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Some multipliers of DBCK-algebras

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Abstract

The purpose of this document is to develop some of the basic theory of the multipliers algebra of dual BCK-algebras. In this study we demonstrate the concept of left bi-multiplier and right bi-multiplier of dual BCK (DBCK-algebra) algebras. Several examples and results pertaining to these multipliers on DBCK-algebras are developed based on these definitions. Then we study the characteristics of the bi-multipliers on DBCK-algebras and obtained some properties of DBCK-algebras. We focused on the behavior of the elements of DBCK-algebras under the concept of left bi-multiplier and right bi-multiplier of DBCK-algebras. We also characterize $Ker_a(X)$ and $Fix_a(X)$ by bi-multipliers on DBCK- algebras. We describe some elementary related properties of these sets.

Keywords: *Multipliers, DBCK-algebras, kernel, fixed set*

DBCK-cebirlerinin bazı çarpanları

Öz

Bu çalışmanın amacı DBCK-cebirlerinin çarpanlarının bazı temel teorilerini geliştirmektir. Bu çalışmada DBCK-cebirlerinin sol ikili-çarpanları ve sağ ikili çarpanları tanıtılmıştır. Bu tanımlardan yola çıkarak DBCK cebirlerinde bu çarpanlara ilişkin çeşitli örnekler ve sonuçlar geliştirilmiştir. Sonrasında DBCK-cebirleri üzerinde ikili çarpanların iligili karakteristik özellikleri çalışılmış ve bazı özellikleri elde edilmiştir. DBCK-cebirlerinin elemanlarının DBCK-cebirleri üzerinde sol ikili-çarpanları ve sağ ikili çarpanları altında görüntüleri çalışılmıştır. DBCK-cebirileri üzerinde $Ker_a(X)$ ve $Fix_a(X)$ kümeleri DBCK-cebirleri üzerinde ikili-çarpanlar aracılığı ile tanımlanmıştır. Bu kümelerin ilgili bazı temel özelliklerine yer verilmiştir.

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Anahtar kelimeler: *Çarpanlar, DBCK-cebirleri, çekirdek, sabit küme*

1. Introduction

The notion of *MV*-algebra was invented by Chang [1] to provide an algebraic proof of the completeness theorem of infinite valued Lukasiewicz propositional calculus. The algebraic theory of *MV*-algebas was deeply studied by [2-5]. The notion of *DBCK*-algebra which is an algebraic system having as models logical systems equipped with implication was introduced by K. H. Kim and Y. H. Yon [6] in 2007. They introduced some characteristics of dual BCK-algebras and MV -algebras, and proved that the MV -algebra is coincided to the bounded commutative dual BCK-algebra. It was also studied and generalized in [7].

A partial multiplier on a commutative semigroup $(A,.)$ was introduced by Larsen [8] as a function F from a nonvoid subset D_F of A into A such that $F(x).y=x. F(y)$ for all x, y in D_F. The concept of multiplier for distributive lattices was defined by Cornish [9]. For a distributive lattice multipliers are used to give a non standard construction of the maximal lattice of quotients [10]. In this study, we establish the notion of left bi-multiplier and right bi-multiplier of dual *BCK* (*DBCK*- algebra) algebras to study the properties of the bi-multipliers on *DBCK*-algebras. We characterize $Ker_a\ (X)$ and $Fix_a\ (X)$ by bi-multipliers on *DBCK* -algebras.

2. Preliminaries

Definition 2.1. [6] A *DBCK*-algebra is an algebraic system $(Y, \Rightarrow, 1)$ that has the following axioms for every $x, y, z \in Y$:

DBCK1. $(x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z)) = 1,$

DBCK2 $x \Rightarrow ((x \Rightarrow y) \Rightarrow y) = 1,$

DBCK3. $x \Rightarrow x = 1,$

DBCK4. $x \Rightarrow y = 1$ and $y \Rightarrow x = 1$ imply $x = y,$

DBCK5. $x \Rightarrow 1 = 1.$

A *DBCK*-algebra is a poset with the binary relation "$\leq$" defined by $x \leq y$ if and only if $x \Rightarrow y=1$, and 1 is the greatest element.

A (meet-) semilattice with a binary operation" $\Rightarrow$" that has the following axiom is called a *Heyting semilattice* (or *implicative semilattice*):

H. $z \wedge x \leq y$ if and only if $z \leq x \Rightarrow y.$

Proposition 2.2. [6] A *DBCK*-algebra Y satisfies the properties given below for every $x, y, z \in Y$:

(1) $x \Rightarrow (y \Rightarrow z) = y \Rightarrow (x \Rightarrow z),$

(2) $y \leq x \rightarrow y$

(3) $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$,

(4) $x \leq y \rightarrow z$ implies $y \leq x \rightarrow z$,

(5) $1 \rightarrow x = x$.

(6) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$

(7) $x \leq (x \rightarrow y) \rightarrow y$,

(8) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$,

(9) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$.

$(x \rightarrow y) \rightarrow y$ is an upper bound of x and y in a *DBCK*-algebra Ɣ by (2) and (7).

If there exists an element 0 in Ɣ where $0 \rightarrow x = 1$ for all $x \in$ Ɣ, a *DBCK*−algebra (Ɣ, $\rightarrow$, 1) is defined as to be *bounded.* The element $x \rightarrow 0$ will be denoted by $x^{\square}$ and $x^{\square\square} = (x^{\square})^{\square}$ for any element x in a bounded *DBCK*−algebra Ɣ. In implicative algebras, implications generate the complementation. Here $x \rightarrow 0 = x^{\square}$ means that $\rightarrow$ induces $^{\square}$.

Proposition 2.3. [6] A bounded *DBCK*-algebra has the properties given below $x, y \in$ Ɣ:

(1) $1^{\square} = 0$ and $0^{\square} = 1$,

(2) $x \leq x^{\square\square}$ and $x^{\square\square\square} = x^{\square}$,

(3) $x \rightarrow y \leq y^{\square} \rightarrow x^{\square}$,

(4) $x \leq y$ implies $y^{\square} \leq x^{\square}$,

(5) $x \rightarrow y^{\square} = y \rightarrow x^{\square}$.

A *DBCK*-algebra Ɣ is said to be *commutative* if it satisfies $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$ for every $x, y \in$ Ɣ.

Proposition 2.4. [6] A bounded commutative *DBCK*-algebra Ɣ has the properties given below for every $x, y \in$ Ɣ:

(1) Ɣ is a lattice with $x \vee y = (x \rightarrow y) \rightarrow y$ and $x \wedge y = (x^{\square} \vee y^{\square})^{\square}$,

(2) $x = x^{\square\square}$,

(3) $x \rightarrow y = y^{\square} \rightarrow x^{\square}$.

3. Multipliers of DBCK-algebras

Definition 3.1. Let Ɣ be a *DBCK*-algebra. A left bi-multiplier of Ɣ is a map $\varkappa$: Ɣ

$\times\ \mathcal{Y} \rightarrow \mathcal{Y}$ satisfying the properties given below: $\forall x, y, z \in \mathcal{Y}$,

(L1) $\varkappa\,(xz, y) = x\,\varkappa\,(z, y)$,

(L2) $\varkappa\,(x, y) = y\,\varkappa\,(x, 1)$.

A right bi-multiplier of $\mathcal{Y}$ is a map $\Psi\!: \mathcal{Y} \times \mathcal{Y} \rightarrow \mathcal{Y}$ satisfying the properties given below: $\forall x, y, z \in \mathcal{Y}$,

(R1) $\Psi(x, yz) = y\,\Psi(x, z)$,

(R2) $\Psi(x, y) = x\,\Psi(1, y)$.

Lemma 3.2. Let $\varkappa$ be a left-multiplier of a *DBCK*-algebra $\mathcal{Y}$. Then it satisfies the undermentioned for all $x, y, z \in \mathcal{Y}$:

(1) $\varkappa\,(1, y) = 1$,

(2) $x \leq \varkappa\,(x, y)$,

(3) $\varkappa\,(xz, y) = \varkappa\,(yz, x)$,

(4) $\varkappa\,(xy, y) = 1$,

(5) $\varkappa\,(x, x) = 1$.

Proof. (1) $\varkappa\,(1, y) = \varkappa\,(\varkappa\,(1, y)1, y) = \varkappa\,(1, y)\,\varkappa\,(1, y) = 1\ \forall y \in \mathcal{Y}$,

(2) $x\,\varkappa\,(x, y) = \varkappa\,(xx, y) = \varkappa\,(1, y) = 1\ \forall x, y \in \mathcal{Y}$, by (L1) and (1) of this lemma, hence $x \leq \varkappa\,(x, y)$.

(3) $\varkappa\,(xz, y) = y\,\varkappa\,(xz, 1) = y(x\,\varkappa\,(z, 1)) = y\,\varkappa\,(z, x) = \varkappa\,(yz, x)\ \forall x, y, z \in \mathcal{Y}$, by (L2) and (L1).

(4) $\varkappa\,(xy, y) = \varkappa\,(yy, x) = \varkappa\,(1, x) = 1\ \forall x, y \in \mathcal{Y}$, by (3) of this lemma.

(5) $\varkappa\,(x, x) = \varkappa\,(1x, x) = 1\ \forall x \in \mathcal{Y}$, by (4) of this lemma.

Lemma 3.3. Let $\varkappa$ be a right-multiplier of a *DBCK*-algebra $\mathcal{Y}$. Then it satisfies the undermentioned for all $x, y, z \in \mathcal{Y}$:

(1) $\varkappa\,(y, 1) = 1$,

(2) $y \leq \varkappa\,(x, y)$,

(3) $\varkappa\,(x, yz) = \varkappa\,(y, xz)$,

(4) $\varkappa\,(x, yx) = 1$,

(5) $\varkappa\,(x, x) = 1$.

Proof. (1) For all $y \in \mathcal{Y}$, $\varkappa\,(y, 1) = \varkappa\,(y, \varkappa\,(1, y)\,1) = \varkappa\,(y, 1)\,\varkappa\,(y, 1) = 1$.

(2) For all $x, y \in \mathcal{Y}$, $y\,\varkappa(x, y) = \varkappa(x, yy) = \varkappa(x, 1) = 1$ by (R1) and (1) of this lemma, hence $y \leq \varkappa(x, y)$.

(3) For all $x, y, z \in \mathcal{Y}$, $\varkappa(x, yz) = x\,\varkappa(1, yz) = x\,(y\,\varkappa(1, z)) = x\,\varkappa(y, z) = \varkappa(y, xz)$ by (R2) and (R1).

(4) For all $x, y \in \mathcal{Y}$, $\varkappa(x, yx) = \varkappa(yy, x) = \varkappa(1, x) = 1$ by (3) of this lemma.

(5) For all $x \in \mathcal{Y}$, $\varkappa(x, x) = \varkappa(1x, x) = 1$ by (4) of this lemma.

Example 3.4. (1) Let $\mathcal{Y}$ be a $DBCK$-algebra. If we define a map $\varkappa: \mathcal{Y} \times \mathcal{Y} \rightarrow \mathcal{Y}$ by

$$\varkappa(x, y) = a\,(yx)$$

for every $x, y \in \mathcal{Y}$ and some $a \in \mathcal{Y}$, then $\varkappa$ is a left-multiplier of $\mathcal{Y}$. In fact, we have $\varkappa(xy, z) = a\,(z\,(xy)) = a\,(x\,(zy)) = x\,(a\,(zy)) = x\,\varkappa(y,z)$, and

$\varkappa(x, y) = a\,(yx) = y\,(ax) = y\,(a\,(1x)) = y\,\varkappa(x, 1)$.

(2) Let $\mathcal{Y}$ be a $DBCK$-algebra. If we define a map $\varkappa: \mathcal{Y} \times \mathcal{Y} \rightarrow \mathcal{Y}$ by $\varkappa(x, y) = a\,(xy)$

for every $x, y \in \mathcal{Y}$ and some $a \in \mathcal{Y}$, then $\varkappa$ is a right-multiplier of $\mathcal{Y}$.

Definition 3.5. Let τ be a left-multiplier of a $DBCK$-algebra $\mathcal{Y}$. Fix $a \in \mathcal{Y}$ and define a set $F_a(\mathcal{Y})$ by $F_a(\mathcal{Y}) := \{x \in \mathcal{Y} \mid \tau(x, a) = x\}$ for all $x \in \mathcal{Y}$.

Lemma 3.6. Let τ be a left-multiplier of a $DBCK$-algebra $\mathcal{Y}$. If $x \in \mathcal{Y}$, $y \in F_a(\mathcal{Y})$ then $x \rightarrow y \in F_a(\mathcal{Y})$.

Proof. Let $x \in \mathcal{Y}$, $y \in F_a(\mathcal{Y})$ and by using the definition of left-multiplier of a $DBCK$-algebra $\mathcal{Y}$ we have $\tau(x \rightarrow y, a) = x \rightarrow \tau(y, a) = x \rightarrow y$. Therefore, $x \rightarrow y \in F_a(\mathcal{Y})$.

Lemma 3.7. Let $\varkappa$ be a left-multiplier of a $DBCK$-algebra $\mathcal{Y}$. If $y \in F_a(\mathcal{Y})$ then $x \vee y \in F_a(\mathcal{Y})$ for a bounded commutative $DBCK$-algebra $\mathcal{Y}$.

Proof. Let $y \in F_a(\mathcal{Y})$ and by using the definition of left-multiplier of a $DBCK$-algebra $\mathcal{Y}$ we have $\varkappa(x \vee y, a) = \varkappa((x \rightarrow y) \rightarrow y, a) = (x \rightarrow y) \rightarrow \varkappa(y, a) = (x \rightarrow y) \rightarrow y = x \vee y$. Therefore, $x \vee y \in F_a(\mathcal{Y})$.

For the rest of the study it is assumed that $\varkappa$ is a left-multiplier of a commutative DBCK- algebra Ɣ unless the contrary is mentioned.

Lemma 3.8. For any $x, y \in$ Ɣ, if $x \leq y$ and $x \in F_a$(Ɣ) then $y \in F_a$(Ɣ).

Proof. Let x, y be any elements in Ɣ where $x \leq y$ and $x \in F_a$(Ɣ) then

$\varkappa(y, a) = \varkappa(1 \Rightarrow y, a) = \varkappa((x \Rightarrow y) \Rightarrow y, a) = \varkappa((y \Rightarrow x) \Rightarrow x, a) = (y \Rightarrow x) \Rightarrow \varkappa(x, a) = (y \Rightarrow x) \Rightarrow x = y.$

So, $y \in F_a$(Ɣ). □

Let Ɣ be a *DBCK*-algebra. A nonempty subset A of Ɣ is said to be $\varkappa$-invariant if $\varkappa(A, A) \subseteq A$ where $\varkappa(A, A) = \{\varkappa(x, x) \mid x \in A\}$.

Lemma 3.9. Every filter Ʈ is $\varkappa$-invariant.

Proof. Let $y \in \varkappa$(Ʈ, Ʈ) then $y = \varkappa(x, z)$ for some $x, z \in$ Ʈ. We have $x \leq \varkappa(x, z)$ from Lemma 3.2 (2). So, $x^{\square} \rightarrow \varkappa(x, z) = 1$. Since Ʈ is a filter, we have $\varkappa(x, z) \subseteq$ Ʈ. So, $y \in$ Ʈ. Therefore, Ʈ is $\varkappa$-invariant.

Lemma 3.10. $\varkappa(x \vee y, z) = \varkappa(x, z) \vee \varkappa(y, z)$ and $\varkappa(x \wedge y, z) = \varkappa(x, z) \wedge \varkappa(y, z)$ for all $x, y, z \in$ Ɣ.

Proof. Let $x, y, z \in$ Ɣ.
In that case

$\varkappa(x \vee y, z) = \varkappa(x^{\square\square} \vee y^{\square\square}, z)$

$= \varkappa((x^{\square} \wedge y^{\square})^{\square}, z)$

$= \varkappa((x^{\square} \wedge y^{\square}) \Rightarrow 0, z)$

$= (x^{\square} \wedge y^{\square}) \Rightarrow \varkappa(0, z)$

$= (x^{\square} \Rightarrow \varkappa(0, z)) \vee (y^{\square} \Rightarrow \varkappa(0, z))$

$= \varkappa(x^{\square} \Rightarrow 0, z) \vee \varkappa(y^{\square} \Rightarrow 0, z)$

$= \varkappa(x^{\square\square}, z) \vee \varkappa(y^{\square\square}, z)$

$= \varkappa(x, z) \vee \varkappa(y, z)$

We can prove the case of meet in the similar way.

Lemma 3.11. $\varkappa$ is isotone left-multiplier, that is if $x_1 \leq x_2$, then $\varkappa(x_1, z) \leq \varkappa(x_2, z)$, for every $z \in Y$.

Proof. Let $x_1 \leq x_2$. Then $\varkappa(x_2, z) = \varkappa(x_1 \vee x_2, z) = \varkappa(x_1, z) \vee \varkappa(x_2, z)$ for all $z \in Y$. This implies that $\varkappa$ is isotone.

Definition 3.12. Let $\hbar$ be a left-multiplier of a *DBCK*-algebra Y. Fix $a \in Y$ and define a set $Ker_a(Y)$ by $Ker_a(Y) := \{x \in Y \mid \hbar(x, a) = 0\}$ for all $x \in Y$.

Lemma 3.13. Let $\varkappa$ be a left-multiplier of a *DBCK*-algebra Y. If $y \in Ker_a(Y)$ and $x \in Y$ then $x \wedge y \in Ker_a(Y)$.

Proof. Let $y \in Ker_a(Y)$. Then $\varkappa(x, a) = 0$.
$\varkappa(x \wedge y, a) = \varkappa(x, a) \wedge \varkappa(y, a) = 0 \wedge \varkappa(y, a) = (1 \vee [\varkappa(y, a)]'')'' = ((1 \Rightarrow [\varkappa(y, a)]'') \Rightarrow [\varkappa(y, a)]'')'' = 1'' = 0$ for all $x \in X$. This implies $x \wedge y \in Ker_a(Y)$.

Lemma 3.14. Let $\varkappa$ be a left-multiplier of a *DBCK*-algebra Y. If $x \in Ker_a(Y)$ and $y \leq x$ then $y \in Ker_a(Y)$.

Proof. Let $x \in Ker_a(Y)$ and $y \leq x$ Then $\varkappa(x, a) = 0$ and $y \Rightarrow x = 0$. Hence
$\varkappa(y, a) = \varkappa(x \wedge y, a) = \varkappa(y, a) \wedge \varkappa(x, a) = \varkappa(y, a) \wedge 0 = (\varkappa(y, a)'' \vee 1)'' = ((\varkappa(y,a)'' \Rightarrow 1) \Rightarrow 1)'' = 0$.
This implies $y \in Ker_a(Y)$.

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