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Research Article

Essential self-adjointness for covariant tri-harmonic operators on manifolds and the separation problem

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Abstract

Consider the tri-harmonic differential expression $L_V^{\nabla} u = (\nabla^+ \nabla)^3 u + Vu$, on sections of a hermitian vector bundle over a complete Riemannian manifold (M, g) with metric g, where ∇ is a metric covariant derivative on bundle E over complete Riemannian manifold, ∇^+ is the formal adjoint of ∇ and V is a self adjoint bundle on E. We will give conditions for L_V^{∇} to be essential self-adjoint in $L^2(E)$. Additionally, we provide sufficient conditions for L_V^{∇} to be separated in $L^2(E)$. According to Everitt and Giertz, the differential operator L_V^{∇} is said to be separated in $L^2(E)$ if for all $u \in L^2(E)$ such that $L_V^{\nabla} u \in L^2(E)$, we have $Vu \in L^2(E)$.

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1. Introduction

Assume that (M, g) be a smooth Riemannian manifold without boundary and $dimM = n$, let M is connected, with metric g, and with Riemannian volume element d_μ. Assume that E be a vector bundle over M. The study of essential self-adjointness for differential operators on R^n has been the central theme of numerous studies, such as [11] and [30]. Gaffney studied essential self-adjointness for differential operators on Riemannian manifolds in [15]. This problem has lead to many works, such as [4, 5, 10, 17, 18, 21, 25]. The study of the separation property for Schrodinger operators on R^n was studied through Everitt and Giertz, see [15]. The operator $-\triangle + V$ in $L^p(R^n)$ is separated if the following condition is satisfied for all $u \in L^p(R^n)$ such that $(\triangle + V)u \in L^p(R^n)$, we have that $\triangle u \in L^p(R^n)$ and $Vu \in L^p(R^n)$. For the separation problem of second and higher order differential operators, see [1, 2, 6, 7, 9, 13, 27, 28]. The separation problem of the differential operator $\triangle_M + V$ on $L^2(M)$ where M is a non-compact Riemannian manifold, $\triangle_M$ is the scalar laplacian on M and $V \in C^1(M)$, was studied in [23]. Milatovic was studied the separation property for $\triangle_M + V$ in $L^p(M)$ in [24]. The separation problem for

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Schrodinger operators on R^n goes back to the work of Everitt and Giertz in [14]. Some authors have studied the separation problem for differential operators on Riemannian manifolds, see [2, 24, 25]. The separation property is linked to the self adjointness in $L^2(M)$, see [26]. Separation problem of differential operators has strong links with the essential self-adjointness of the underlying operator. In this article, we will give the conditions for essentially self-adjointness of $\Delta_A^3 + V$ on $C_c^\infty(M)$, where Δ_A^3 be the (non-negative) magnetic tri-Laplacian (with a smooth magnetic field A) on a geodescially complete Riemannian manifold M and $0 \le V \in C^2(M)$. Additionally, we provide sufficient conditions for L_V^∇ to be separated in $L^2(E)$.

2. General notations

In this article we consider the differential operator $(\nabla^+\nabla)^3 u + Vu$, where ∇ is a metric covariant derivative on a hermitian bundle E over a Riemannian manifold M, ∇^+ its formal adjoint and V is a linear self-adjoint bundle map over E. In general, the symbols $C^\infty(E)$ and $C_c^\infty(E)$ and $\Omega^1(M)$ denote sections of E and compactly supported sections of E, and complex-valued smooth 1-forms on M respectively. We call $L^p(E)$, $1 \le p < \infty$, indicates the space of p-integrable sections of E with the norm $\|u\|_p := (\int_M |u(x)|^p \, d\mu)^{1/p}$. In the special case $p = 2$, we have the Hilbert space $L^2(E)$ and we use $(.,.)$ to denote the corresponding inner product in $L^2(M)$ and the pairing (linear in the first and anti-linear in the second slot) between $L^p(M)$ and $L^q(M)$ with $1/p + 1/q = 1$. For local Sobolev spaces of sections we use the notation $W_{loc}^{k,p}(E)$, with p and k indicating the corresponding L^p spaces and the highest order of derivatives, respectively. For $k = 0$ we use $L_{loc}^p(E)$. In the case $E = M \times C$, we denote the corresponding function spaces by $C^\infty(M)$, $C_c^\infty(M)$, $L^p(M)$, $L_{loc}^p(M)$ and $W_{loc}^{k,p}(M)$. In this paper, $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$ stands for a smooth metric covariant derivative on E, and $\nabla^+ : C^\infty(T^*M \otimes E) \to C^\infty(E)$ indicates the formal adjoint of ∇ with respect to $(.,.)$. The covariant derivative ∇ on E induces the covariant derivative ∇^{End} on the bundle of endomorphisms $End\ E$, making $\nabla^{End} V$ a section of the bundle $T^*M \otimes (End\ E)$. Working in the space $L^2(E)$ only, we find it convenient to indicate by $(.,.)$ and $\|.\|$ the inner product and the norm in the spaces $L^2(E)$ and $L^2(T^*M \otimes E)$. We study the separation property on $L^2(E)$ that can be seen as an extension of the work mentioned in [23]. We define a set $D_2^\nabla := \{u \in L^2(E) : L_V^\nabla u \in L^2(E\}$, it is not true that for all $u \in D_2^\nabla$, we have $(\nabla^+\nabla)^3 u \in L^2(E)$ and $Vu \in L^2(E)$ simultaneously, using the terminology of Everitt and Giertz [15] we will say that $L_V^\nabla = (\nabla^+\nabla)^3 + V$ is separated in $L^2(E)$ when the following statement holds true for all $u \in D_2^\nabla$, we have $Vu \in L^2(E)$, to make the notations less cumbersome, the symbols $(.,.)$ and $\| \, . \, \|_p$will also be used when referring to $L^p(\Lambda^1 T^* M)$, the space of p-integrable 1-forms on M, we only consider the space $L^2(M)$, we will use $\| \, . \, \|$ instead of $\| \, . \, \|_2$to indicate the norm. In a magnetic field $A \in \Omega^1(M)$, where A be real-valued form, the operator $d_A : C^\infty(M) \to \Omega^1(M)$ stands for the magnetic differential where $d_A u = du + iu\mathrm{A}$ where $d : \ C^\infty(M) \to \Omega^1(M)$ be the standard differential and $i = \sqrt{-1}$. We denote the formal adjoint of d_A with respect to $(\cdot, \cdot)$ by d_A^+, the (non-negative) magnetic Laplacian on M by $\Delta_A := d_A^+ d_A$, and the magnetic tri-Laplacian by $\Delta_A^3 := \left(d_A^+ d_A\right)^3$. We start recalling some abstract terminology concerning m-accretive operators on Banach spaces. A linear operator S on a Banach space $\varkappa$ is called accretive if $\|(\xi + S)u\|_\varkappa \ge \xi \|u\|_\varkappa$, for all $\xi > 0$ and all $u \in Dom(S)$. In [12], a densely defined accretive operator S is close and its closure $S^\sim$ is also accretive. An operator S on $\varkappa$ is called m-accretive if it is accretive and $\xi + S$ is surjective for all $\xi > 0$. An operator S on $\varkappa$ is named essentially m-accretive if it is accretive and $S^\sim$ is m-accretive. We use the relation between m-accretivity and self-adjointness of operators on Hilbert spaces which stated in the paper [22] that the operator S is a self-adjoint and non-negative operator if and only if S is symmetric, closed and m-accretive. We mentioned some results on the essential m-accretivity of operators in $L^2(E)$ used in this paper with L_V^{∇} with $0 \leq V \in L_{loc}^{\infty}(End\ E)$ and $0 \leq v \in L_{loc}^{\infty}(M)$, we define an operator $H_{2.V}^{\nabla}$ as $H_{2.V}^{\nabla}u := L_V^{\nabla}u$ with the domain $D_2^{\nabla} := \{u \in L^2(E) : L_V^{\nabla}u \in L^2(E\}$ and an operator $H_{2.v}^d$ as $H_{2.v}^d u := S_v^d u$ for all $u \in D_2^d$, where $D_2^d := \left\{u \in L^2(M) : S_v^d u \in L^2(M)\right\}$. Atia was studied the separation problem of bi-harmonic differential operators on Riemannian manifolds in [3, 4].

3. Essential self-adjointness result for a perturbation of Δ_A^3

Let $p(t, x, y)$ be the heat kernel of M. For a Borel function $f : M \to R$, define

$$F(t) := \sup_{x\in M} \int_0^t \int_M p(s, x, y)|f(y)|d\mu(y)ds,$$

where $f \in D(M)$ if there exists $t > 0$ such that $F(t) < 1$. We say that f belongs to the Kato class $K(M)$ of M if $F(t) \to 0$ as $t \to 0+$. So $K(M) \subset D(M)$, see Theorem 7.13 in [16].

Now, we remind our main result, in this section.

Theorem 3.1. *Let M is a geodesically complete connected Riemannian manifold. Let $U = U_1 + U_2$ with $0 \leq U_1 \in L_{loc}^2(M)$ and $0 \leq U_2 \in L_{loc}^2(M) \cap D(M)$. Furthermore, let $W \in W_{loc}^{3,2}(M)$ and $W \geq 0$. Also there exist constants $c_1 \geq 0$ and $c_2 \geq 0$ such that*

$$|dW(x)|^2 \leq c_1 + c_2 W(x), \tag{3.1}$$

for all almost $x \in M$. Then $\Delta_A^3 + U - W$ is essentially self-adjoint on $C_c^{\infty}(M)$.

Now, we explain some notations used in subsequent results.

We define operators $T_{\Delta_A}^{(p)}$ and $T_{\Delta_A^3}^{(p)}$ in $L^p(M)$ where $1 < p < \infty$ by the formulas

$$T_{\Delta_A}^{(p)}u := \Delta_A u,\ u \in Dom\left(T_{\Delta_A}^{(p)}\right) := \{u \in L^p(M) : \Delta_A u \in L^p(M)\}, \tag{3.2}$$

and

$$T_{\Delta_A^3}^{(p)}u := \Delta_A^3 u,\ u \in Dom\left(T_{\Delta_A^3}^{(p)}\right) := \left\{u \in L^p(M) : \Delta_A^3 u \in L^p(M)\right\}. \tag{3.3}$$

We now state the result of Okazawa, see [29].

Lemma 3.2. *Let S and G be nonnegative symmetric operators in a Hilbert space H with inner product $(.,.)_H$ and norm $\|\ .\ \|_H$. We assume D be a linear subspace of H on which $S+G$ is essentially self-adjoint. Assume that the following inequalities hold for all $u \in D$:*

$$\|Su\|_H + \|Gu\|_H \leq a_1 \|u\|_H + a_2 \|(S+G)u\|_H \tag{3.4}$$

and

$$|\mathrm{Im}\,(Gu, Su)_H| \leq \widetilde{a_1} \|u\|_H^2 + \widetilde{a_2} \|(S+G)u\|_H \|u\|_H, \tag{3.5}$$

where a_1, a_2, $\widetilde{a_1}$ and $\widetilde{a_2} \geq 0$ are constants. Then $S - G$ is essentially self-adjoint on D.

The following lemma is a direct consequence of Lemma $VI.7$ in [20] and Corollary 2.5 in [19]:

Lemma 3.3. *Let M be a geodesically complete connected Riemannian manifold with Riemannian volume element $d\mu$. Let $0 \geq U_2 \in L_{loc}^2(M) \cap D(M)$. Then, there exist constants $0 \leq \delta < 1$ and $\xi \geq 0$ such that*

$$\int_M |U_2||u|^2 d\mu \leq \delta \int_M |d_A u|^2 d\mu + \xi \|u\|^2, \tag{3.6}$$

for all $u \in W_A^{3,2}(M)$.

We will introduce the following lemma which will be used in the proof of the main theorem in this section,

Lemma 3.4. *Let M be a geodesically complete connected Riemannian manifold with Riemannian volume element $d\mu$. Let $U = U_1 + U_2$ with $0 \le U_1 \in L^2_{loc}(M)$ and $0 \ge U_2 \in L^2_{loc}(M) \cap D(M)$. Additionally, let $0 \le W \in W^{3,2}_{loc}(M)$ is a function satisfying* (3.1). *Then we have*

$$2\,\mathrm{Re}\left((T_W + 1)\,u, \left(T_{\Delta^3_A} + T_U + \xi\right) u\right) \ge -c_3 \left\| d^+_A d_A u \right\|^2, \tag{3.7}$$

for all $u \in C^\infty_c(M)$, where ξ is as in (3.6) *and $c_3 := (c_1 + c_2)/2$.*

Proof. We will use the integration by parts and the product rule

$$d_A(fv) = f d_A v + (df)\,v,$$

where f and v are functions on M, also we define

$$W^{1,3}_A(M) := \left\{u \in L^2(M) : d_A u \in L^2\left(\Lambda^1 T^* M\right)\right\}$$

$$\begin{aligned}
\mathrm{Re}(T_{\Delta^3_A} u, (T_W + 1)u) &= \mathrm{Re}(\Delta^3_A u, (W + 1)u) \\
&= \mathrm{Re}(\left(d^+_A d_A\right)^3 u, (W + 1)u) \\
&= \mathrm{Re}\left(d_A\left(d^+_A d_A\right)^2 u, d_A\left(\sqrt{W+1}\sqrt{W+1}u\right)\right) \\
&= \mathrm{Re}\left(d_A\left(d^+_A d_A\right)^2 u, \sqrt{W+1}\, d_A\left(\sqrt{W+1}u\right)\right) \\
&+ \mathrm{Re}\left(d_A\left(d^+_A d_A\right)^2 u, \left(d\sqrt{W+1}\right)\left(\sqrt{W+1}u\right)\right) \\
&= \mathrm{Re}\left(\sqrt{W+1}\, d_A\left(d^+_A d_A\right)^2 u, d_A\left(\sqrt{W+1}u\right)\right) \\
&+ \mathrm{Re}\left(d_A\left(d^+_A d_A\right)^2 u, \frac{dW}{2\sqrt{W+1}}\sqrt{W+1}u\right) \\
&= \mathrm{Re}\left(\sqrt{W+1}\, d_A\left(d^+_A d_A\right)^2 u, d_A\left(\sqrt{W+1}u\right)\right) \\
&+ \frac{1}{2}\mathrm{Re}\left(d_A\left(d^+_A d_A\right)^2 u, dWu\right) \\
&= \mathrm{Re}\left(\begin{array}{c} d_A\left(\sqrt{W+1}\left(d^+_A d_A\right)^2 u\right) \\ -d\left(\sqrt{W+1}\right)\left(d^+_A d_A\right)^2 u, d_A\left(\sqrt{W+1}u\right) \end{array}\right) \\
&+ \frac{1}{2}\mathrm{Re}\left(d_A\left(d^+_A d_A\right)^2 u, dWu\right) \\
&= \mathrm{Re}\left(d_A\left(\sqrt{W+1}\left(d^+_A d_A\right)^2 u\right), d_A\left(\sqrt{W+1}u\right)\right) \\
&- \mathrm{Re}\left(d\left(\sqrt{W+1}\right)\left(d^+_A d_A\right)^2 u, d_A\left(\sqrt{W+1}u\right)\right) \\
&+ \frac{1}{2}\mathrm{Re}\left(d_A\left(d^+_A d_A\right)^2 u, dWu\right) \\
&= \mathrm{Re}\left(\sqrt{W+1}\left(d^+_A d_A\right)^2 u, d^+_A d_A\left(\sqrt{W+1}u\right)\right) \\
&- \mathrm{Re}\left(\frac{dW}{2\sqrt{W+1}}\left(d^+_A d_A\right)^2 u, d_A\left(\sqrt{W+1}u\right)\right) \\
&+ \frac{1}{2}\mathrm{Re}\left(d_A\left(d^+_A d_A\right)^2 u, dWu\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{W+1}\,\mathrm{Re}\left(u, \left(d_A^+ d_A\right)^3 \left(\sqrt{W+1}u\right)\right) \\
&- \frac{1}{2}\,\mathrm{Re}\left(dW \left(d_A^+ d_A\right)^2 u\,(W+1)^{-1/2}, d_A\left(\sqrt{W+1}u\right)\right) \\
&+ \frac{1}{2}\,\mathrm{Re}\left(d_A\left(d_A^+ d_A\right)^2 u, dWu\right) \\
&= \mathrm{Re}\left(\left(d_A^+ d_A\right)^{3/2}\left(\sqrt{W+1}u\right), \left(d_A^+ d_A\right)^{3/2}\left(\sqrt{W+1}u\right)\right) \\
&- \frac{1}{2}\,\mathrm{Re}\left(dW \left(d_A^+ d_A\right)^2 u\,(W+1)^{-1/2}, \sqrt{W+1}d_A u + d\sqrt{W+1}u\right) \\
&+ \frac{1}{2}\,\mathrm{Re}\left(d_A\left(d_A^+ d_A\right)^2 u, dWu\right) \\
&= \left\|\left(d_A^+ d_A\right)^{3/2}\left(\sqrt{W+1}u\right)\right\|^2 \\
&- \frac{1}{2}\,\mathrm{Re}\left(dW \left(d_A^+ d_A\right)^2 u\,(W+1)^{-1/2}, (W+1)^{1/2} d_A u\right) \\
&- \frac{1}{2}\,\mathrm{Re}\left(dW \left(d_A^+ d_A\right)^2 u\,(W+1)^{-1/2}, d\sqrt{W+1}u\right) \\
&+ \frac{1}{2}\,\mathrm{Re}\left(d_A\left(d_A^+ d_A\right)^2 u, dWu\right) \\
&= \left\|\left(d_A^+ d_A\right)^{3/2}\left(\sqrt{W+1}u\right)\right\|^2 \\
&- \frac{1}{2}\,\mathrm{Re}\left(dW \left(d_A^+ d_A\right)^2 u, d_A u\right) \\
&- \frac{1}{2}\,\mathrm{Re}\left(dW \left(d_A^+ d_A\right)^2 u\,(W+1)^{-1/2}, \frac{dW}{2\sqrt{W+1}}u\right) \\
&+ \frac{1}{2}\,\mathrm{Re}\left(d_A\left(d_A^+ d_A\right)^2 u, dWu\right) \\
&= \left\|\left(d_A^+ d_A\right)^{3/2}\left(\sqrt{W+1}u\right)\right\|^2 \\
&- \frac{1}{2}\,\mathrm{Re}\left(dW \left(d_A^+ d_A\right)^2 u, d_A u\right) \\
&- \frac{1}{4}\,\mathrm{Re}\left(\frac{dW\left(d_A^+ d_A\right)^2 u}{\sqrt{W+1}}, \frac{dW}{\sqrt{W+1}}u\right) \\
&+ \frac{1}{2}\,\mathrm{Re}\left(d_A\left(d_A^+ d_A\right)^2 u, dWu\right) \\
&= \left\|\left(d_A^+ d_A\right)^{3/2}\left(\sqrt{W+1}u\right)\right\|^2 \\
&- \frac{1}{4}\int_M \frac{\left|d_A^+ d_A u\right|^2}{W+1}\,|dW|^2\,d\mu \\
&- \frac{1}{2}\,\mathrm{Re}\left(dW \left(d_A^+ d_A\right)^2 u, d_A u\right) \\
&+ \frac{1}{2}\,\mathrm{Re}\left(d_A\left(d_A^+ d_A\right)^2 u, dWu\right) \\
&= \left\|\left(d_A^+ d_A\right)^{3/2}\left(\sqrt{W+1}u\right)\right\|^2 - \frac{1}{4}\int_M \frac{\left|d_A^+ d_A u\right|^2}{W+1}\,|dW|^2\,d\mu.
\end{aligned}
$$

We use our assumptions on W, we get $\left(\sqrt{W+1}u\right) \in W_A^{3,2}(M)$, we combine the last equality, (3.1) and (3.6) we get

$$
\begin{aligned}
&\operatorname{Re}\left((T_W+1)\,u, \left(T_{\Delta_A^3}+T_U\right)u\right) \\
&= \operatorname{Re}\left((T_W+1)\,u, T_{\Delta_A^3}u\right) + \operatorname{Re}\left((T_W+1)\,u, T_U u\right) \\
&= \operatorname{Re}\left(T_{\Delta_A^3}u, (T_W+1)\,u\right) + \operatorname{Re}\int_M U_1 \left|u\sqrt{W+1}\right|^2 d\mu \\
&+ \operatorname{Re}\int_M U_2 \left|u\sqrt{W+1}\right|^2 d\mu \\
&\geq \left\|\left(d_A^+ d_A\right)^{3/2}\left(\sqrt{W+1}u\right)\right\|^2 - \frac{1}{4}\int_M \frac{\left|d_A^+ d_A u\right|^2}{W+1}\left|dW\right|^2 d\mu \\
&- \delta\left\|\left(d_A^+ d_A\right)^{3/2}\left(\sqrt{W+1}u\right)\right\|^2 - \xi\left\|\sqrt{W+1}u\right\|^2 \\
&= (1-\delta)\left\|\left(d_A^+ d_A\right)^{3/2}\left(\sqrt{W+1}u\right)\right\|^2 - \frac{1}{4}\int_M \frac{\left|d_A^+ d_A u\right|^2}{W+1}\left|dW\right|^2 d\mu \\
&- \xi\left\|\sqrt{W+1}u\right\|^2 \\
&\geq -\xi\left\|\sqrt{W+1}u\right\|^2 - \frac{c_1+c_2}{4}\left\|d_A^+ d_A u\right\|^2.
\end{aligned}
$$

So we proved the lemma. □

3.1. Proof of theorem 3.1

We assume that the hypotheses of Theorem 3.1 are satisfied, we assume that M is a geodesically complete connected Riemannian manifold with Riemannian volume element d_μ. Let $U = U_1 + U_2$ with $0 \leq U_1 \in L^2_{loc}(M)$ and $0 \geq U_2 \in L^2_{loc}(M) \cap D(M)$, let $0 \leq W \in W^{3,2}_{loc}(M)$ is a function satisfying (3.1). We get $\left(T_{\Delta_A^3}+T_U+\xi\right)_{C_c^\infty(M)}$ is a non-negative symmetric operator. By the assumptions on W, it follows that $(T_W+1)_{C_c^\infty(M)}$ is a non-negative symmetric operator. Since $0 \leq (U_1+W) \in L^2_{loc}(M)$ also $U_2 \in L^2_{loc}(M) \cap D(M)$ by using Theorem X.1 in [21] to conclude that the operator $\left(T_{\Delta_A^3}+T_U+T_W+1+\xi\right)$ is essentially self-adjoint on $C_c^\infty(M)$. By using (3.7) we get

$$
\begin{aligned}
&\left\|(T_{\Delta_A^3}+T_U+T_W+1+\xi)u\right\|^2 + c_3\left\|u\right\|^2 \\
&= \left((T_{\Delta_A^3}+T_U+T_W+1+\xi)u, (T_{\Delta_A^3}+T_U+T_W+1+\xi)u\right) \\
&+ c_3\left\|u\right\|^2 \\
&= \left\|\left(T_{\Delta_A^3}+T_U+\xi\right)u\right\|^2 + 2\operatorname{Re}\left((T_W+1)\,u, \left(T_{\Delta_A^3}+T_U+\xi\right)u\right) \\
&+ \left\|(T_W+1)u\right\|^2 + c_3\left\|u\right\|^2 \\
&\geq \left\|\left(T_{\Delta_A^3}+T_U+\xi\right)u\right\|^2 + \left\|(T_W+1)u\right\|^2,
\end{aligned}
$$

for all $u \in C_c^\infty(M)$. Applying hypothesis (3.4) of Lemma 3.2 is satisfied with $S = T_{\Delta_A^3} + T_U + \xi, G = T_W + 1$ and $D = C_c^\infty(M)$, we will use the integration by parts and the product rule, for all $u \in C_c^\infty(M)$ we have

$$\begin{aligned}\operatorname{Im}\left(T_W u, T_{\Delta_A^3} u\right) &= \operatorname{Im}\left(Wu, \left(d_A^+ d_A\right)^3 u\right)\\ &= \operatorname{Im}\left(d_A(Wu), d_A\left(d_A^+ d_A\right)^2 u\right)\\ &= \operatorname{Im}\left(W d_A u + dWu, d_A\left(d_A^+ d_A\right)^2 u\right)\\ &= \operatorname{Im}\left(W d_A u, d_A\left(d_A^+ d_A\right)^2 u\right) + \operatorname{Im}\left(dWu, d_A\left(d_A^+ d_A\right)^2 u\right)\\ &= \operatorname{Im}\left(dWu, d_A\left(d_A^+ d_A\right)^2 u\right). \end{aligned} \tag{3.8}$$

From (3.1) and (3.8) we obtain

$$\begin{aligned}\left|\operatorname{Im}\left(T_W u, T_{\Delta_A^3} u\right)\right| &= \left|\operatorname{Im}\left(dWu, d_A\left(d_A^+ d_A\right)^2 u\right)\right|\\ &\le \frac{1}{2}\int_M |dW|^2 |u|^2 d\mu + \frac{1}{2}\int_M \left|d_A\left(d_A^+ d_A\right)^2 u\right|^2 d\mu\\ &\le \frac{c_1}{2}\|u\|^2 + \frac{c_2}{2}\int_M W|u|^2 d\mu + \frac{1}{2}\left\|d_A\left(d_A^+ d_A\right)^2 u\right\|^2\\ &\le \frac{c_1}{2}\|u\|^2 + \frac{c_2}{2}(u, T_W u) + \frac{1}{2}\left(u, T_{\Delta_A^5} u\right)\\ &\le \frac{c_1}{2}\|u\|^2 + \frac{c_{2+1}}{2}\left(u, \left(T_{\Delta_A^5} + T_W\right) u\right)\end{aligned}$$

For all $u \in C_c^\infty(M)$ we get

$$\begin{aligned}\left|\operatorname{Im}\left(T_W u, \left(T_{\Delta_A^3} + T_U + \xi\right) u\right)\right| &= \left|\operatorname{Im}\left(T_W u, T_{\Delta_A^3} u\right)\right|\\ &\le \frac{c_1}{2}\|u\|^2 + \frac{c_{2+1}}{2}\left(u, \left(T_{\Delta_A^5} + T_W\right) u\right). \end{aligned} \tag{3.9}$$

From (3.6) we obtain

$$\begin{aligned}(u, T_U u) = (u, (U_1 + U_2) u) &\ge (u, U_2 u)\\ &\ge -\delta(u, \Delta_A u) - \xi\|u\|^2,\end{aligned}$$

for all $u \in C_c^\infty(M)$ we get

$$\begin{aligned}\left(u, \left(T_{\Delta_A^3} + T_U\right) u\right) + \xi\|u\|^2 &= \left(u, T_{\Delta_A^3} u\right) + (u, T_U u) + \xi\|u\|^2\\ &= \left(u, T_{\Delta_A^3} u\right) + \xi\|u\|^2 + (u, Uu)\\ &\ge \left(u, T_{\Delta_A^3} u\right) + \xi\|u\|^2 - \delta\int |d_A u|^2 d\mu - \xi\|u\|^2\\ &= \left(u, T_{\Delta_A^3} u\right) - \delta\left(u, \left(d_A^+ d_A\right)^3 u\right)\\ &= (1-\delta)\left(u, T_{\Delta_A^3} u\right), \end{aligned} \tag{3.10}$$

for all $u \in C_c^\infty(M)$ from (3.9), (3.10) and since

$$\operatorname{Im}\left(u, \left(T_{\Delta_A^3} + T_U + \xi\right) u\right) = 0.$$

We get

$$
\begin{aligned}
\left|\operatorname{Im}\left(\left(T_W+1\right)u,\left(T_{\Delta_A^3}+T_U+\xi\right)u\right)\right| &= \left|\operatorname{Im}\left(T_W u,\left(T_{\Delta_A^3}+T_U+\xi\right)u\right)\right| \\
&\leq \frac{c_1}{2}\|u\|^2+\frac{c_{2+1}}{2}\left(u,\left(T_{\Delta_A^5}+T_W\right)u\right) \\
&= \frac{c_1}{2}\|u\|^2+\frac{c_{2+1}}{2}\left(u,T_W u\right)+\frac{c_{2+1}}{2}\left(u,T_{\Delta_A^5}u\right) \\
&\leq \frac{c_1}{2}\|u\|^2+\frac{c_{2+1}}{2}\left(u,T_W u\right) \\
&+\frac{c_{2+1}}{2\left(1-\delta\right)}\left(u,\left(T_{\Delta_A^5}+T_U\right)u\right) \\
&+\left(\frac{c_{2+1}}{2}\right)\left(\frac{\xi}{1-\delta}\right)\|u\|^2 \\
&\leq \left(\frac{c_1}{2}+\left(\frac{c_{2+1}}{2}\right)\left(\frac{\xi}{1-\delta}\right)\right)\|u\|^2 \\
&+\frac{c_{2+1}}{2\left(1-\delta\right)}\left(u,\left(T_{\Delta_A^5}+T_U+T_W+1+\xi\right)u\right).
\end{aligned}
$$

Hence the assumptions of (3.5) of Lemma 3.2 is satisfied with $S=T_{\Delta_A^3}+T_U+\xi$ and $G=T_W+1$ and $D=C_c^{\infty}\left(M\right)$, Thus by Lemma 3.2 it follows that $S-G=T_{\Delta_A^3}+T_U+\xi-T_W-1$ is essentially self-adjoint on $C_c^{\infty}\left(M\right)$, since $\xi-1$ is a constant, then Δ_A^3+U-W is essentially self-adjoint on $C_c^{\infty}\left(M\right)$.

4. The separation problem result

We will introduce our main theorem in this section.

Theorem 4.1. *Let (M,g) be a complete connected Riemannian manifold without boundary, let E be a vector bundle over M with a metric covariant derivative ∇. We assume $V\in C^1\left(End\ E\right)$, $V\left(x\right)\geq 0$, for all $x\in M$, where the inequality is understood in the sense of linear operators $E_x\to E_x$ and*

$$\left|\left(\nabla^{End}V\left(x\right)\right)\right|\leq\sigma\left(V\left(x\right)\right)^{3\backslash 2},\ 0\leq\sigma<1. \tag{4.1}$$

Then

$$\left\|\left(\nabla^{+}\nabla\right)^{3}u\right\|+\|Vu\|\leq C\ \left[\left\|L_V^{\nabla}u\right\|+\|u\|\right], \tag{4.2}$$

for all $u\in D_2^{\nabla}$, where $C\geq 0$ be a constant, that is L_V^{∇} is separated in $L^2\left(E\right)$.

Lemma 4.2. *Under the hypothesis of the Theorem 3.1, then the following inequalities are valid for all $u\in C_c^{\infty}\left(E\right)$,*

$$\left\|\left(\nabla^{+}\nabla\right)^{3}u\right\|+\|Vu\|\leq C_1\left\|L_V^{\nabla}u\right\|, \tag{4.3}$$

and

$$\left\|V^{1/2}\left(\nabla^{+}\nabla\right)^{3/2}u\right\|\leq C_1\left\|L_V^{\nabla}u\right\|, \tag{4.4}$$

where $V^{1/2}$ is the square root of the operator $V\left(x\right):E_x\to E_x$ and C_1 is a constant depending on $n=\dim M$, $m=\dim E_x$ and σ.

Proof. By the definition of L_V^∇, for all $\delta > 0$ and all $u \in C_c^\infty(E)$ we obtain

$$\begin{aligned}
\left\|L_V^\nabla u\right\|^2 &= \left\|\left(\nabla^+\nabla\right)^3 u + Vu\right\|^2 \\
&= \left(\left(\nabla^+\nabla\right)^3 u + Vu, \left(\nabla^+\nabla\right)^3 u + Vu\right) \\
&= \|Vu\|^2 + \left\|\left(\nabla^+\nabla\right)^3 u\right\|^2 + 2\operatorname{Re}\left(\left(\nabla^+\nabla\right)^3 u, Vu\right) \\
&= \|Vu\|^2 + \left\|\left(\nabla^+\nabla\right)^3 u\right\|^2 + 2\operatorname{Re}\left(\left(\nabla^+\nabla\right)^3 u, Vu\right) + \delta\left\|\left(\nabla^+\nabla\right)^3 u\right\|^2 \\
&- \delta\left\|\left(\nabla^+\nabla\right)^3 u\right\|^2 \\
&= \|Vu\|^2 + \delta\left\|\left(\nabla^+\nabla\right)^3 u\right\|^2 + (1-\delta)\left\|\left(\nabla^+\nabla\right)^3 u\right\|^2 \\
&+ 2\operatorname{Re}\left(\left(\nabla^+\nabla\right)^3 u, Vu\right) \\
&= \|Vu\|^2 + \delta\left\|\left(\nabla^+\nabla\right)^3 u\right\|^2 + (1-\delta)\operatorname{Re}\left(\nabla^+\nabla\right)^3 u, L_V^\nabla u - Vu) \\
&+ 2\operatorname{Re}\left(\nabla^+\nabla\right)^3 u, Vu) \\
&= \|Vu\|^2 + \delta\left\|\left(\nabla^+\nabla\right)^3 u\right\|^2 + (1-\delta)\operatorname{Re}\left(\nabla^+\nabla\right)^3 u, L_V^\nabla u) \\
&+ (1+\delta)\operatorname{Re}(\left(\nabla^+\nabla\right)^3 u, Vu)).
\end{aligned}$$

By the product rule $\nabla(Vu) = \left(\nabla^{End}V\right)u + V\nabla u$, so for all $u \in C_c^\infty(E)$, we have

$$\begin{aligned}
\operatorname{Re}(\left(\nabla^+\nabla\right)^3 u, Vu) &= \operatorname{Re}(\left(\nabla^+\nabla\right)^2 u, \left(\nabla^+\nabla\right)(Vu)) \\
&= \operatorname{Re}(\left(\nabla^+\nabla\right)^2 u, \nabla^+(\nabla^{End}V)u + \nabla^+V\nabla u) \\
&= \operatorname{Re}(\left(\nabla^+\nabla\right)^2 u, \nabla^+(\nabla^{End}V)u) + \operatorname{Re}(\left(\nabla^+\nabla\right)^2 u, \nabla^+V\nabla u) \\
&= \operatorname{Re}(Z) + W, \qquad (4.5)
\end{aligned}$$

where $Z := ((\nabla^+\nabla)^2 u, \nabla^+(\nabla^{End}V)u)$ and $W := \left(V^{1/2}(\nabla^+\nabla)^{3/2}u, V^{1/2}(\nabla^+\nabla)^{3/2}u\right)$, then, we obtain

$$(1+\delta)\operatorname{Re}(\left(\nabla^+\nabla\right)^3 u, Vu) = (1+\delta)\operatorname{Re}Z + (1+\delta)W \geq -(1+\delta)|Z| + (1+\delta)W. \quad (4.6)$$

By Cauchy-Schwartz $2ab \leq ka^2 + k^{-1}b^2$, where k, a and b are positive real numbers and the condition (4.1) we obtain

$$\begin{aligned}
|Z| &\leq (\sigma+1)\int_M \left|\left(\nabla^+\nabla\right)^{3/2} V^{1/2}u\right|_{(T^*M\otimes E)_x} |Vu|_{E_x}\, d\mu, \\
|Z| &\leq \frac{\delta\alpha}{2}\left\|V^{1/2}\left(\nabla^+\nabla\right)^{3/2}u\right\|^2 + \frac{(\sigma+1)^2}{2\delta\alpha}\|Vu\|^2, \qquad (4.7)
\end{aligned}$$

for all $\alpha > 0$, we use Cauchy-Schwartz again, we obtain

$$\left|\operatorname{Re}(\left(\nabla^+\nabla\right)^3 u, L_V^\nabla u)\right| \leq |\,(\left(\nabla^+\nabla\right)^3 u, L_V^\nabla u)\,| \leq \frac{\gamma}{2}\left\|\left(\nabla^+\nabla\right)^3 u\right\|^2 + \frac{1}{2\gamma}\left\|L_V^\nabla u\right\|^2, \quad (4.8)$$

for all $\gamma > 0$, we obtain

$$\begin{aligned}\left\|L_V^{\nabla}u\right\|^2 \geq \|Vu\|^2 + \delta\left\|\left(\nabla^{+}\nabla\right)^3 u\right\|^2 \\ - \frac{(1+\delta)\,\delta\alpha}{2}\left\|V^{1/2}\left(\nabla^{+}\nabla\right)^{3/2}u\right\|^2 - \frac{(1+\delta)\,(\sigma+1)^2}{2\delta\alpha}\|Vu\|^2 \\ + (1+\delta)\left\|V^{1/2}\left(\nabla^{+}\nabla\right)^{3/2}u\right\|^2 - \frac{\mid 1-\delta \mid \gamma}{2}\left\|\left(\nabla^{+}\nabla\right)^3 u\right\|^2 \\ - \frac{\mid 1-\delta \mid}{2\gamma}\left\|L_v^{\nabla}u\right\|^2,\end{aligned}$$

from this, we obtain

$$\begin{aligned}\left(1+\frac{\mid 1-\delta \mid}{2\gamma}\right)\left\|L_V^{\nabla}u\right\|^2 \geq \left(1-\frac{(1+\delta)\,(\sigma+1)^2}{2\delta\alpha}\right)\|Vu\|^2 \\ + \left(\delta - \frac{\mid 1-\delta \mid \gamma}{2}\right)\left\|\left(\nabla^{+}\nabla\right)^3 u\right\|^2 \\ + \left((1+\delta) - \frac{(1+\delta)\,\delta\alpha}{2}\right)\left\|V^{1/2}\left(\nabla^{+}\nabla\right)^{3/2}u\right\|^2.\end{aligned}$$

Now the inequalities (4.3) and (4.4) holds if

$$|1-\delta| < \frac{2\delta}{\gamma}, \delta\alpha < 2 \text{ and } (1+\delta)\,(\sigma+1)^2 < 4. \tag{4.9}$$

Since, from $0 \leq \sigma < 1$, there exist $\delta > 0$, $\gamma > 0$ and $\alpha > 0$ such that the inequalities (4.10) hold. □

4.1. Proof of theorem 4.1

As M be a geodesically complete manifold it is known that $\left(L_V^d|_{C_c^\infty(M)}\right)^\sim$ in $L^2(M)$, be m-accretive and it coincides with $H_{2.V}^d$. Also, from the assumption on M, the operator $\left(L_V^{\nabla}|_{C_c^\infty(E)}\right)^\sim$ in $L^2(E)$, is m-accretive and it coincides with $H_{2.V}^{\nabla}$. Both of these statements are proven in [31]. From the strategy of Milatovic employs in [25], then the operator $L_V^{\nabla}|_{C_c^\infty(E)}$ is essentially self-adjoint and $\left(L_V^{\nabla}|_{C_c^\infty(E)}\right)^\sim = H_{2.V}^{\nabla}$. We prove (4.3) and (4.4) for all $u \in D_2^{\nabla} = Dom\left(H_{2.V}^{\nabla}\right)$, from which (4.2) follows directly. Since $H_{2.V}^{\nabla}$ is a closed operator, there exists a sequence $\{u_k\}$ in $C_c^\infty(E)$ such that $u_k \to u$ and $L_V^{\nabla}u_k \to H_{2.V}^{\nabla}u$ in $L^2(E)$, by the previous lemma the sequence $\{u_k\}$ satisfies (4.3) and (4.4), hence $\{(\nabla^{+}\nabla)^3 u_k\}, \{Vu_k\}$ and $\{V^{1/2}(\nabla^{+}\nabla)^{3/2}u_k\}$ are Cauchy sequences in the space $L^2(E)$. Furthermore, $\{\nabla u_k\}$ is a Cauchy sequence in $L^2(T^*M \otimes E)$ as

$$\|\nabla u_k\|^2 = (\nabla u_k, \nabla u_k) = \left(\nabla^{+}\nabla u_k, u_k\right) \leq \left\|\nabla^{+}\nabla u_k\right\|\|u_k\|,$$

and

$$\begin{aligned}\left\|\left(\nabla^{+}\nabla\right)^{3/2}u_k\right\|^2 &= \left(\left(\nabla^{+}\nabla\right)^{3/2}u_k, \left(\nabla^{+}\nabla\right)^{3/2}u_k\right) \\ &= \left(\left(\nabla^{+}\nabla\right)^3 u_k, u_k\right) \leq \left\|\left(\nabla^{+}\nabla\right)^3 u_k\right\|\|u_k\|.\end{aligned}$$

We will prove that $\left(\nabla^{+}\nabla\right)^3 u_k \to \left(\nabla^{+}\nabla\right)^3 u, Vu_k \to Vu$, $V^{1/2}\left(\nabla^{+}\nabla\right)^{3/2}u_k \to V^{1/2}\left(\nabla^{+}\nabla\right)^{3/2}u$, $\nabla u_k \to \nabla u$ and $\left(\nabla^{+}\nabla\right)^{3/2}u_k \to \left(\nabla^{+}\nabla\right)^{3/2}u$. As the essential self-adjointness of $\nabla^{+}\nabla|_{C_c^\infty(E)}$ and $\left(\nabla^{+}\nabla\right)^3|_{C_c^\infty(E)}$ we obtain $\left(\nabla^{+}\nabla\right)^{3/2}u_k \to \left(\nabla^{+}\nabla\right)^{3/2}u$ and $\left(\nabla^{+}\nabla\right)^3 u_k \to \left(\nabla^{+}\nabla\right)^3 u$ in $L^2(E)$. As $\{\nabla u_k\}$ is a Cauchy sequences in $L^2(T^*M \otimes E)$, it follows that ∇u_k convergent to some elements $z \in L^2(T^*M \otimes E)$, respectively then for all $\Psi \in C_c^\infty(T^*M \otimes E)$ we have $0 = (\nabla u_k, \Psi) - (u_k, \nabla^+ \Psi) \to (z, \Psi) - (u, \nabla^+ \Psi) = (z, \Psi) - (\nabla u, \Psi)$, as $Dom\left(H_{2.V}^{\nabla}\right) \subset W_{loc}^{2,2}(E)$ (see, Lemma 8.8 in [8]). With the convergence relations, we take the limit as $k \to \infty$ in all terms in (4.3) and (4.4) with u replaced by u_k then (4.3) and (4.4) hold for all $u \in D_2^{\nabla} = Dom\left(H_{2.V}^{\nabla}\right)$.

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