k) = \max_{t>0} I(tu_k,tv_k) \quad \textit{for each } k.$$

Hence there exists a constant $K > 0$ such that,

$$\begin{aligned} C_{\mathscr{N}} + o_k(1) &= I(u_k,v_k) \\ &\geq I(K\overline{u}_k, K\overline{v}_k) = \int_{\Omega\times\Omega} A_1\big(Kh_{\overline{u}_k}\big)d\mu + \int_{\Omega\times\Omega} A_2\big(Kh_{\overline{v}_k}\big)d\mu \\ &- \int_\Omega H(x,K\overline{u}_k,K\overline{v}_k)dx. \end{aligned} \tag{42}$$

Combining (41), (42) and using Lemma 3.1 we get

$$C_{\mathscr{N}} + o_k(1) \geq 2\xi_0(K) + o_k(1). \tag{43}$$

Passing to the limit in (43) we get

$$C_{\mathscr{N}} \geq 2\xi_0(K) > 0, \qquad K > 0.$$

which is impossible. Therfore $[\overline{u} \neq 0] \neq 0$ and $[\overline{v} \neq 0] \neq 0$. Remember we are assuming that $||(u_k, v_k)|| \to +\infty$ and $I(u_k, v_k) \to C_{\mathcal{N}}$. Hence

$$\frac{I(u_k, v_k)}{||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2}} = o_k(1).$$

which implies that,

$$\begin{aligned}\int_\Omega \frac{H(x, u_k, v_k)}{||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2}} dx &= \frac{1}{||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2}} \int_{\Omega\times\Omega} A_1\big(h_{u_k}\big) d\mu \\ &+ \frac{1}{||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2}} \int_{\Omega\times\Omega} A_2\big(h_{v_k}\big) d\mu + o_k(1).\end{aligned}$$

Then by applying Lemma 3.12, we have

$$\begin{aligned}\int_\Omega \frac{H(x, u_k, v_k)}{||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2}} dx &\leq \frac{||u_k||^{n_1}_{s,A_1}}{||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2}} + \frac{||v_k||^{n_2}_{s,A_2}}{||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2}} + o_k(1) \\ &= 1 + o_k(1).\end{aligned}$$

Going to the limit of the last inequality, we deduce that

$$\lim_{k\to\infty} \sup \int_\Omega \frac{H(x, u_k, v_k)}{||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2}} dx \leq 1.$$

On the other hand, applying Fatou's Lemma and the fact that $[\overline{u} \neq 0] \neq 0$ and $[\overline{v} \neq 0] \neq 0$, we have

$$\begin{aligned}\lim_{k\to\infty} \inf \int_\Omega \frac{H(x, u_k, v_k)}{||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2}} dx &\geq \int_\Omega \lim_{k\to\infty} \inf \left\{ \frac{H(x, u_k, v_k)}{||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2}} \right\} dx \\ &\geq \int_\Omega \lim_{k\to\infty} \inf \left\{ \frac{H(x, u_k, v_k)}{|u_k|^{n_1} + |v_k|^{n_2}} \times \frac{|\overline{u}_k|^{n_1} |\overline{v}_k|^{n_2}}{\max\{|\overline{u}_k|^{n_1}, |\overline{v}_k|^{n_2}\}} \right\} dx.\end{aligned}$$

It's clair that $\dfrac{|\overline{u}_k|^{n_1} |\overline{v}_k|^{n_2}}{\max\{|\overline{u}_k|^{n_1}, |\overline{v}_k|^{n_2}\}} \to \min\{|\overline{u}|^{n_1}, |\overline{v}|^{n_2}\} \neq 0$ when $k \to +\infty$. Then by using (H_2) and last inequalities just above we conclude

$$\lim_{k\to\infty} \inf \int_\Omega \frac{H(x, u_k, v_k)}{||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2}} dx = +\infty,$$

which is impossible.
Case2. Suppose that $||u_k||_{s,A_1} \leq C$ or $||v_k||_{s,A_2} \leq C$ for some $C > 0$ and all $k \in \mathbb{N}$. Without loss of generality, we assume that $||u_k||_{s,A_1} \to +\infty$ and $||v_k||_{s,A_2} \leq C$, for some $C > 0$ and for all $k \in \mathbb{N}$. Let $\overline{u}_k = u_k ||u_k||^{-1}_{s,A_1}$ and $\overline{v}_k = v_k ||u_k||^{-1}_{s,A_1}$ then $||\overline{v}_k||_{s,A_2} \to 0$ and $||\overline{u}_k||_{s,A_1} \to 1$. By Remark 3, there exists a point $(\overline{u}, \overline{v}) \in X$ such that:
(c) $\overline{u}_k \rightharpoonup \overline{u}$ in $X^{s,A_1}_0(\Omega)$ and $\overline{u}_k \to \overline{u}$ in a.e in Ω,
(d) $\overline{v}_k \rightharpoonup \overline{v}$ in $X^{s,A_2}_0(\Omega)$; and $\overline{v}_k \to \overline{v}$ in a.e in Ω.
By similar argument in Case 1, we prove that $[\overline{u} \neq 0] \neq 0$. Hence $||(u_k, v_k)|| \to +\infty$ and $I(u_k, v_k) \to C_{\mathcal{N}}$. Then

$$\frac{I(u_k, v_k)}{||u_k||^{n_1}_{s,A_1} + C^{n_1}} = o_k(1).$$

Applying Lemma 3.12, we get

$$\int_\Omega \frac{H(x, u_k, v_k)}{||u_k||^{n_1}_{s,A_1} + C^{n_1}} dx \leq 1 + o_k(1).$$

Passing to the limit in the last inequalities we have

$$\lim_{k\to\infty} \sup \int_\Omega \frac{H(x, u_k, v_k)}{||u_k||^{n_1}_{s,A_1} + C^{n_2}} dx \leq 1.$$

In other way, using Fatou's Lemma and the fact $[\overline{u} \neq 0] \neq 0$, we have

$$\begin{aligned}\liminf_{k\to\infty} \int_\Omega \frac{H(x,u_k,v_k)}{||u_k||_{s,A_1}^{n_1} + C^{n_2}} dx &\geq \int_\Omega \liminf_{k\to\infty} \Bigg\{ \frac{H(x,u_k,v_k)}{|u_k|^{n_1} + |v_k|^{n_2}} \\ &\times \frac{||u_k||_{s,A_1}^{n_2} |\overline{u}_k|^{n_1} |v_k|^{n_2}}{\max\{||u_k||_{s,A_1}^{n_2} |v_k|^{n_2}, C^{n_2}|\overline{u}_k|^{n_1}\}} \Bigg\} dx.\end{aligned}$$

We can see that, when k is large enough we have

• If $\max\{||u_k||_{s,A_1}^{n_2} |v_k|^{n_2}, C^{n_2}|\overline{u}_k|^{n_1}\} = ||u_k||_{s,A_1}^{n_2} |v_k|^{n_2}$, then

$$\frac{||u_k||_{s,A_1}^{n_2} |\overline{u}_k|^{n_1} |v_k|^{n_2}}{\max\{||u_k||_{s,A_1}^{n_2} |v_k|^{n_2}, C^{n_2}|\overline{u}_k|^{n_1}\}} = |u_k|^{n_1} \neq 0. \tag{44}$$

• If $\max\{||u_k||_{s,A_1}^{n_2} |v_k|^{n_2}, C^{n_2}|\overline{u}_k|^{n_1}\} = C^{n_2}|\overline{u}_k|^{n_1}$, remembre that $|v_k| = |\overline{v}_k| ||u_k||_{s,A_1}$ and $||u_k||_{s,A_1} \to 1$ then we have

$$\frac{||u_k||_{s,A_1}^{n_2} |\overline{u}_k|^{n_1} |v_k|^{n_2}}{\max\{||u_k||_{s,A_1}^{n_2} |v_k|^{n_2}, C^{n_2}|\overline{u}_k|^{n_1}\}} = \frac{||u_k||_{s,A_1}^{n_2} |v_k|^{n_2}}{C^{n_2}} = \frac{||u_k||_{s,A_1}^{n_2+1} |\overline{v}_k|^{n_2}}{C^{n_2}} \to \frac{|\overline{v}|^{n_2}}{C^{n_2}} \; for\ k \to +\infty. \tag{45}$$

According to (H_2), (44), (45) and last inequalitities just above we conclude

$$\liminf_{k\to\infty} \int_\Omega \frac{H(x,u_k,v_k)}{||u_k||_{s,A_1}^{n_1} + C^{n_2}} dx = +\infty.$$

which is impossible. Thus (u_k, v_k) is bounded in X. The proof is complete. □

Proposition 7. *Assume* $(\phi_1) - (\phi_3)$, $(H_0) - (H_3)$. *Then* $\mathscr{N}$ *is a* C^1*-submanifold of* X. *i.e, any critical point of* $I_{/\mathscr{N}}$ *is a critical point of* I.

Proof. Consider the functional $J_t : W \to \mathbb{R}$ defined by

$$J_t(u,v) = I_t(u,v) - \int_\Omega H_u(x,tu,tv)u dx - \int_\Omega H_v(x,tu,tv)v dx,$$

where

$$I_t(u,v) = \int_{\Omega\times\Omega} ta_1(th_u)h_u^2 d\mu + \int_{\Omega\times\Omega} ta_2(th_v)h_v^2 d\mu.$$

According to the Lemma 3.8 we can see that $I_t \in C^1$ and by using equation (40) we show that

$$\begin{aligned}\langle I_t'(u,v), (\overline{u},\overline{v})\rangle &= \int_{\Omega\times\Omega} \big(2a_1(|th_u|) + a_1'(|th_u|)|th_u|\big) h_u h_{\overline{u}} d\mu \\ &+ \int_{\Omega\times\Omega} \big(2a_2(|th_v|) + a_2'(|th_v|)|th_u|\big) h_v h_{\overline{v}} d\mu, \quad \overline{u},\overline{v} \in X, t \in \mathbb{R}.\end{aligned}$$

One also shows that

$$\begin{aligned}\Phi_{u,v}''(t) &= \int_{\Omega\times\Omega} \Big[\big(a_1(|th_u|) + a_1'(|th_u|)|th_u|\big)h_u^2 + \big(a_2(|th_v|) + a_2'(|th_v|)|th_v|\big)h_v^2\Big] d\mu \\ &- \int_\Omega (H_u(x,tu,tv))' u dx - \int_\Omega (H_v(x,tu,tv))' v dx.\end{aligned} \tag{46}$$

Now set

$$B(u,v) = \langle I'(u,v), (u,v)\rangle \qquad (u,v) \in X.$$

By using Lemma 3.8, it follows that $B \in C^1$. Furthermore Proposition 5 infer that $t = 1$ is the global maximum of $\Phi_{u,v}$, also in the prove of Lemma 3.9 we see that

$$\langle B'(u,v), (u,v)\rangle = \Phi_{u,v}''(1) < 0, \qquad (u,v) \in \mathscr{N}. \tag{47}$$

Using the fact that $\mathcal{N} = B^{-1}(0)$ and 0 is a regular value for B, the set $\mathcal{N}$ is a C^1-submanifold of X. Now assume that $(u,v) \in \mathcal{N}$ is a critical point of $I_{/\mathcal{N}}$. According to Theorem of Lagrange multiplier, there exist a real constant λ such that

$$I'(u,v) = \lambda B'(u,v),$$
$$\langle I'(u,v),(u,v)\rangle = \lambda \langle B'(u,v),(u,v)\rangle = 0.$$

Using (47) we infer that $\lambda = 0$. Therefore $I(u,v) \equiv 0$, so that (u,v) is a free critical point of I. This completes the proof. □

Lemma 3.11. *Assume that* (ϕ_1)-(ϕ_3), (H_0)-(H_3). *Then there exists* $(u,v) \in \mathcal{N}$ *such that*

$$C_{\mathcal{N}} = I(u,v) > 0.$$

Proof. Let $(u_k,v_k) \in \mathcal{N}$ be a minimizing sequence of I over the Nehari manifold $\mathcal{N}$. By Lemma 3.10, there exists $(u,v) \in X$ such that
(c) $u_k \rightharpoonup u$ in $X_0^{s,A_1}(\Omega)$ and $v_k \rightharpoonup v$ in $X_0^{s,A_2}(\Omega)$.
• We claim that $(u,v) \neq (0,0)$. Assume on the contrary that $(u,v) = (0,0)$. The fact that $(u_k,v_k) \in \mathcal{N}$ and by using $(\phi_3)'$ we have

$$\begin{aligned} 0 &\leq \int_{\Omega\times\Omega} A_1(h_{u_k})d\mu + \int_{\Omega\times\Omega} A_1(h_{u_k})d\mu \\ &\leq \frac{1}{l_1}\int_{\Omega\times\Omega} a_1(h_{u_k})h_{u_k}^2 d\mu + \frac{1}{l_2}\int_{\Omega\times\Omega} a_2(h_{v_k})h_{v_k}^2 d\mu \\ &\leq \max\{\frac{1}{l_1},\frac{1}{l_2}\}\int_\Omega (H_u(x,u_k,v_k)u_k + H_v(x,u_k,v_k)v_k)dx. \end{aligned} \tag{48}$$

Applying Lemma 3.5, we get

$$\int_\Omega (H_u(x,u_k,v_k)u_k + H_v(x,u_k,v_k)v_k)dx = o_k(1).$$

As a consequence of Lemma 3.1, $||u_k,v_k|| \to 0$ which contradicting Lemma 3.15. So the claim is proven. According to the Lemma 3.8 we have $(u,v) \in X \mapsto \langle I'(u,v),(u,v)\rangle$ is weakly lower continuous. Hence

$$\langle I'(u,v),(u,v)\rangle \leq \liminf_{k\to+\infty}\langle I'(u_k,v_k),(u_k,v_k)\rangle = 0.$$

Recall that $\Phi'_{u,v}(1) = \langle I'(u,v),(u,v)\rangle \leq 0$. By Lemma 3.9 there is $t \in (0,\infty)$ such that $\Phi'_{u,v}(tu,tv) = 0$. Hence $(tu,tv) \in \mathcal{N}$.
• Claim that $t = 1$ so that (u,v) is in $\mathcal{N}$.
Furthermore, by contrary we assume that $t \in (0;1)$. Then

$$\begin{aligned} C_{\mathcal{N}} &\leq I(tu,tv) = I(tu,tv) - \frac{1}{G}\langle I'(tu,tv),(tu,tv)\rangle \\ &= \int_{\Omega\times\Omega}\Big(A_1(th_u) - \frac{1}{G}a_1(th_u)(th_u)^2\Big)d\mu + \int_{\Omega\times\Omega}\Big(A_2(th_v) - \frac{1}{G}a_2(th_v)(th_v)^2\Big)d\mu \\ &+ \int_\Omega\big(\frac{1}{G}H_u(x,tu,tv)tu - \overline{H}_u(x,tu,tv)\big)dx + \int_\Omega\big(\frac{1}{G}H_v(x,tu,tv)tv - \overline{H}_v(x,0,tv)\big)dx, \end{aligned} \tag{49}$$

where $\overline{H}_u$ and $\overline{H}_v$ is given in Lemma 3.6. The fact that, the functions

$$t \mapsto A_i(t) - \frac{1}{G}a_i(|t|)t^2 \quad \text{are increasing in } (0;\infty),$$

and by using Lemma 3.6, we conclude that

$$\begin{aligned} C_{\mathcal{N}} &< \int_{\Omega\times\Omega}\Big(A_1(h_u) - \frac{1}{G}a_1(h_u)(h_u)^2\Big)d\mu + \int_{\Omega\times\Omega}\Big(A_2(h_v) - \frac{1}{G}a_2(h_v)(h_v)^2\Big)d\mu \\ &+ \int_\Omega\big(\frac{1}{G}H_u(x,u,v)u - \overline{H}_u(x,u,v)\big)dx + \int_\Omega\big(\frac{1}{G}H_v(x,u,v)v - \overline{H}_v(x,0,v)\big)dx. \end{aligned}$$

Now using the weak lower continuity of the functions

$$u \mapsto \int_{\Omega\times\Omega} \Big(A_i(h_u) - \frac{1}{K} a_i(h_u)(h_u)^2\Big) d\mu$$

and (H_0) we infer that

$$\begin{aligned} C_{\mathcal{N}} &< \liminf_{k\to+\infty} \Big[\int_{\Omega\times\Omega} \Big(A_1(h_{u_k}) - \frac{1}{G} a_1(h_{u_k})(h_{u_k})^2\Big) d\mu + \int_{\Omega\times\Omega} \Big(A_2(h_{v_k}) - \frac{1}{G} a_2(h_{v_k})(h_{v_k})^2\Big) d\mu \\ &+ \int_\Omega \big(\frac{1}{} H_u(x,u_k,v_k)u_k - \overline{H}_u(x,u_k,v_k)\big) dx + \int_\Omega \big(\frac{1}{G} H_v(x,u_k,v_k)v_k - \overline{H}_v(x,0,v_k)\big) dx \Big] \\ &= \lim_{k\to\infty} \big(I(u_k,v_k) - \frac{1}{G}\langle I'(u_k,v_k),(u_k,v_k)\rangle\big) = C_{\mathcal{N}}. \end{aligned}$$

Which impossible. Thus $t = 1$, then $(u, v) \in \mathcal{N}$. This finishes the proof. Now we have all tools to prove our Theorem 3.4.

Proof of Theorem 3.4

Let $(u_k, v_k) \in \mathcal{N}$ be a minimizing sequence for I over $\mathcal{N}$. By the proof of Lemma 3.10 there is $(u, v) \in \mathcal{N} \subset W$ such that

$$u_k \rightharpoonup u \quad in\ X_0^{s,A_1}(\Omega) \quad and \quad v_k \rightharpoonup v \quad in\ X_0^{s,A_2}(\Omega).$$

Applying Lemma 3.3 we get

$$u_k \to u \quad in\ X_0^{s,A_1}(\Omega) \quad and \quad v_k \to v \quad in\ X_0^{s,A_2}(\Omega).$$

Since $I \in C^1(X, \mathbb{R})$, it follows that $I'(u_k, v_k) \to I'(u, v)$. By Lemma 3.11, $(u, v) \in \mathcal{N}$ and

$$C_{\mathcal{N}} = I(u,v) = \min_{\mathcal{N}} I > 0.$$

By Proposition 7 the set $\mathcal{N}$ is a C^1-submanifold of X so that (u, v) is a critical point of $I_{/\mathcal{N}}$. Again, the proposition 7 shows that (u, v) is a critical point of I. □

Appendix.

In this section we give some inequalities which will be used in our proofs. the proof is given in [18].

Lemma 3.12. *Let $\xi_0(t) = \min\{t^{l_i}, t^{n_i}\}$, $\xi_1(t) = \max\{t^{l_i}, t^{n_i}\}$ and A_i is an N-function then these assertions are equivalent:*
1)

$$1 < l_i := \inf_{t>0} \frac{ta_i(t)}{A_i(t)} \le \sup_{t>0} \frac{ta_i(t)}{A_i(t)} := n_i < N. \tag{50}$$

2)

$$\xi_0(t)A_i(\rho) \le A_i(\rho t) \le \xi_1(t)A_i(\rho), \qquad \forall t,\rho \ge 0. \tag{51}$$

3) A_i satisfies a Δ_2-condition.

Lemma 3.13. *If A_i is an N-function satisfies (50) then we have*

$$\xi_0(||u||_{A_i}) \le \int_\Omega A_i(|u|)dx \le \xi_1(||u||_{A_i}), \qquad \forall u \in L_{A_i}(\Omega). \tag{52}$$

Lemma 3.14. *Let $\overline{A}_i$ be the complement of A_i and $\xi_2(t)=\min\{t^{\bar{l}_i},t^{\overline{n}_i}\}$, $\xi_3(t)=\max\{t^{\bar{l}_i},t^{\overline{n}_i}\}$, $t\geq 0$ where $\bar{l}_i=\frac{l_i}{l_i-1}$ and $\overline{n}_i=\frac{n_i}{n_i-1}$, If A_i is an N-function and (50) hold, then $\overline{A}_i$ satisfies:*
1)

$$\xi_2(t)\overline{A}_i(r)\leq \overline{A}_i(rt)\leq \xi_3(t)\overline{A}_i(r),\qquad \forall t,r\geq 0. \tag{53}$$

2)

$$\xi_2(||u||_{\overline{A}_i})\leq \int_\Omega \overline{A}_i(|u|)dx\leq \xi_3(||u||_{\overline{A}_i}),\qquad \forall u\in L_{\overline{A}_i}(\Omega). \tag{54}$$

Proposition 8. *Assume that $(\phi_3)'$ holds. If $||(u_k,v_k)||\leq \frac{1}{k}$ with a large k, then we have*

$$\lim_{k\to\infty}\frac{||u_k||_{s,A_1}^{l_{\Psi_1}}+||v_k||_{s,A_2}^{l_{\Psi_2}}}{||u_k||_{s,A_1}^{n_1}+||v_k||_{s,A_2}^{n_2}}+\frac{||u_k||_{s,A_1}^{\bar{l}_{\Psi_1}}+||v_k||_{s,A_2}^{\bar{l}_{\Psi_2}}}{||u_k||_{s,A_1}^{n_1}+||v_k||_{s,A_2}^{n_2}}=0.$$

Proof. The fact that $||(u_k,v_k)||=||u_k||_{s,A_1}+||v_k||_{s,A_2}\leq\frac{1}{k}$ implies

$$\frac{||u_k||_{s,A_1}^{l_{\Psi_1}}+||v_k||_{s,A_2}^{l_{\Psi_2}}}{||u_k||_{s,A_1}^{n_1}+||v_k||_{s,A_2}^{n_2}}\leq\frac{\frac{1}{k^{l_{\Psi_1}}}+\frac{1}{k^{l_{\Psi_2}}}}{\frac{1}{k^{n_1}}+\frac{1}{k^{n_2}}}=\frac{k^{l_{\Psi_1}}+k^{l_{\Psi_2}}}{k^{n_1}+k^{n_2}}\times\frac{k^{n_1+n_2}}{k^{l_{\Psi_1}+l_{\Psi_2}}} \tag{55}$$

and

$$\frac{||u_k||_{s,A_1}^{\bar{l}_{\Psi_1}}+||v_k||_{s,A_2}^{\bar{l}_{\Psi_2}}}{||u_k||_{s,A_1}^{n_1}+||v_k||_{s,A_2}^{n_2}}\leq\frac{\frac{1}{k^{\bar{l}_{\Psi_1}}}+\frac{1}{k^{\bar{l}_{\Psi_2}}}}{\frac{1}{k^{n_1}}+\frac{1}{k^{n_2}}}=\frac{k^{\bar{l}_{\Psi_1}}+k^{\bar{l}_{\Psi_2}}}{k^{n_1}+k^{n_2}}\times\frac{k^{n_1+n_2}}{k^{\bar{l}_{\Psi_1}+\bar{l}_{\Psi_2}}}. \tag{56}$$

Combining (55) and (56) we get

$$\begin{aligned}\frac{k^{l_{\Psi_1}}+k^{l_{\Psi_2}}}{k^{n_1}+k^{n_2}} &\times\frac{k^{n_1+n_2}}{k^{l_{\Psi_1}+l_{\Psi_2}}}+\frac{k^{\bar{l}_{\Psi_1}}+k^{\bar{l}_{\Psi_2}}}{k^{n_1}+k^{n_2}}\times\frac{k^{n_1+n_2}}{k^{\bar{l}_{\Psi_1}+\bar{l}_{\Psi_2}}}=\frac{k^{n_1+n_2}}{k^{n_1}+k^{n_2}}\Big(\frac{k^{l_{\Psi_1}}+k^{l_{\Psi_2}}}{k^{l_{\Psi_1}+l_{\Psi_2}}}\\ &+\frac{k^{\bar{l}_{\Psi_1}}+k^{\bar{l}_{\Psi_2}}}{k^{\bar{l}_{\Psi_1}+\bar{l}_{\Psi_2}}}\Big)\\ &=\frac{1}{k^{-n_1}+k^{-n_2}}\Big(k^{-l_{\Psi_1}}+k^{-l_{\Psi_2}}+k^{-\bar{l}_{\Psi_1}}+k^{-\bar{l}_{\Psi_2}}\Big)\\ &=\frac{1}{k^{l_{\Psi_1}-n_1}+k^{l_{\Psi_1}-n_2}}+\frac{1}{k^{l_{\Psi_2}-n_1}+k^{l_{\Psi_2}-n_2}}+\frac{1}{k^{\bar{l}_{\Psi_1}-n_1}+k^{\bar{l}_{\Psi_1}-n_2}}\\ &+\frac{1}{k^{\bar{l}_{\Psi_2}-n_1}+k^{\bar{l}_{\Psi_2}-n_2}}.\end{aligned} \tag{57}$$

The fact that $n_i<l_{\Psi_i},\bar{l}_{\Psi_i}$ we get the result when $k\to\infty$. □

Lemma 3.15. *Assume that $(\phi_1)-(\phi_3)$, $(H_1)-(H_3)$. Then there exists a constant $C>0$ such that $||u,v||\geq C$ for each $(u,v)\in\mathscr{N}$.*

Proof. The prove is arguing by contradiction, suppose that there exists a subsequence denoted by $(u_k,v_k)\in\mathscr{N}$, such that $||(u_k,v_k)||\leq\frac{1}{k}$, for each integer $k\geq 1$. Using (29) with $t=1$ and $(\phi_3)'$ then we have

$$\begin{aligned}\int_{\Omega\times\Omega}A_1(|h_{u_k}|)d\mu &+\int_{\Omega\times\Omega}A_2(|h_{v_k}|)d\mu\\ &\leq\frac{1}{l_1}\int_{\Omega\times\Omega}a_1(h_{u_k})h_{u_k}^2d\mu+\frac{1}{l_2}\int_{\Omega\times\Omega}a_2(h_{v_k})h_{v_k}^2d\mu\\ &\leq\max\{\frac{1}{l_1},\frac{1}{l_2}\}\int_\Omega(H_u(x,u_k,v_k)u_k+H_v(x,u_k,v_k)v_k)dx\\ &\leq c_2\Big[\max\{(n_{\Psi_1}+1)\xi_1(||u_k||_{\Psi_1}),(n_{\Psi_2}+1)\xi_1(||v_k||_{\Psi_2})\}\\ &+\max\{\xi_3(||u_k||_{\overline{\Psi}_1}),\xi_3(||v_k||_{\overline{\Psi}_2})\}\Big],\end{aligned}$$

where $c_2 = 2c_1 \max\left\{\frac{1}{l_1}, \frac{1}{l_2}\right\}$. According to the Lemma 3.12, Remark 3 and last inequality we have for k large enough

$$\begin{aligned} ||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2} &\leq c_2 \max\{(n_{\Psi_1}+1)||u_k||^{l_{\Psi_1}}_{\Psi_1}, (n_{\Psi_2}+1)||v_k||^{l_{\Psi_2}}_{\Psi_2}\} \\ &+ \max\{||u_k||^{\bar{l}_{\Psi_1}}_{\overline{\Psi}_1}, ||v_k||^{\bar{l}_{\Psi_2}}_{\overline{\Psi}_2}\} \\ &\leq c_2 \max\{C_{\Psi_1}(n_{\Psi_1}+1)||u_k||^{l_{\Psi_1}}_{s,A_1}, C_{\Psi_2}(n_{\Psi_2}+1)||v_k||^{l_{\Psi_2}}_{s,A_2}\} \\ &+ \max\{C_{\overline{\Psi}_1}||u_k||^{\bar{l}_{\Psi_1}}_{s,A_1}, C_{\overline{\Psi}_2}||v_k||^{\bar{l}_{\Psi_2}}_{s,A_2}\}. \\ &\leq c_2 \max\{C_{\Psi_1}(n_{\Psi_1}+1), C_{\Psi_2}(n_{\Psi_2}+1)\}\big(||u_k||^{l_{\Psi_1}}_{s,A_1} + ||v_k||^{l_{\Psi_2}}_{s,A_2}\big) \\ &+ c_2 \max\{C_{\overline{\Psi}_1}, C_{\overline{\Psi}_2}\}\big(||u_k||^{\bar{l}_{\Psi_1}}_{s,A_1} + ||v_k||^{\bar{l}_{\Psi_2}}_{s,A_2}\big). \end{aligned} \tag{58}$$

Dividing the last expression by $||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2}$ we get to

$$\begin{aligned} 1 &\leq c_3 \frac{||u_k||^{l_{\Psi_1}}_{s,A_1} + ||v_k||^{l_{\Psi_2}}_{s,A_2}}{||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2}} + c_4 \frac{||u_k||^{\bar{l}_{\Psi_1}}_{s,A_1} + ||v_k||^{\bar{l}_{\Psi_2}}_{s,A_2}}{||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2}} \\ &\leq c_5 \Big(\frac{||u_k||^{l_{\Psi_1}}_{s,A_1} + ||v_k||^{l_{\Psi_2}}_{s,A_2}}{||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2}} + \frac{||u_k||^{\bar{l}_{\Psi_1}}_{s,A_1} + ||v_k||^{\bar{l}_{\Psi_2}}_{s,A_2}}{||u_k||^{n_1}_{s,A_1} + ||v_k||^{n_2}_{s,A_2}} \Big), \end{aligned} \tag{59}$$

where $c_3 = c_2 \max\{C_{\Psi_1}(n_{\Psi_1}+1), C_{\Psi_2}(n_{\Psi_2}+1)\}$, $c_4 = c_2 \max\{C_{\overline{\Psi}_1}, C_{\overline{\Psi}_2}\}$ and $c_5 = \max\{c_3, c_4\}$. The fact that $n_i < l_{\Psi_i}, \bar{l}_{\Psi_i}$ and by applying Proposition 8, we get for a large k,

$$1 \leq c_5 o_k(1),$$

which is a contarduction. □

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